Matrix characterizations of Lipschitz operators on Banach spaces over Krull valued fields

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Abstract

Let K be a complete infinite rank valued field and E a K-Banach space with a countable orthogonal base. In [9] and [10] we have studied bounded (called Lipschitz) operators on E and introduced the notion of a strictly Lipschitz operator. Here we characterize them, as well as compact and nuclear operators, in terms of their (infinite) matrices. This results provide new insights and also useful criteria for constructing operators with given properties.

Introduction

Functional Analysis over fields other than $\mathbb{R}$ or $\mathbb{C}$ has been developing for quite some time. First $\mathbb{R}$ and $\mathbb{C}$ have been replaced by a field K with a non-archimedean valuation $|\ |\colon\ K \to [0,\infty) \subset \mathbb{R}$ satisfying (i) $|\lambda| = 0$ if and only if $\lambda = 0$, (ii) $|\lambda\mu| = |\lambda||\mu|$, (iii) $|\lambda + \mu| \leq \max\,(|\lambda|, |\mu|)$. See [11] for a good background account on this.

In the so called strong triangle inequality (iii) addition of real numbers no longer plays a role, so that only ordering and multiplication are used. This observation leads to the introduction of the 'Krull valued fields K'. They have a valuation $|\ |\colon\ K \to G \cup \{0\}$, where G is an arbitrary linearly ordered multiplicative group augmented with a smallest element called 0. In doing this, finer structures of the order of G appear. As a consequence it may happen that for $\lambda \in K$, $|\lambda| \leq 1$ the sequence $1, \lambda, \lambda^2, \ldots$ does not tend to 0!

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A norm $\| \ \|$ on a K-vector space E should satisfy

(i) $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|\lambda x\| = |\lambda|\|x\|$,

(iii) $\|x + y\| \leq \max\,(\|x\|, \|y\|)$.

These requirements make sense if we ask the range of the norm to be in $G \cup \{0\}$ but that turns out to be too restrictive ([3] and [4]). The essential features of a suitable 'home' for norm values were captured by the concept of a G-module (introduced in [7]), a linearly ordered set with an action of G on it, see also Section 1.

In this setup one can define Banach spaces and analogues of Hilbert spaces in a natural way [7]. The theory of linear operators on spaces over Krull valued fields differs markedly from the classical one as well as from the theory in [11]. For example, a continuous linear operator may not be bounded ([9] 2.1.13), there exist self-adjoint operators without proper closed invariant subspaces (refer to [5]), in a wide class of spaces linear isometries are surjective (see [12]), and so on.

In this paper we study in Section 3 the bounded (= Lipschitz) operators on Banach spaces with a countable orthogonal base, and characterize them and express their norms in terms of their matrices. The results are not only interesting in their own right but can generate concrete (counter) examples of operators having certain desired properties, and provide useful tests to decide whether an operator is bounded or not.

In the course of this development the new notion of a strictly Lipschitz operator A ($\|Ax\| < g\|x\|$ for all non-zero x) appeared naturally. Not every Lipschitz operator is strictly Lipschitz, a striking fact! ([9] 2.1.17). We characterize the strictly Lipschitz operators in the above spirit as well as the compact and nuclear operators. Surprinsigly the trace function defined for nuclear operators is in general not Lipschitz continuous but is strictly Lipschitz continuous.

For the reader's convenience we summarize in Sections 1 and 2, with complete references, the needed and already established results.

1 Preliminaries

Throughout G is a linearly ordered abelian group, written multiplicatively, with unit 1. A subset $H \subset G$ is called *convex* if $g_1, g_2 \in H,\ g_1 \leq g_2$ implies $\{x \in G : g_1 \leq x \leq g_2\} \subset H$. We shall assume everywhere that G is the union of a strictly increasing sequence of convex subgroups, so G has a cofinal sequence. We augment G with an element 0 and define $0 < g$, $0 \cdot g :=\ 0 \cdot 0 :=\ 0$ for all $g \in G$. A **Krull valuation** on a field K with **value group** G is a surjective map $| \ | : K \to G \cup \{0\}$ such that, for all $\lambda, \mu \in K$ (i) $|\lambda| = 0$ if and only if $\lambda = 0$ (ii) $|\lambda + \mu| \leq \max(|\lambda|, |\mu|)$ (iii) $|\lambda\mu| = |\lambda||\mu|$. We denote the Dedekind completion of G by $G^\#$.

From now on in this paper K is a field with a Krull valuation $| \ |$ and value group G, therefore metrizable. We also assume that $(K, | \ |)$ is complete with respect to the valuation.

Inspired by the terminology in rank 1 valuation theory we introduce the following.

G is called **quasidiscrete** if $\min\{g \in G : g > 1\}$ exists; otherwise G is called **quasidense**.

From [7] 1.1.1 it follows that G is quasidense if and only if $\inf\{g \in G : g > 1\} = 1$.

Remark. The reason for using the prefix 'quasi' lies in the fact that, contrary to the rank 1 case, a quasidiscrete group may have quasidense subgroups.

In order to define norms on K-vector spaces we introduce the following important notion. A linearly ordered set X is called a G-**module** if there exists an action of G on X, written $(g, x) \mapsto gx$ and called **multiplication**, such that for all $g, h \in G$ and all $x, y \in X$ we have that $g \geq h$ and $x \geq y$ imply $gx \geq hy$ and also that there is a $j \in G$ with $jx < y$.

Remark. It is clear that G itself is a G-module, with the action defined by group multiplication.

Let X be a G-module augmented with a smallest element also denoted by 0 (see [7] Section 2 for details), let E be a K-vector space. An X-**norm** on E is a map $\| \, \| : E \to X \cup \{0\}$ such that for all $x, y \in E, \lambda \in K$, (i) $\|x\| = 0$ if and only if $x = 0$ (ii) $\|\lambda x\| = |\lambda| \, \|x\|$ (iii) $\|x + y\| \leq \max(\|x\|, \|y\|)$.

1.1 Topological types

We shall study with greater detail the relationship between the group G and a G-module X.

The action of G on X yields a partition of X into orbits; if $s \in X$ then its orbit $Gs = \{gs : g \in G\}$ will be called the **algebraic type** of s. The **stabilizer** of s, $\mathrm{Stab}(s) := \{g \in G : gs = s\}$, is a convex subgroup of G.

Definition 1.1.1 $s \in X$ is called **faithful** if $\mathrm{Stab}(s) = \{1\}$. If each element of X is faithful then X is called **faithful**.
X is called **almost faithful** if there is a proper convex subgroup H of G such that $\mathrm{Stab}(s) \subset H$ for each $s \in X$.

We shall denote the Dedekind completion of the G-module X by $X^{\#}$.

Fix an element $t \in X$. It is easy to see that the subsets $L := \{gs : gs \leq t\}$ and $U := \{gs : gs \geq t\}$ of Gs are not empty (see [7] 1.5.1 (v)). For each $s \in X$, let $\tau_l(s) := \sup_{X^{\#}} L$ and $\tau_u(s) := \inf_{X^{\#}} U$. Clearly $\tau_l(s) \leq \tau_u(s)$ and frequently the inequality is a strict one.

Clearly if $h = 1 \in G$ we have that $\tau_l(s) \leq ht \leq \tau_u(s)$. It turns out that a crucial object in this study is the subset of G defined by such a property.

Definition 1.1.2 ([7] 1.6.1) Let $s \in X$. Its **topological type** with respect to $t \in X$ is

$$\tau(s;t) := \{h \in G : \tau_l(s) \leq ht \leq \tau_u(s)\},$$

A useful characterization is given in the next Proposition.

Proposition 1.1.3 *Let $s, t \in X$. Then if $s \in Gt$ then $\tau(s;t) = \mathrm{Stab}(s) = \mathrm{Stab}(t)$, otherwise $\tau(s;t)$ is the largest convex subgroup H of G for which* $\mathrm{conv}_X(Ht) \cap Gs = \varnothing$, where $\mathrm{conv}_X(Ht) := \{x \in X : h_1 t \leq x \leq h_2 t$ *for some* $h_1,\ h_2 \in H\}$.

Proof. See [7] 1.6.2 and [9] 1.5.2.

Now we study the topological types $\tau(s;t)$ and $\tau(t;s)$ as a function of two variables.

Theorem 1.1.4 *Let $s, t, u \in X$.*

(i) *If $s' \in Gs,\ t' \in Gt$ then $\tau(s;t) = \tau(s';t')$.*

(ii) $\mathrm{Stab}(s) \cup \mathrm{Stab}(t) \subset \tau(s;t)$.

(iii) $\tau(s;t) = \tau(t;s)$.

(iv) $\tau(s;u) \subset \tau(s;t) \cup \tau(t;u)$.

(v) *If $\tau(s;t) \neq \tau(t;u)$ then $\tau(s;u) = \tau(s;t) \cup \tau(t;u)$.*

Proof. See [9] 1.5.3.

Theorem 1.1.5 *Let $s, t \in X$. Set*

$$u := \inf_{G^\#}\{g \in G : s \leq gt\} \in G^\#$$

$$u^\sim := \inf_{G^\#}\{g \in G : s < gt\} \in G^\#.$$

Then $\mathrm{Stab}(u) = \mathrm{Stab}(u^\sim) = \tau(t;s)$.

Proof. See [9] 1.5.4.

Our assumptions on G make the following concepts interesting.

Definition 1.1.6 ([7] 1.6.4) Let X be a G module, let $s_1, s_2, \ldots$ be a sequence in X.

(i) We say that $s_1, s_2, \ldots$ satisfies the **type condition** if, for any sequence $g_1, g_2, \ldots$ in G, boundedness above of $\{g_1 s_1, g_2 s_2, \ldots\}$ implies $\lim_{n\to\infty} g_n s_n = 0$.

(ii) Let t be an element of X. We say that $\lim_{n\to\infty}\tau(s_n, t) = \infty$ if for each proper convex subgroup H of G we have $\tau(s_n, t) \supsetneqq H$ for large n.

The link between (i) and (ii) above is given by the next Proposition.

Proposition 1.1.7 *Let X be a G-module. Then, for a sequence $s_1, s_2, \ldots$ in X the following are equivalent:*

(α) *$s_1, s_2, \ldots$ satisfies the type condition.*

(β) For any $t \in X$, $\lim_{n\to\infty}\tau(s_n;t) = \infty$.

Proof. See [7] 1.6.6.

1.2 The extension of the operations

Let the group G and the G-module X be embedded in their completions $G^{\#}$ and $X^{\#}$ respectively. We want to define:

a) a multiplication in $G^{\#}$ that extends the multiplication of G, making $G^{\#}$ into a semigroup,

b) an action of G on $X^{\#}$ that extends the action of G on X.

We can cover both cases by the more general procedure of extending the action $G \times X \to X$ at once to a multiplication $G^{\#} \times X^{\#} \to X^{\#}$ as follows.

Definition 1.2.1 For $s \in G^{\#}$, $r \in X^{\#}$ we set
$$s * r := \inf_{X^{\#}}\{gu : g \in G,\ u \in X,\ g \geq s,\ u \geq r\}.$$
In the case $g \in G$ and $r \in X^{\#}$ we shall write gr instead of $g * r$. Therefore,
$$gr = \inf_{X^{\#}}\{gu : u \in X,\ u \geq r\}.$$

Remarks.

1. By [9] 1.1.3 we have that $s * r = \inf_{X^{\#}}\{gr : g \in G,\ g \geq s\}$.
2. The map $(s, r) \mapsto s * r$ is increasing in both variables.
3. The map $(g, r) \mapsto gr$, $g \in G$, $r \in X^{\#}$ defines a natural G-module structure on $X^{\#}$.

By the last remark we have the important fact that $G^{\#}$ is a G-module. In this case the topological type of an element $s \in G^{\#}$ with respect to $t := 1$ is particularly simple, in fact $\tau(s; 1) = \mathrm{Stab}(s)$, see [12] 3.1.

1.3 Continuous G-modules

In this section we sum up the results of [9] Section 1.6.

Let X be a G-module embedded in its completion $X^{\#}$, let $r \in X$.

Definition 1.3.1 We say that X is **continuous at** $r \in X$ if for every $W \subset G$ for which $\inf_G W$ exists we have
$$(\inf_G W)r = \inf_X Wr.$$

By saying that X is continuous we mean that X is continuous at each $r \in X$.

The dual property is also true.

Proposition 1.3.2 *If X is continuous then for every $r \in X$ and for every $W \subset G$ for which $\sup_G W$ exists we have*
$$(\sup_G W)r = \sup_X Wr.$$

Proof. [9] 1.6.2.

See [9] 1.6, 1.7 for an example of a non-continuous G-module and also for basic facts on continuous G-modules, from which we need here only the following facts.

Proposition 1.3.3

(i) *If G is quasidiscrete, then each G-module X is continuous.*

(ii) *The completion of a continuous G-module is again continuous. In particular $G^{\#}$ is continuous.*

2 Spaces of continuous linear maps

Here we summarize the operator theory of [9] Chapter 2,3 inasmuch as necessary for our purposes.
IN THIS SECTION X IS A G-MODULE AND E, F ARE X-NORMED SPACES OVER K.

The set of all continuous linear operators $E \to F$ is denoted by $L(E, F)$. We write $L(E) := L(E, E)$ and $E' := L(E, K)$. Under pointwise addition and scalar multiplication the set $L(E, F)$ is a K-vector space. In addition, the space $L(E)$ is a K-algebra under composition as multiplication with the identity map I as a unit.

2.1 Lipschitz and strictly Lipschitz operators

Definition 2.1.1 A linear operator $A : E \to F$ is called **Lipschitz** (or, in most literature, **bounded**) if there is a $g \in G$ such that $\|Ax\| \leq g\|x\|$ for all $x \in E$. The set of all such Lipschitz operators is denoted by $Lip(E, F)$.
A linear operator $A : E \to F$ is called **strictly Lipschitz** if there is a $g \in G$ such that $\|Ax\| < g\|x\|$ for all nonzero $x \in E$. The set of all such strictly Lipschitz operators is denoted by $Lip^{\sim}(E, F)$. We write $Lip(E) := Lip(E, E)$ and $Lip^{\sim}(E) := Lip^{\sim}(E, E)$.

Remark. Under pointwise addition and scalar multiplication the set $Lip(E, F)$ is a K-vector space having $Lip^{\sim}(E, F)$ as a subspace. In addition $Lip(E)$ is a subalgebra of $L(E)$ with unit I. It is easily seen that $Lip^{\sim}(E)$ is a two-sided ideal in $Lip(E)$.

Next we introduce natural norms on $Lip(E, F)$ and $Lip^{\sim}(E, F)$.

Definition 2.1.2 For $A \in Lip(E, F)$ put $\Gamma_A := \{g \in G : \|Ax\| \leq g\|x\| \text{ for all } x \in E\}$ and $\|A\| := \inf_{G^{\#} \cup \{0\}} \Gamma_A$. Similarly, for $B \in Lip^{\sim}(E, F)$ put $\Gamma_B^{\sim} := \{g \in G : \|Bx\| < g\|x\| \text{ for all nonzero } x \in E\}$ and $\|B\|^{\sim} := \inf_{G^{\#} \cup \{0\}} \Gamma_B^{\sim}$. We call $\| \ \|$ the **Lipschitz norm** and $\| \ \|^{\sim}$ the **strict Lipschitz norm**.

Convention. To avoid complicated notations we will write henceforth, for a subset V of G, $\inf V$ in place of $\inf_{G^{\#} \cup \{0\}} V$; similarly for sup.

Proposition 2.1.3 *Let* $A \in Lip(E,F)$, $B \in Lip^{\sim}(E,F)$. *Then* $\|A\|, \|B\|^{\sim} \in G^{\#} \cup \{0\}$, $\|B\| \leq \|B\|^{\sim}$ *and*

(i) $\{g \in G : \ g > \|A\|\} \subset \Gamma_A \subset \{g \in G : \ g \geq \|A\|\}$,

(ii) $\{g \in G : \ g > \|B\|^{\sim} \subset \Gamma_B^{\sim} \subset \{g \in G : \ g \geq \|B\|^{\sim}\}$.

Proof. Straightforward.

Proposition 2.1.4 $\| \ \|$ *is a norm on* $Lip(E,F)$; $\| \ \|^{\sim}$ *is a norm on* $Lip^{\sim}(E,F)$. *If* F *is a Banach space then* $(Lip(E,F), \ \| \ \|)$ *and* $(Lip^{\sim}(E,F), \ \| \ \|^{\sim})$ *are Banach spaces.*

Proof. [9] 2.2.4 and 2.2.5.

Proposition 2.1.5 *Let* $A, B \in Lip(E)$. *Then*

$$\|AB\| \leq \|A\| * \|B\|.$$

Let $A \in Lip^{\sim}(E), B \in Lip(E)$. *Then*

$$\max(\|AB\|^{\sim}, \|BA\|^{\sim}) \leq \|A\|^{\sim} * \|B\|.$$

If, in addition, X *is continuous and complete, then, for each* $A \in Lip(E,F)$ *we have*

$$\|Ax\| \leq \|A\| * \|x\| \ (x \in E)$$

Proof. [9] 2.2.16 and 2.2.17.

Remark. From Proposition 2.1.4 we infer that if E is a Banach space then so is $Lip(E)$ and the inequality $\|AB\| \leq \|A\| * \|B\|$ of Proposition 2.1.5 shows that $Lip(E)$ deserves the qualification 'Banach algebra'.

Definition 2.1.6 A sequence $e_1, e_2, \ldots$ in a normed space is called an **orthogonal base** if for every $x \in E$ there are $\lambda_1, \lambda_2, \ldots \in K$ such that $x = \sum_{n=1}^{\infty} \lambda_n e_n$ and $\|x\| = \max_n \|\lambda_n e_n\|$.

It follows easily that the above representation is unique. In the same spirit we have the notions of an **orthogonal set** ([7] 2.4.7), an **orthogonal complement** ([7] 2.4.2), and an **orthogonal projection** ([7] 2.4.2).

Proposition 2.1.7 *Let* E *have an orthogonal base* $e_1, e_2, \ldots$. *Then for* $A \in Lip(E,F)$, $B \in Lip^{\sim}(E,F)$ *we have*

$$\|A\| = \inf\{g \in G : \|Ae_n\| \leq g\|e_n\| \text{ for each } n\}$$

$$\|B\|^{\sim} = \inf\{g \in G : \|Be_n\| < g\|e_n\| \text{ for each } n\}.$$

Conversely, let $g \in G$ *and* $y_1, y_2, \ldots \in F$ *such that* $\|y_n\| \leq g\|e_n\|$ (*resp.* $\|y_n\| < g\|e_n\|$) *for all* n. *Then* $e_n \mapsto y_n$ $(n \in \mathbb{N})$ *extends uniquely to a Lipschitz operator* (*resp. strictly Lipschitz operator*) $E \to F$.

Proof. Straightforward.

2.2 The trace function and compact operators

We start this section by considering finite rank operators. The definition is classical.

Definition 2.2.1 An element $A \in L(E,F)$ is said to be **of finite rank** if AE is finite-dimensional. The set of all such finite rank operators is denoted by $FR(E,F)$. We write $FR(E) := FR(E,E)$.

Remark. Under pointwise addition and scalar multiplication the set $FR(E,F)$ is a subspace of $L(E,F)$. In addition, it is easily seen that $FR(E)$ is a two-sided ideal in the K-algebra $L(E)$. In [9] 2.1.3 even the following Proposition is proved.

Proposition 2.2.2 *$FR(E,F) \subset Lip^{\sim}(E,F)$. In particular, if F is one-dimensional each continuous linear map $E \to F$ is strictly Lipschitz.*

The construction of the trace function below is well-known but we include it here for reference.

For $f \in E',\ a \in E$ let $A_{f,a}(x) := f(x)a\ (x \in E)$. The map $E' \times E \to FR(E)$ given by $(f,a) \mapsto A_{f,a}$ is bilinear, so by the universal property of the tensor product it induces a linear map $\varphi : E' \otimes E \to FR(E)$ which is in fact, a bijection.

The bilinear map $E' \times E \to K$ given by $(f,a) \mapsto f(a)$ induces a linear map $\tau : E' \otimes E \to K$ given by $\tau(f \otimes a) = f(a)\ (f \in E',\ a \in E)$.

Definition 2.2.3 For $A \in FR(E)$ let $\mathrm{tr}(A) :=\ (\tau \circ \varphi^{-1})(A)$.

We have the usual properties.

Proposition 2.2.4

(i) tr *is a linear map $FR(E) \to K$.*

(ii) *If $A \in FR(E)$ and*
$Ax = \sum_{i=1}^{n} f_i(x)a_i \quad (x \in E)\ (f_1,\dots,f_n \in E',\ a_1,\dots,a_n \in E)$
is any representation of A then $\mathrm{tr}(A) = \sum_{i=1}^{n} f_i(a_i)$.

(iii) *For $A \in FR(E)$, $B \in L(E)$,* $\quad \mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Proof. See [9] 2.1.23.

We now consider continuity properties of the trace.

From now on E is a normed Banach space with an orthogonal base.

Theorem 2.2.5 *For each $A \in FR(E)$ we have $|\mathrm{tr}(A)| \le \|A\|^{\sim}$.*

Proof. [9] 2.4.1.

The trace may not be Lipschitz continuous, as can be seen from the next result (see [9] 2.4.2).

Theorem 2.2.6 *The trace function* $\mathrm{tr} : FR(E) \to K$ *is Lipschitz continuous if and only if $\|E\| \setminus \{0\}$ is almost faithful .*

We now introduce compact operators. For a subset V of $Lip(E)$ we denote by $\overline{V}$ its closure with respect to $\| \,\|$. For a subset W of $Lip^{\sim}(E)$, let $\overline{W}^{\sim}$ be its closure with respect to $\| \,\|^{\sim}$.

Definition 2.2.7 Let $C(E) := \overline{FR(E)}$. An element of $C(E)$ is called **compact** (supercompact in [8] 3.3). Similarly, let $C^{\sim}(E) := \overline{FR(E)}^{\sim}$. An element of $C^{\sim}(E)$ is called **nuclear** or **of trace class**.

Clearly $C^{\sim}(E) \subset C(E)$, $C(E)$ is a two-sided ideal in $Lip(E)$, $C^{\sim}(E)$ is a two-sided ideal in $Lip^{\sim}(E)$. Applying Theorem 2.2.5 the trace function in $FR(E)$ can uniquely be extended to a continuous linear function, again denoted tr, in $C^{\sim}(E)$ and we have $|\mathrm{tr}(A)| \leq \|A\|^{\sim}$ for all $A \in C^{\sim}(E)$.

Proposition 2.2.8

(i) *$C^{\sim}(E)$ is a two-sided ideal in $Lip(E)$.*

(ii) *If $A \in C^{\sim}(E)$, $B \in Lip(E)$ then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.*

(iii) *Let $A \in C(E)$, $B \in Lip^{\sim}(E)$. Then AB and BA are in $C^{\sim}(E)$, and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.*

(iv) *$Lip^{\sim}(E)$ is dense in $Lip(E)$.*

(v) *$C^{\sim}(E)$ is dense in $C(E)$.*

Proof [9] 2.4.4, 2.4.5 and 3.2.1.

3 Matrix characterizations of operators on spaces with an orthogonal base

THROUGHOUT THIS SECTION X WILL BE A G-MODULE AND E AN X-NORMED BANACH SPACE WITH AN ORTHOGONAL BASE $e_1, e_2, \ldots$.

3.1 Matrix characterizations.

Each $A \in Lip(E)$ has a matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \cdots & \\ \vdots & & \end{pmatrix}$$

with respect to the given base $e_1, e_2, \ldots$. Of special interest are the 'building blocks' P_{mn} $(m, n \in \mathbb{N})$ given by the formula

$$P_{mn}(e_k) = \delta_{kn} e_m \quad (k \in \mathbb{N}).$$

Clearly $P_{mn} \in FR(E)$ and its matrix has zero entries except for a one in the nth column and the mth row. With this in mind it is natural to compare A with $\{a_{mn} P_{mn} : m, n \in \mathbb{N}\}$.

Lemma 3.1.1

(i) *For each $m, n \in \mathbb{N}$ we have*

$$\|P_{mn}\| = \inf\{g \in G : \ \|e_m\| \leq g\|e_n\|\}$$

$$\|P_{mn}\|^{\sim} = \inf\{g \in G : \ \|e_m\| < g\|e_n\|\}.$$

(ii) $\{P_{mn} : \ m, n \in \mathbb{N}\}$ *is an orthogonal set with respect to* $\| \ \|$ *and* $\| \ \|^{\sim}$.

Proof. (i) By Proposition 2.1.7

$$\|P_{mn}\| = \inf \{g \in G : \ \|P_{mn}(e_k)\| \leq g\|e_k\| \text{ for all } k \in \mathbb{N}\}.$$

Now $P_{mn}(e_k) = 0$ for $k \neq n$ so we get

$$\|P_{mn}\| = \inf \{g \in G : \ \|e_m\| \leq g\|e_n\|\}.$$

The formula for $\|P_{mn}\|^{\sim}$ is proved in the same way.
(ii) Let $A := \sum_{m,n=1}^{k} \lambda_{mn} P_{mn}$ be a finite linear combination of the P_{mn}. Let $g \in \Gamma_A$ (resp. $g \in \Gamma_A^{\sim}$); we show that for $m, n \in \mathbb{N}$, $\|\lambda_{mn} P_{mn}\| \leq g$ (resp. $\|\lambda_{mn} P_{mn}\|^{\sim} \leq g$). To this end we may assume $\lambda_{mn} \neq 0$. We have $g\|e_n\| \geq$ (resp. $>$) $\|Ae_n\| = \|\sum_{i,j=1}^{k} \lambda_{ij} P_{ij}(e_n)\| = \|\sum_{i=1}^{k} \lambda_{in} P_{in}(e_n)\| = \|\sum_{i=1}^{k} \lambda_{in} e_i\| \geq \|\lambda_{mn} e_m\|$. Hence $\|e_m\| \leq$ (resp. $<$) $|\lambda_{mn}^{-1}| \, g \, \|e_n\|$, showing that $|\lambda_{mn}^{-1}| \, g \geq \|P_{mn}\|$ (resp. $\|P_{mn}\|^{\sim}$) and we are done.

Corollary 3.1.2

(i) *For each m, n we have*

$$\mathrm{Stab}(\|P_{mn}\|^{\sim}) = \mathrm{Stab}(\|P_{mn}\|) = \tau(\|e_n\|; \|e_m\|).$$

(ii) *For each n we have*

$$\|P_{nn}\| = \inf \mathrm{Stab}(\|e_n\|), \ \|P_{nn}\|^{\sim} = \sup \mathrm{Stab}(\|e_n\|).$$

(iii) *Let $A \in Lip(E)$ have the matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & \dots & \\ \vdots & & \end{pmatrix}$$

with respect to $e_1, e_2, \ldots$. Then

$$A \in \overline{[P_{mn} : m, n \in \mathbb{N}]} \ \textit{if and only if} \ \lim_{m+n \to \infty} \|a_{mn} P_{mn}\| = 0;$$
$$A \in \overline{[P_{mn} : m, n \in \mathbb{N}]}^{\sim} \ \textit{if and only if} \ \lim_{m+n \to \infty} \|a_{mn} P_{mn}\|^{\sim} = 0.$$

Proof. Straightforward (for (i) use Theorem 1.1.5).

Lemma 3.1.3 *Let $a_{mn} \in K$ $(m, n \in \mathbb{N})$. The following are equivalent.*

(α) *For each n,* $\lim_{m\to\infty} \quad a_{mn}e_m = 0.$

(β) *For each n,* $\lim_{m\to\infty} \quad \|a_{mn}P_{mn}\| = 0.$

(γ) *For each n,* $\lim_{m\to\infty} \quad \|a_{mn}P_{mn}\|^\sim = 0.$

Proof. $(\alpha) \Rightarrow (\gamma)$. Let $n \in \mathbb{N}$, $g \in G$. There is an m_0 such that for $m \geq m_0$ we have $\|a_{mn}e_m\| < g\|e_n\|$ i.e. $\|a_{mn}P_{mn}(e_n)\| < g\|e_n\|$. Thus we have $\|(a_{mn}P_{mn})(e_j)\| < g\|e_j\|$ for each j, each $m \geq m_0$. It follows that$\|a_{mn}P_{mn}\|^\sim \leq g$ for $m \geq m_0$; in other words we proved (γ). The implication $(\gamma) \Rightarrow (\beta)$ is trivial, so we prove $(\beta) \Rightarrow (\alpha)$. Let $n \in \mathbb{N}$, $\varepsilon \in X$. Choose a $g \in G$ with $g\|e_n\| < \varepsilon$. There is an m_0 such that for $m \geq m_0$ we have $\|a_{mn}P_{mn}\| < g$. Then, by Proposition 2.1.3 (i), $\|a_{mn}P_{mn}(x)\| \leq g\|x\|$ for all $x \in E$ and $m \geq m_0$. By taking $x = e_n$ we find $\|a_{mn}e_m\| \leq g\|e_n\| < \varepsilon$ for $m \geq m_0$ and we are done.

Theorem 3.1.4 (*Characterization of Lipschitz operators by matrices*)

(i) *Let $A \in Lip(E)$ have the matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \cdots & \\ \vdots & & \end{pmatrix}$$

with respect to $e_1, e_2, \ldots$. Then, for each n, $\lim_{m\to\infty}\|a_{mn}P_{mn}\| = 0$ *and*

$$\|A\| = \sup\{\|a_{mn}P_{mn}\| : m, n \in \mathbb{N}\}.$$

(ii) *Conversely, let*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \cdots & \\ \vdots & & \end{pmatrix}$$

be a matrix with entries in K, such that, for each n, $\lim_{m\to\infty}\|a_{mn}P_{mn}\| = 0$ *and such that $(m, n) \mapsto \|a_{mn}P_{mn}\|$ is bounded above. Then the matrix represents a Lipschitz operator.*

Proof. (i) Let $n \in \mathbb{N}$. Then $Ae_n = \sum_{m=1}^\infty a_{mn}e_m$, so $\lim_{m\to\infty}\|a_{mn}e_m\| = 0$, so by the previous Lemma we have $\lim_{m\to\infty}\|a_{mn}P_{mn}\| = 0$. Next we prove $\|a_{mn}P_{mn}\| \leq \|A\|$ for each $m, n \in \mathbb{N}$. Let $g \in \Gamma_A$. We have $\|a_{mn}e_m\| \leq \|\sum_{j=1}^\infty a_{jn}e_j\| = \|Ae_n\| \leq g\|e_n\|$. Thus (assuming $a_{mn} \neq 0$) $\|e_m\| \leq |a_{mn}|^{-1}g\|e_n\|$, so, by Lemma 3.1.1, $\|P_{mn}\| \leq |a_{mn}|^{-1}g$.
To complete the proof of (i) we suppose that

$$s := \sup\{\|a_{mn}P_{mn}\| : m, n \in \mathbb{N}\} < \|A\|$$

and derive a contradiction. (The proof looks rather overnice; we would welcome proposals for a more direct proof.) First assume that X is continuous. (Then $\Gamma_B = \{g \in G : g \geq \|B\|\}$ for each $B \in Lip(E)$.) There is a $g \in G$ with $s \leq g < \|A\|$. Thus $g \notin \Gamma_A$ so there is an n such that $\|Ae_n\| > g\|e_n\|$ and since $Ae_n = \sum_{j=1}^{\infty} a_{jn}e_j$ there is an m such that $\|a_{mn}e_m\| > g\|e_n\|$. So $|a_{mn}|^{-1}g \notin \Gamma_{P_{mn}}$ and by assumption $|a_{mn}|^{-1}g < \|P_{mn}\|$ or $g < |a_{mn}|\ \|P_{mn}\|$ conflicting $g \geq s$.
Now suppose that X is not continuous. Then G is quasidense (Proposition 1.3.3 (i)) which implies the existence of a $g \in G$ such that $s < g < \|A\|$ (quasidenseness is used when $s, \|A\| \in G$.) By the same reasoning as above we find m, n such that $|a_{mn}|^{-1}g \notin \Gamma_{P_{mn}}$, so $g \notin \Gamma_{a_{mn}P_{mn}}$ conflicting $g > s$.
(ii) Let $x \in E$ have expansion $\sum_{n=1}^{\infty} \xi_n e_n$. Set

$$t_{mn} := \xi_n a_{mn} e_m \quad (m, n \in \mathbb{N}).$$

We have 1 and 2 below.

1. For each n, $\lim_{m\to\infty} a_{mn}e_m = 0$ (Lemma 3.1.3) so that $\lim_{m\to\infty} t_{mn} = 0$ for each n.

2. Let $g \in G,\ g > \|a_{mn}P_{mn}\|$ for each m, n. Then $g \in \Gamma_{a_{mn}P_{mn}}$, so $\|t_{mn}\| = \|\xi_n a_{mn} P_{mn}(e_n)\| \leq \|\xi_n e_n\| g$, so

$$\lim_{n\to\infty} t_{mn} = 0 \quad \text{uniformly in } m.$$

Together 1 and 2 imply unconditional summability of t_{mn}, so the formula

$$Ax = \sum_{n=1}^{\infty} \xi_n \sum_{m=1}^{\infty} a_{mn} e_m$$

defines a map $A : E \to E$. Direct verification tells that A is linear and that its matrix is the given one. To see that A is Lipschitz, let $g \in G$ be as in 2, and $x \in E$. Then

$$\|Ax\| \leq \sup\{\|t_{mn}\| : \ m, n \in \mathbb{N}\} \leq g \max\{\|\xi_n e_n\| : \ n \in \mathbb{N}\} = g\|x\|.$$

In the same vein we have

Theorem 3.1.5 (*Characterization of strictly Lipschitz operators by matrices*)

(i) *Let $A \in Lip^{\sim}(E)$ have the matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & \cdots & \\ \vdots & & \end{pmatrix}$$

with respect to $e_1, e_2, \ldots$. Then, for each n, $\lim_{m\to\infty} \|a_{mn}P_{mn}\|^{\sim} = 0$ and

$$\|A\|^{\sim} = \sup\{\|a_{mn}P_{mn}\|^{\sim} : m, n \in \mathbb{N}\}.$$

(ii) *Conversely, let*

$$\begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & \dots & \\ \vdots & & \end{pmatrix}$$

be a matrix with entries in K*, such that, for each* n*,* $lim_{m\to\infty}\|a_{mn}P_{mn}\|^{\sim} = 0$ *and such that* $(m,n) \mapsto \|a_{mn}P_{mn}\|^{\sim}$ *is bounded above. Then the matrix represents a strictly Lipschitz operator.*

Proof. Straightforward adaptation of the proof of Theorem 3.1.4. We leave the details to the reader.

Now we characterize compact and nuclear operators (see Definition 2.2.7).

Theorem 3.1.6 *Let* $A \in Lip(E)$ *have the matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & \dots & \\ \vdots & & \end{pmatrix}$$

with respect to $e_1, e_2, \dots$*. Then*

(i) $A \in C(E)$ *if and only if* $\lim_{m\to\infty}\|a_{mn}P_{mn}\| = 0$ *uniformly in* $n \in \mathbb{N}$,

(ii) $A \in C^{\sim}(E)$ *if and only if* $\lim_{m\to\infty}\|a_{mn}P_{mn}\|^{\sim} = 0$ *uniformly in* $n \in \mathbb{N}$.

Proof. Suppose $\lim_{m\to\infty}\|a_{mn}P_{mn}\| = 0$ uniformly in n. Let $\varepsilon \in G$. There is an m such that $\|a_{kn}P_{kn}\| < \varepsilon$ for all $k > m$, all n. The matrix decomposition

$$\begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & \vdots & \dots \\ \vdots & \vdots & \\ a_{m1} & a_{m2} & \dots \\ a_{m+1,1} & \dots & \\ \vdots & & \\ \vdots & & \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & \vdots & \dots \\ \vdots & \vdots & \\ a_{m1} & a_{m2} & \dots \\ 0 & 0 & \dots \\ 0 & \dots & \\ \vdots & & \end{pmatrix} + \begin{pmatrix} 0 & 0 & \dots \\ 0 & \vdots & \dots \\ \vdots & \vdots & \\ 0 & 0 & \dots \\ a_{m+1,1} & a_{m+1,2} & \dots \\ a_{m+2,1} & \dots & \\ \vdots & & \end{pmatrix}$$

corresponds to a decomposition $A = A_1 + A_2$; where $A_1, A_2 \in Lip(E)$. Clearly $A_1 \in FR(E)$ and $\|A_2\| = \sup\ \{\|a_{kn}P_{kn}\| : k > m, n \in \mathbb{N}\} \le \varepsilon$. We see that $\|A - A_1\| \le \varepsilon$. Thus $A \in C(E)$. A similar proof goes for the 'if' part of (ii).
To prove the 'only if' parts observe that

$$\{B \in Lip(E) : \lim_{m\to\infty}\|a_{mn}P_{mn}\| = 0 \text{ uniformly in } n\ \}$$

is a $\|\ \|$-closed subspace of $Lip(E)$ and that

$$\{B \in Lip^{\sim}(E) : \lim_{m\to\infty}\|a_{mn}P_{mn}\|^{\sim} = 0 \text{ uniformly in } n\ \}$$

is a $\| \|^{\sim}$-closed subspace of $Lip^{\sim}(E)$. So we are done as soon as the latter set contains $B : x \mapsto f(x)a$ $(f \in E',\ a \in E)$ which we shall prove now. There is an $s_0 \in X$ such that $s_0|f(e_n)| < \|e_n\|$ for all $n \in \mathbb{N}$ (Proposition 2.2.2). Let $\varepsilon \in G$, let a have an expansion $\sum_{i=1}^{\infty} \xi_i e_i$. (Note that then B has a matrix (b_{mn}), with $b_{mn} = \xi_m f(e_n)$). There is an m_0 such that $\|\xi_m e_m\| \leq \varepsilon s_0$ for $m \geq m_0$. Then for $m \geq m_0$ and $n \in \mathbb{N}$ we have $\|(b_{mn}P_{mn})(e_n)\| = \|b_{mn}e_m\| = \|\xi_m f(e_n)e_m\| \leq \varepsilon s_0|f(e_n)| < \varepsilon\|e_n\|$. Thus $\|b_{mn}P_{mn}\|^{\sim} \leq \varepsilon$ for those m, n and we are done.

We also have the following expected formula for the trace.

Theorem 3.1.7 *Let $A \in C^{\sim}(E)$ have the matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & \dots & \\ \vdots & & \end{pmatrix}$$

with respect to $e_1, e_2, \dots$ Then $\lim_{n\to\infty} a_{nn} = 0$ and $\mathrm{tr}(A) = \sum_{n=1}^{\infty} a_{nn}$.

Proof. From Theorem 3.1.6 (ii) we get $\lim_{n\to\infty}\|a_{nn}P_{nn}\|^{\sim} = 0$.
Now $\|P_{nn}\|^{\sim} \geq 1$ (Corollary 3.1.2 (ii)), so $\lim_{n\to\infty} a_{nn} = 0$, thus $\sum_{n=1}^{\infty} a_{nn}$ exists.
Clearly the conclusion of Theorem 3.1.7 holds for operators in $FR(E)$ whose matrices have the form

$$\begin{pmatrix} a_{11} & a_{12} & \dots \\ a_{21} & \vdots & \dots \\ \vdots & \vdots & \\ a_{m1} & a_{m2} & \dots \\ 0 & 0 & \dots \\ 0 & \dots & \\ \vdots & & \end{pmatrix}$$

(by using Proposition 2.2.4). Those operators form a dense subspace of $C^{\sim}(E)$ on which the continuous maps $A \mapsto \sum_{n=1}^{\infty} a_{nn}$ and $A \mapsto \mathrm{tr}(A)$ coincide, hence they coincide on $C^{\sim}(E)$.

Remark. Since the choice of the orthogonal base was arbitrary we can conclude that the formula for the trace is 'independent of the choice of orthogonal base' in the sense that, if $b_1, b_2, \dots$ is a second orthogonal base and $A \in C^{\sim}(E)$ has matrix (c_{mn}) with respect to $b_1, b_2, \dots$ then $\mathrm{tr}(A) = \sum_{n=1}^{\infty} c_{nn}$.

3.2 Matrix properties of subclasses

Among the spaces with a countable orthogonal base we select the so-called Norm Hilbert Spaces which are of particular interest and have been studied in [6], [8] [12] and [10].

Definition 3.2.1 An X-normed Banach space over K is called a **Norm Hilbert space** (NHS) if each closed subspace has an orthogonal complement.

The following Proposition characterizes NHS. For further equivalent statements see [10] Theorem 3.2.1.

Proposition 3.2.2 *Let X be a G-module, let E be an infinite-dimensional X-normed Banach space. Then the following are equivalent.*

(α) *E is Norm Hilbert space.*

(β) *For each closed subspace D there is an orthogonal projection $P : E \to E$ with $PE = D$.*

(γ) *Each orthogonal system can be extended to an orthogonal base.*

(δ) *Each maximal orthogonal system is an orthogonal base.*

(ε) *For each orthogonal base $e_1, e_2, \dots$ the sequence $n \mapsto \|e_n\|$ satisfies the type condition.*

Proof. See [10] 3.2.1, 3.2.3.

Theorem 3.2.3 *Let E be a Norm Hilbert space, let $A \in L(E)$ have the matrix (a_{mn}) with respect to $e_1, e_2, \dots$.*
Then the following are equivalent.

(α) $A \in Lip(E)$.

(β) $(m, n) \mapsto \|a_{mn}P_{mn}\|$ *is bounded.*

(γ) $(m, n) \mapsto \|a_{mn}P_{mn}\|$ *is bounded. For each* $m \in \mathbb{N}$, $\lim_{n\to\infty}\|a_{mn}P_{mn}\| = 0$. *For each* $n \in \mathbb{N}$, $\lim_{m\to\infty}\|a_{mn}P_{mn}\| = 0$.

Similarly, the following are equivalent.

$(\alpha)^\sim$ $A \in Lip^\sim(E)$.

$(\beta)^\sim$ $(m, n) \mapsto \|a_{mn}P_{mn}\|^\sim$ *is bounded.*

$(\gamma)^\sim$ $(m, n) \mapsto \|a_{mn}P_{mn}\|^\sim$ *is bounded. For each* $m \in \mathbb{N}$, $\lim_{n\to\infty}\|a_{mn}P_{mn}\|^\sim = 0$. *For each* $n \in \mathbb{N}$, $\lim_{m\to\infty}\|a_{mn}P_{mn}\|^\sim = 0$.

Proof. We only prove the equivalence of (α), (β), (γ) leaving the other case to the reader.
The implications (α) $\Rightarrow$ (β) and (γ) $\Rightarrow$ (α) follow from Theorem 3.1.4, so we prove (β) $\Rightarrow$ (γ). By Corollary 3.1.2 (i) we have $\mathrm{Stab}(\|P_{mn}\|) = \tau(\|e_n\|; \|e_m\|)$. Since E is a Norm Hilbert space we have, by Theorem 3.2.2 (ε) and Proposition 1.1.7, $\lim_{n\to\infty}\tau(\|e_n\|; \|e_m\|) = \infty$ for each m. Thus $n \mapsto \|P_{mn}\|$ and (since $\mathrm{Stab}(\|P_{mn}\|) = \mathrm{Stab}(\|P_{nm}\|)$) $m \mapsto \|P_{mn}\|$ satisfy the type condition, so (β) implies $\lim_{n\to\infty}\|a_{mn}P_{mn}\| = 0$, $\lim_{m\to\infty}\|a_{mn}P_{mn}\| = 0$.

Corollary 3.2.4 *Let E be a Norm Hilbert space, let $A \in Lip(E)$ have the matrix (a_{mn}) with respect to $e_1, e_2, \dots$. Then $A \in C(E)$ if and only if* $lim_{m+n\to\infty}\|a_{mn}P_{mn}\| =$

0; and the P_{mn} form an orthogonal base of $C(E)$. Similarly, $A \in C^{\sim}(E)$ if and only if $\lim_{m+n\to\infty} \|a_{mn}P_{mn}\|^{\sim} = 0$; and the P_{mn} form an orthogonal base of $C^{\sim}(E)$.

Proof. Combine Theorems 3.1.6, 3.2.3, Lemma 3.1.1 (ii) and Corollary 3.1.2 (iii).

The properties of Theorem 3.2.3 and Corollary 3.2.4 also characterize Norm Hilbert spaces, as is shown by the following.

Theorem 3.2.5 *Let E be not a Norm Hilbert space. Then there exists an $A \in FR(E)$ with matrix (a_{mn}) such that not $\lim_{n\to\infty}\|a_{1n}P_{1n}\| = 0$.*

Proof. There are a subsequence $e_{n_1}, e_{n_2}, \ldots$ of $e_1, e_2, \ldots,$ $\lambda_1, \lambda_2, \ldots \in K$ and $c_1, c_2 \in X$ such that

$$c_1 \le |\lambda_i|\ \|e_{n_i}\| \le c_2 \quad (i \in \mathbb{N}).$$

For all $x \in E$ with expansion $\sum_{i=1}^{\infty} \xi_i e_i$ set

$$f(x) = \sum_{i=1}^{\infty} \lambda_i^{-1}\xi_{n_i},$$

(f is easily seen to be in E') and put

$$Ax := f(x)e_1.$$

Then $A \in FR(E)$ with matrix (a_{1n}) such that $a_{1n} = f(e_n)$ for all n. We will show that the sequence $i \mapsto \|a_{1n_i}P_{1n_i}\|$ does not tend to 0. We have for each i

$$\|a_{1n_i}P_{1n_i}\| = \|\lambda_i^{-1}P_{1n_i}\| = \inf V_i$$

where

$$V_i = \{g \in G : \|e_1\| \le g|\lambda_i|\ \|e_{n_i}\|\}.$$

Now let $g \in V_i$. Then $\|e_1\| \le gc_2$. Choose $g_1 \in G$ such that $\|e_1\| > g_1c_2$. Then $g > g_1$, so g_1 is a lower bound of V_i for each i and we have $\|a_{1n_i}P_{1n_i}\| \ge g_1$ for each i.

To a classical Functional analyst the following feature will appear surrealistic.

Theorem 3.2.6 *Let E be infinite-dimensional. Then the following are equivalent.*

(α) $C(E) = Lip(E)$.

(β) $C^{\sim}(E) = Lip^{\sim}(E)$.

(γ) $\lim_{n\to\infty} \mathrm{Stab}(\|e_n\|) = \infty$ (i.e. for every proper convex subgroup H of G we have $\mathrm{Stab}(\|e_n\|) \supsetneqq H$ for large n).

Proof. (α) $\Rightarrow$ (β). Let $A \in Lip^{\sim}(E)$, $\varepsilon \in G$. Choose $\delta \in G$ such that $\delta\|A\|^{\sim} < \varepsilon$. By assumption there is a $B \in FR(E)$ with $\|I - B\| < \delta$. Then (Proposition 2.1.5) $\|A - BA\|^{\sim} \le \|I - B\| * \|A\|^{\sim} \le \delta\|A\|^{\sim} < \varepsilon$.

(β) $\Rightarrow$ (α). Let $A \in Lip(E)$, $\varepsilon \in G$. By Proposition 2.2.8 (iv) there is a $B \in Lip^{\sim}(E)$ with $\|A - B\| < \varepsilon$. By assumption there is a $C \in FR(E)$ with $\|B - C\|^{\sim} < \varepsilon$, hence $\|B - C\| < \varepsilon$. Then $\|A - C\| \le \max(\|A - B\|, \|B - C\|) < \varepsilon$.

$(\alpha) \Rightarrow (\gamma)$. We have that $I \in C(E)$. For its matrix entries we have $a_{mn} = \delta_{mn}$, so by Theorem 3.1.6 (i), $\lim_{n\to\infty}\|P_{nn}\| = 0$. But $\|P_{nn}\| = \inf \mathrm{Stab}(\|e_n\|) \to 0$, so (γ) follows.

$(\gamma) \Rightarrow (\alpha)$. From (γ) we obtain $\|P_{nn}\| \to 0$. Then $I = \sum_{n=1}^{\infty} P_{nn} \in C(E)$.

Remark. Since $\mathrm{Stab}(\|e_n\|) \subseteq \tau(t; \|e_n\|) = \tau(\|e_n\|; t)$ (Theorem 1.1.4 (iii)), we have that the orthogonal base $e_1, e_2, \ldots$ satisfies the type condition (Proposition 1.1.7). Therefore, by [10] 3.2.1 (κ), condition (γ) implies that E is a Norm Hilbert space. For a concrete example of a space E satisfying $(\alpha) - (\gamma)$, see [7] 4.2.2.

3.3 Type separating spaces

We conclude this paper by describing a class of Norm Hilbert spaces thereby generalizing the results of [6] considerably.

Definition 3.3.1 Let us call E **type-separating** if it is a Norm Hilbert space and if there exists an $s_0 \in X$ such that $n \neq m$ implies

$$\tau(\|e_n\|; s_0) \neq \tau(\|e_m\|; s_0).$$

Examples of such spaces can be found in [2], [3], [4]. The fact that E is a Norm Hilbert space implies that $\lim_{n\to\infty} \tau(\|e_n\|; s_0) = \infty$ (Proposition 3.2.2).

For an $A \in Lip(E)$ with matrix (a_{mn}) the matrix decomposition

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & & \\ a_{21} & a_{22} & & & \\ \vdots & & & & \\ & & & & \\ & & & & \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 & \ldots & \\ 0 & a_{22} & & & \\ 0 & & \cdot & & \\ \vdots & & & \cdot & \\ & & & & \cdot \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & \ldots & & \\ a_{21} & 0 & & & \\ \vdots & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \end{pmatrix}$$

represents a decomposition $A = D + S$ (which we will call henceforth the **standard decomposition**), where $D, S \in Lip(E)$, D has diagonal matrix, S has zero diagonal.

Proposition 3.3.2 *Let E be type-separating. Let $D + S$ be the standard decomposition of an $A \in Lip(E)$. Then S is nuclear.*

Proof. From Corollary 3.1.2 (i) and Theorem 1.1.4 (v), we infer $\mathrm{Stab}(\|P_{mn}\|) = \tau(\|e_n\|; \|e_m\|) = \tau(\|e_n\|; s_0) \cup \tau(\|e_m\|; s_0)$ whenever $m \neq n$ (Theorem 1.1.4 (i)). We see that $\{\|P_{mn}\| : \ m \neq n\}$ satisfies the type condition and therefore

$$\lim_{m+n\to\infty \ (m\neq n)} \|a_{mn} P_{mn}\| = 0,$$

showing (Corollary 3.2.4) that S is compact. But, since the algebraic types of $\|e_m\|$ and $\|e_n\|$ must differ whenever $m \neq n$, (Theorem 1.1.4 (iii)), we have $\|P_{mn}\| = \|P_{mn}\|^{\sim}$ according to Lemma 3.1.1 (i), hence $\lim_{m+n\to\infty \ (m\neq n)} \|a_{mn} P_{mn}\|^{\sim} = 0$, i.e. S is nuclear (Corollary 3.2.4).

The following consequences are obtained (compare [8], 3.8 and 4.3).

Theorem 3.3.3 *Let E be type separating.*

(i) *If $A, B \in Lip(E)$ then $AB - BA \in C^{\sim}(E)$ and* $\mathrm{tr}(AB - BA) = 0$.

(ii) *Let $A \in Lip(E)$. Then $A \in C(E)$ if and only if $\lim_{n\to\infty}\|a_{nn}P_{nn}\| = 0$, and $A \in C^{\sim}(E)$ if and only if $\lim_{n\to\infty}\|a_{nn}P_{nn}\|^{\sim} = 0$.*

(iii) *The Calkin algebra $Lip(E)/C(E)$ is commutative.*

(iv) *If $A \in Lip(E)$, $n \in \mathbb{N}$, $A^n \in C(E)$ then $A \in C(E)$.*

(v) *If $A \in Lip(E), n \in \mathbb{N}, A^n \in C^{\sim}(E)$ then $A \in C^{\sim}(E)$.*

Proof. From Proposition 3.3.2 it follows that an $A \in Lip(E)$ is compact (nuclear) if its "diagonal part" is compact (nuclear). This, together with Corollary 3.2.4, yields (ii). To prove (i), let $A = D_1 + S_1$, $B = D_2 + S_2$ be the standard decomposition. Then $D_1D_2 = D_2D_1$ so that

$$AB - BA = (S_1D_2 - D_2S_1) + (D_1S_2 - S_2D_1) + (S_1S_2 - S_2S_1)$$

According to Proposition 3.3.2 the operators S_1 and S_2 are in $C^{\sim}(E)$. From Proposition 2.2.8 (i) and (ii) it then follows that each one of the three bracketed operators above is in $C^{\sim}(E)$ and has zero trace, and we have (i). Statement (iii) follows directly from (i). To prove (iv) and (v) it suffices to consider the case $n = 2$. So, let $A \in Lip(E)$ have standard decomposition $A = D + S$. Then

$$A^2 = D^2 + DS + SD + S^2$$

and we know from Propositions 3.3.2 and 2.2.8 (i) that DS, SD and S^2 are in $C^{\sim}(E)$. Now we treat two cases (a) and (b).
(a) Suppose $A^2 \in C(E)$. Then $D^2 \in C(E)$, so by (ii) $\lim_{n\to\infty}|a_{nn}^2|\|P_{nn}\| = 0$. Suppose not $\lim_{n\to\infty}\|a_{nn}P_{nn}\| = 0$; we arrive at a contradiction. There exist a $g \in G$ and an increasing sequence $i \mapsto n_i$ such that for all i

$$\|a_{n_in_i}P_{n_in_i}\| \geq g.$$

Now, since $\|P_{nn}\| \leq 1$ for all n (Lemma 3.1.1(i)) we have $|a_{n_in_i}| \geq g$ for all i implying

$$\|a_{n_in_i}^2 P_{n_in_i}\| = |a_{n_in_i}|\ \|a_{n_in_i}P_{n_in_i}\| \geq g^2,$$

a contradiction. This proves (iv).
(b) Suppose $A^2 \in C^{\sim}(E)$. Then $D^2 \in C^{\sim}(E)$, so by (ii) $\lim_{n\to\infty}\|a_{nn}^2 P_{nn}\|^{\sim} = 0$. Suppose not $\lim_{n\to\infty}\|a_{nn}P_{nn}\|^{\sim} = 0$; we arrive at a contradiction. There exist a $g \in G$ and an increasing sequence $i \mapsto n_i$ such that for all i
(*)
$$\|a_{n_in_i}P_{n_in_i}\|^{\sim} > g.$$
Now, if $i \mapsto \|P_{n_in_i}\|^{\sim}$ were bounded above by, say, $g_1 \in G$ then $|a_{n_in_i}| \geq gg_1^{-1}$ and, since, again by Lemma 3.1.1 (i), $\|P_{nn}\|^{\sim} \geq 1$ for all n, we obtain $|a_{n_in_i}^2|\ \|P_{n_in_i}\|^{\sim} \geq (gg_1^{-1})^2$ for all i, a contradiction.

Thus $i \mapsto \|P_{n_i n_i}\|^\sim$ is unbounded, so, by taking a suitable subsequence, we may assume that $\lim_{i\to\infty}\|P_{n_i n_i}\|^\sim = \infty$.
Let $s_i := \sup \mathrm{Stab}(\|e_{n_i}\|)$. Then by Corollary 3.1.2 (ii), $s_i = \|P_{n_i n_i}\|^\sim$ for all i.
Then we may assume that g is in $\mathrm{Stab}(\|e_{n_i}\|)$ for each i, so by multiplying $(*)$ by g^{-1} we find that $(*)$ holds for $g := 1$. Now we have $\lim_{i\to\infty}|a^2_{n_i n_i}|s_i = 0$ implying that for large i

$$|a^2_{n_i n_i}|s_i \leq 1.$$

By Lemma 3.3.4 below it then follows that for those i

$$|a_{n_i n_i}|s_i \leq 1.$$

conflicting (*).

Lemma 3.3.4 *Let H be a proper convex subgroup of G, $s := \sup H$, $g \in G$. Then $gs \leq 1 \Leftrightarrow g^2 s \leq 1$.*

Proof. We may assume $H \neq \{1\}$. Then $s > 1$. If $gs \leq 1$ then g must be less or equal to 1 so $g^2 s = g \cdot gs \leq gs \leq 1$. Conversely, let $g^2 s \leq 1$. Then $g^2 \leq 1$ hence $g \leq 1$. If $g \in H = \mathrm{Stab}(s)$ then $g^2 s = s > 1$. Thus it follows that $g < \inf H$. It suffices to prove that $gs \leq \inf H$. Suppose not. Then there exists an $h_1 \in H$ such that $gs > h_1$, i.e. $s > g^{-1}h_1$. So there exists an $h_2 \in H$ such that $s > h_2 \geq g^{-1}h_1$ and we find $g \geq h_2^{-1}h_1 > \inf H$, a contradiction.

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