A stability theorem for the index of sphere bundles

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Abstract

We prove that the index of every m-dimensional vector bundle over B is equal to m if $m \geq 2 \dim B$. We also determine the smallest integer k for which every m-dimensional vector bundle with $m \geq k$ is I-stable in the cases $B = FP^n$ and $B = S^n$.

1 Introduction

Let α be a finite-dimensional real vector bundle over a CW complex B, and let $S(\alpha)$ be its sphere bundle with respect to some metric on α. We regard $S(\alpha)$ as a $\mathbb{Z}/2$ -space by the antipodal map on each fibre. The index of α, denoted $\operatorname{ind}\alpha$, is defined to be the largest integer k for which there exists a $\mathbb{Z}/2$ -map from S^{k-1} to $S(\alpha)$ [CF1, CF2, T1]. Here, S^{k-1} also is regarded as a $\mathbb{Z}/2$ -space by the antipodal map. From the inclusion of the fibre, we clearly have $\operatorname{ind}\alpha \geq \dim \alpha$. It is also clear that $\operatorname{ind}\alpha \leq \operatorname{ind}(\alpha \oplus 1)$.

We describe α as *I-stable* if the equality $\operatorname{ind}(\alpha \oplus k) = \operatorname{ind}\alpha + k$ holds for any positive integer k. Here, we abuse notation and denote the k-dimensional trivial bundle simply by k. Our definition of the stability is slightly different from that in [CF1] in the sense that we consider the fibrewise suspension. If α is trivial, then $\operatorname{ind}\alpha = \dim\alpha$ and α is I-stable. The tangent bundle τ_M of a closed manifold M also has this property ; $\operatorname{ind}\tau_M = \dim\tau_M$ and τ_M is I-stable (see [T2, Theorem 4.6]). For the canonical line bundle η_F over the projective space FP^n ($F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$), $\eta_F \oplus dn$ has the above property but $\eta_F \oplus \ell$ ($0 \leq \ell < dn$) does not, where $d = \dim_{\mathbb{R}} F$ and η_F is considered as a real bundle (see [T2, Theorem 4.2, 4.4]).

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In this paper, we prove the following theorem.

Theorem 1.1. *Let* α *be a vector bundle over a finite complex* B *with* $\dim B = n$. *If* $\dim\alpha \geq 2n$, *then* $\operatorname{ind}\alpha = \dim\alpha$ *and* α *is I-stable.*

From this theorem, the following corollary follows immediately.

Corollary 1.2. *For any vector bundle* α *over a finite complex* B, *there exists an integer* k *such that* $\operatorname{ind}(\alpha\oplus k) = \dim(\alpha\oplus k)$ *and* $\alpha\oplus k$ *is I-stable. Moreover, such* k *can be taken so that* $k \leq 2\dim B - \dim\alpha$.

The condition $\dim\alpha \geq 2n$ in Theorem 1.1 is best possible at least when $B = S^1, S^2, S^4$ and S^8. Likewise, the condition $k \leq 2\dim B - \dim\alpha$ in Corollary 1.2 is best possible as a general estimate.

If we define $\operatorname{ind}^s(\alpha)$, the stable index of α, by

$$\operatorname{ind}^s(\alpha) = \lim_{k\to\infty}\{\operatorname{ind}(\alpha\oplus k) - \dim(\alpha\oplus k)\},$$

the above corollary can be restated as follows.

Corollary 1.3. *For any vector bundle* α *over a finite complex, we have* $\operatorname{ind}^s(\alpha) = 0$.

The stable co-index $\text{co-ind}^s(\alpha)$ is similarly defined using the co-index which is the dual of the index. We note that the stable co-index satisfies $0 \leq \text{co-ind}^s(\alpha) \leq \dim B$ in general and, in the case $B = \mathbb{R}P^n$, every integer k such that $0 \leq k \leq \dim B$ can be realized actually as the stable co-index of some vector bundle over B (see [T3, Corollary 1.6]).

The condition $\dim\alpha \geq 2n$ in Theorem 1.1 can be made more strict for individual spaces B. For $B = FP^n$, we obtain the following result.

Theorem 1.4. *Let* α *be a vector bundle over* FP^n. *If* $\dim\alpha \geq \dim FP^n + d$, *then* $\operatorname{ind}\alpha = \dim\alpha$ *and* α *is I-stable. This condition is best possible; there is a vector bundle* α *over* FP^n *with* $\dim\alpha = \dim FP^n + d - 1$ *such that* $\operatorname{ind}\alpha \neq \dim\alpha$, *nor is* α *I-stable.*

By this result, for $B = FP^n$, the smallest integer k such that every m-dimensional vector bundle with $m \geq k$ is I-stable is equal to $dn + d$. The smallest integer k such that the equality $\operatorname{ind}\alpha = \dim\alpha$ holds for every vector bundle α with $\dim\alpha \geq k$ is also equal to $dn + d$.

For $B = S^n$, we obtain the following result.

Theorem 1.5. *If* $n \neq 1, 2, 4, 8$, *then* $\operatorname{ind}\alpha = \dim\alpha$ *and* α *is I-stable for any vector bundle* α *over* S^n. *If* $n = 1, 2, 4$ *or* 8, *there is a vector bundle* α *over* S^n *with* $\dim\alpha = 2n - 1$ *such that* $\operatorname{ind}\alpha \neq \dim\alpha$, *nor is* α *I-stable.*

By this result and Theorem 1.1, the smallest integer k, for $B = S^n$, such that every m-dimensional vector bundle with $m \geq k$ is I-stable, is equal to $2n$ if $n = 1, 2, 4$ or 8, and equal to 0 otherwise. The smallest integer k such that the equality $\operatorname{ind}\alpha = \dim\alpha$ holds for every vector bundle α with $\dim\alpha \geq k$ is the same as above.

2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We use the following result in [T1].

Proposition 2.1. [T1, Proposition 2.4] *Let α be an m-dimensional real vector bundle over B. If B satisfies* $\mathrm{Hom}(\widetilde{H}^*(B), \widetilde{H}^*(\mathbb{R}P^m)) = 0$*, then* $\operatorname{ind}\alpha = m$.

Here, the cohomology has coefficients $\mathbb{Z}/2$ and $\mathrm{Hom}(\cdot\,,\cdot)$ consists of all homomorphisms (of degree 0) as graded algebra over the Steenrod algebra mod 2.

Sketch proof of Proposition 2.1. The inequality $\operatorname{ind}\alpha \geq m$ is obvious by the inclusion of the fibre. Assume $\operatorname{ind}\alpha > m$. Then there is a $\mathbb{Z}/2$-map $f : S^m \longrightarrow S(\alpha)$ and it induces $\tilde{f} : \mathbb{R}P^m \longrightarrow P(\alpha)$. Here $P(\alpha)$ denotes the associated projective bundle of α. Let $e(\in H^1(P(\alpha)))$ denote the $\mathbb{Z}/2$-Euler class of the line bundle $\alpha \to P(\alpha)$, and let $t(\in H^1(\mathbb{R}P^m))$ denote the $\mathbb{Z}/2$-Euler class of the canonical line bundle over $\mathbb{R}P^m$. Then we have $\tilde{f}^*(e^m) = t^m \neq 0 \in H^m(\mathbb{R}P^m)$. Let $\bar{f} : \mathbb{R}P^m \longrightarrow B$ be the composition of $\tilde{f}$ with the projection $p : P(\alpha) \longrightarrow B$. Then $\bar{f}^*$ is the zero homomorphism since $\mathrm{Hom}(\widetilde{H}^*(B), \widetilde{H}^*(\mathbb{R}P^m)) = 0$. Using the relation $e^m = \sum_{i=0}^{m-1} w_{m-i}e^i$, where w_i denotes the ith Stiefel-Whitney class of α, we have $\tilde{f}^*(e^m) = \tilde{f}^*(\sum_{i=0}^{m-1} w_{m-i}e^i) = \sum_{i=0}^{m-1} \bar{f}^*(w_{m-i})t^i = 0$. This contradicts $\tilde{f}^*(e^m) \neq 0$. ■

From the above proposition, we have the following theorem.

Theorem 2.2. *Suppose that a finite complex B satisfies the condition*

$$\mathrm{Hom}(\widetilde{H}^*(B), \widetilde{H}^*(\mathbb{R}P^\ell)) = 0$$

for some integer ℓ with $\ell \geq \dim B$. Then, for any real vector bundle α over B with $\dim\alpha \geq \ell$, $\operatorname{ind}\alpha = \dim\alpha$ *and α is I-stable.*

Proof. Suppose that B satisfies $\mathrm{Hom}(\widetilde{H}^*(B), \widetilde{H}^*(\mathbb{R}P^\ell)) = 0$ with $\ell \geq \dim B$, and let α be an m-dimensional vector bundle over B with $m \geq \ell$. We prove that $\operatorname{ind}(\alpha \oplus k) = \dim(\alpha \oplus k)$ for all $k \geq 0$. In view of Proposition 2.1, it suffices to prove that $\mathrm{Hom}(\widetilde{H}^*(B), \widetilde{H}^*(\mathbb{R}P^{m+k})) = 0$. Consider the diagram

$$\begin{array}{ccc} \widetilde{H}^*(B) & \xrightarrow{\varphi} & \widetilde{H}^*(\mathbb{R}P^{m+k}) \\ & {\scriptstyle i^* \circ \varphi} \searrow & \downarrow i^* \\ & & \widetilde{H}^*(\mathbb{R}P^\ell) \end{array}$$

where i is the inclusion $\mathbb{R}P^\ell \hookrightarrow \mathbb{R}P^{m+k}$. Since $\dim B \leq \ell$ and i^* is an isomorphism for $* \leq \ell$, we have $\varphi = 0$ if $i^* \circ \varphi = 0$. Thus, $\mathrm{Hom}(\widetilde{H}^*(B), \widetilde{H}^*(\mathbb{R}P^\ell)) = 0$ implies $\mathrm{Hom}(\widetilde{H}^*(B), \widetilde{H}^*(\mathbb{R}P^{m+k})) = 0$. This proves the theorem. ■

By the above theorem, Theorem 1.1 follows immediately from the following Lemma.

Lemma 2.3. *Let B be a finite complex with* $\dim B = n$. *Then,* $\mathrm{Hom}(\widetilde{H}^*(B), \widetilde{H}^*(\mathbb{R}P^{2n})) = 0$.

Proof. Let $\varphi : \widetilde{H}^*(B) \longrightarrow \widetilde{H}^*(\mathbb{R}P^{2n})$ be a homomorphism. For any $x \in \widetilde{H}^i(B)$ $(1 \le i \le n)$, we put $\varphi(x) = \epsilon t^i$, where t is the generator of $\widetilde{H}^*(\mathbb{R}P^{2n})$ and $\epsilon = 0$ or 1. Now, choose an integer j so that $n < ij \le 2n$. Then we have $x^j = 0$ because of the dimension reason, and so we have $\varphi(x^j) = 0$. On the other hand, we have $\varphi(x^j) = (\varphi(x))^j = \epsilon t^{ij}$. Hence, ϵ must be zero since t^{ij} is not zero. Therefore, $\varphi(x) = 0$ for any $x \in \widetilde{H}^i(B)$ and we conclude that φ is the zero homomorphism. ∎

3 Proof of Theorem 1.4 and 1.5

In this section, we prove Theorem 1.4 and Theorem 1.5.

First, we consider the case $B = FP^n$. Theorem 1.4 is actually proved in [T2], but we reconsider it to emphasize that the first half of it follows as an immediate corollary of Theorem 2.2. In fact, it is easy to see that $\mathrm{Hom}(\widetilde{H}^*(FP^n), \widetilde{H}^*(\mathbb{R}P^{d(n+1)})) = 0$. Therefore, for any real vector bundle α over FP^n with $\dim \alpha \ge d(n+1)$, we see that $\mathrm{ind}\, \alpha = \dim \alpha$ and α is I-stable from Theorem 2.2. For the latter half of Theorem 1.4, we recall that $\mathrm{ind}(m\eta_F \oplus \ell) = \max\{d(n+1), dm+\ell\}$ (see [T2, Theorem 4.2, 4.4]). This has been shown as follows. It is enough to consider the case where $dm + \ell < d(n+1)$. First, it is shown that $\mathrm{ind}(m\eta_F \oplus \ell) \le d(n+1)$ by an analogous argument as in the proof of Proposition 2.1 calculating $\tilde{f}^*(e^{dm+\ell})$ with the fact $\mathrm{Hom}(\widetilde{H}^*(FP^n), \widetilde{H}^*(\mathbb{R}P^{d(n+1)})) = 0$. On the other hand, a $\mathbb{Z}/2$ -map $S^{d(n+1)-1} \longrightarrow S(m\eta_F \oplus \ell)$ is given by the composition $S^{d(n+1)-1} \equiv S(\eta_F) \hookrightarrow S(m\eta_F \oplus \ell)$. Thus, $\mathrm{ind}(m\eta_F \oplus \ell) = d(n+1)$ when $dm + \ell < d(n+1)$.

From this result, for such a bundle α over FP^n with $\dim \alpha = d(n+1) - 1$ as $\eta_F \oplus (dn-1)$ or $n\eta_F \oplus (d-1)$, we have $\mathrm{ind}\, \alpha = \mathrm{ind}(\alpha \oplus 1) = \dim(\alpha \oplus 1) = d(n+1)$, so that $\mathrm{ind}\, \alpha \ne \dim \alpha$ and α is not I-stable either.

Next, we consider the case $B = S^n$. If we intend to utilize Theorem 2.2, we will see that $\mathrm{Hom}(\widetilde{H}^*(S^n), \widetilde{H}^*(\mathbb{R}P^\ell)) = 0$ if $\ell \ge n + 2^a$ (and of course if $\ell < n$), where a is the integer defined by $n = 2^a(2b+1)$. In fact, let $\varphi : \widetilde{H}^*(S^n) \longrightarrow \widetilde{H}^*(\mathbb{R}P^\ell)$ be a homomorphism and put $\varphi(x) = \epsilon t^n$ for $x \in H^n(S^n)$ just as in the proof of Lemma 2.3. Since $Sq^{2^a} x = 0$ by the dimension reason, we have $\varphi(Sq^{2^a} x) = 0$. On the other hand, we have $\varphi(Sq^{2^a} x) = Sq^{2^a} \varphi(x) = \epsilon\, Sq^{2^a}(t^n) = \epsilon \binom{n}{2^a} t^{n+2^a}$. Since $\binom{n}{2^a} \equiv 1 \pmod 2$ and $t^{n+2^a} \ne 0$ (because $\ell \ge n + 2^a$), we obtain $\epsilon = 0$ and we conclude that φ is the zero homomorphism. Therefore, by Theorem 2.2, it is seen that a vector bundle α over S^n such that $\dim \alpha \ge n + 2^a$ has the property that $\mathrm{ind}\, \alpha = \dim \alpha$ and α is I-stable. However, $\mathrm{Hom}(\widetilde{H}^*(S^n), \widetilde{H}^*(\mathbb{R}P^\ell))$ is not zero at least for $\ell = n$ (for any positive integer n), so that the above method does not seem to be adequate enough for our purpose.

Let $W(\alpha)$ be the total Stiefel-Whitney class of α. By Proposition 2.2 in [T1], if $W(\alpha) = 1$, then $\mathrm{ind}\, \alpha = \dim \alpha$. This has been shown in the same context just as in the proof of Proposition 2.1 observing $e^m = \sum_{i=0}^{m-1} w_{m-i} e^i = 0$ $(m = \dim \alpha)$. Since $W(\alpha \oplus k) = W(\alpha)$ for any positive integer k, we can improve this proposition as follows.

Proposition 3.1. *Let* α *be a real vector bundle over* B. *If* $W(\alpha) = 1$, *then* $\operatorname{ind}\alpha = \dim\alpha$ *and* α *is I-stable.*

In view of the above proposition, the first half of Theorem 1.5 follows from the following theorem, which was originally proved by Milnor.

Theorem 3.2. [M, Theorem 1] *If* $n \neq 1, 2, 4, 8$, *then* $W(\alpha) = 1$ *for any vector bundle* α *over* S^n.

Proof. Milnor proved this theorem, first by using Wu's formula of Steenrod squares on Stiefel-Whitney classes (see [W]) for the case $n \neq 2^r$, and next by using Bott's theorem on Pontrjagin classes. Here we give an alternative proof, which is more straightforward for the case $n \neq 2^r$ and related to the Hopf invariant one problem.

It is obvious that $W(\alpha) = 1$ if $\dim\alpha < n$. If $\dim\alpha > n$, α can be written as $\alpha = \beta \oplus k$ ($k \in \mathbb{Z}$) for some n-dimensional vector bundle β and $W(\alpha) = W(\beta)$. Therefore it suffices to prove the theorem in the case $\dim\alpha = n$. Let α be an n-dimensional vector bundle over S^n ($n > 1$) and assume that $w_n(\alpha) \neq 0$, where w_n is the nth Stiefel-Whitney class. We look at the associated projective bundle $P(\alpha)$ of α. If we denote by e the $\mathbb{Z}/2$-Euler class of the line bundle $\lambda : \alpha \to P(\alpha)$, $H^*(P(\alpha))$ can be written as $H^*(P(\alpha)) = H^*(S^n)\{1, e, e^2, \cdots, e^{n-1}\}$ as a $H^*(S^n)$-module. Moreover, in $H^*(P(\alpha))$, we have the relation $e^n = w_n(\alpha) + w_{n-1}(\alpha)e + w_{n-2}(\alpha)e^2 + \cdots + w_1(\alpha)e^{n-1}$. Let s denote the generator of $H^*(S^n)$. Then, $w_n(\alpha) = s$ by the assumption $w_n(\alpha) \neq 0$, and we have the relation $e^n = s$. Applying the total squaring operation Sq, we have $Sq(e^n) = Sq(s)$. Clearly, $Sq(s) = s$. On the other hand, $Sq(e^n) = (Sq(e))^n = (e + e^2)^n = e^n(1 + e)^n$. Hence, we obtain $\binom{n}{k}e^{n+k} = 0$ for $k \geq 1$. Now we remark that, in $H^*(P(\alpha))$, $e^{2n-1} = se^{n-1} \neq 0$ because $e^n = s$, so that $e^{n+k} \neq 0$ for $1 \leq k \leq n-1$. Therefore, we obtain $\binom{n}{k} \equiv 0 \pmod 2$ for $1 \leq k \leq n-1$. This implies that n is a power of 2.

In the case where n is a power of 2, a considerably deeper argument would be necessary. So we reduce it to the problem of nonexistence of elements of Hopf invariant one. Let $U \in H^n(D(\alpha), S(\alpha))$ denote the Thom class of α, where $D(\alpha)$ is the disk bundle of α. Let $\phi : H^*(S^n) \xrightarrow{\simeq} H^*(D(\alpha), S(\alpha))$ be the Thom isomorphism. Then we have $Sq^nU = \phi(w_n(\alpha))$. Since we have assumed $w_n(\alpha) \neq 0$, Sq^nU is not zero. Let $T(\alpha)$ be the Thom space of α. Then the operation Sq^n is not trivial on $\widetilde{H}^*(T(\alpha)) \cong H^*(D(\alpha), S(\alpha))$. As is well-known, $T(\alpha)$ is homotopy-equivalent to $S^n \cup_{J\alpha} e^{2n}$, where $J\alpha : S^{2n-1} \longrightarrow S^n$ is a map obtained by the Hopf-Whitehead construction from α considered as a map $S^{n-1} \longrightarrow SO(n)$ (e.g. see [A, Lemma 10.1]). Since Sq^n is not trivial on $H^n(T(\alpha))$, $J\alpha$ is a map of Hopf invariant one. By the Adams' theorem, it follows that $n = 2, 4$ or 8. ∎

For the latter half of Theorem 1.5, we consider the Hopf bundle. Let $d = 1, 2$ or 4 and consider S^d as FP^1, where $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ respectively. As shown in the proof of Theorem 1.4, if we put $\alpha = \eta_F \oplus (d-1)$, then $\dim\alpha = 2d - 1$ and $\operatorname{ind}\alpha = \operatorname{ind}(\alpha \oplus 1) = \dim(\alpha \oplus 1) = 2d$, so that $\operatorname{ind}\alpha \neq \dim\alpha$ and α is not I-stable either.

In the case $d = 8$, we should be a little more careful. As is well-known, there is a S^7-bundle $S^{15} \longrightarrow S^8$ with group $O(8)$ which is obtained by using Cayley numbers (e.g. [S, p109]). This bundle can be extended to a 8-dimensional real vector bundle

σ with $S(\sigma)$ identified with S^{15}. If we have constructed a $\mathbb{Z}/2$ -map $S^{15} \longrightarrow S(\sigma)$, we will have $16 \le \operatorname{ind}\sigma \le \operatorname{ind}(\sigma \oplus k) \le \operatorname{ind}(\sigma \oplus 8)$ for all k with $0 \le k \le 8$. Since $\operatorname{ind}(\sigma \oplus 8) = \dim(\sigma \oplus 8) = 16$ by Theorem 1.1, we will obtain $\operatorname{ind}(\sigma \oplus k) = 16$ for all k with $0 \le k \le 8$. Thus, if we put $\alpha = \sigma \oplus 7$, then $\dim\alpha = 15$ and $\operatorname{ind}\alpha = \operatorname{ind}(\alpha \oplus 1) = \dim(\alpha \oplus 1) = 16$, so that $\operatorname{ind}\alpha \neq \dim\alpha$ and α is not I-stable either.

Finally, we construct a $\mathbb{Z}/2$ -map $S^{15} \longrightarrow S(\sigma)$. Let T denote the involution of $S(\sigma)$, which by definition is the antipodal map on each fibre. We consider the covering projection $S(\sigma) \longrightarrow S(\sigma)/T$. First, we choose a map $f : \mathbb{R}P^1 = S^1 \longrightarrow S(\sigma)/T$ so that f represents the generator of $\pi_1(S(\sigma)/T) = \mathbb{Z}/2$. Since $2f$ represents zero in $\pi_1(S(\sigma)/T)$ and $\pi_i(S(\sigma)/T) = 0$ for $2 \le i \le 14$, f can be extend to a map $g : \mathbb{R}P^{15} \longrightarrow S(\sigma)/T$. Moreover, g can be covered by a map $\tilde{g} : S^{15} \longrightarrow S(\sigma)$ by the lifting theorem. Then, $\tilde{g}$ has the property either $\tilde{g}(-x) = T\tilde{g}(x)$ or $\tilde{g}(-x) = \tilde{g}(x)$ (for all $x \in S^{15}$). If $\tilde{g}(-x) = \tilde{g}(x)$, then g has a lift, which contradicts our choice of f. Therefore, $\tilde{g}$ is a $\mathbb{Z}/2$ -map.

Remark. Since $\operatorname{ind}\sigma = 16 \neq \dim\sigma$, it follows from Proposition 3.1 that $w_8(\sigma) \neq 0$.

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