Maximum modulus principles for radial solutions of quasilinear and fully nonlinear singular P.D.E's

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Abstract

We obtain maximum modulus principles for solutions to some quasilinear and fully nonlinear ODEs and discuss their applications to quasilinear PDEs involving p-Laplacian. Our approach is convenient to deal with singular PDEs. Its idea can be tracked back to the old theory by Szegö on orthogonal polynomials.

1 Introduction

The aim of this paper is to study maximum modulus principles for solutions of the O.D.Es

$$A\left(\tau, u(\tau), u'(\tau)\right) u''(\tau)+B\left(\tau, u(\tau), u'(\tau)\right)+f\left(\tau, u(\tau), u'(\tau), u''(\tau)\right) = 0, \;\; \tau \in (\alpha, \beta), \tag{1}$$

where f is nonnegative or nonpositive and either $u \in C([\alpha,\beta]) \cap W^{2,1}((\alpha,\beta))$ or $u \in C([\alpha,\beta]) \cap C^2((\alpha,\beta))$. This is a fully nonlinear equation which becomes quasilinear when $f \equiv 0$. The typical reprezentants of equations like (1) are some of the hypergeometric equations of Gauss, like for example the equation defining the Legendre polynomial

$$(1-\tau^2)u''(\tau) - 2\tau u'(\tau) + n(n+1)u(\tau) = 0, \quad \tau \in (-1,1). \tag{2}$$

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In general their solutions are not monotonic functions. Therefore the maximum principles of our interest include oscillating functions as well. We consider three variants of the maximum principles: left- and right-hand side maximum principles asserting that the maximum of the modulus of the solution of (1) is achieved at the left or right endpoint of an interval $[\alpha, \beta]$ respectively, and the maximum principle asserting that the maximum modulus of u solving (1) is achieved at either α or β. We are interested in both: the classical and the distributional solutions in the convenient Sobolev space.

The theory of maximum principles is a broad discipline of mathematics (we refer for example to monographs [11, 13, 18], or to the papers [19, 23] and to their references).

Most of the maximum principles considered in this theory asserts the monotonicity property of the solution to the PDE (see also [16] for the nonclassical solutions). Roughly speaking, the monotonicity property asserts that if u is the solution to the given PDE defined on Ω and Ω_1, Ω_2 are (sufficiently regular) subsets of Ω such that $\Omega_1 \subseteq \Omega_2$ then $\max_{\partial\Omega_1}|u| \leq \max_{\partial\Omega_2}|u|$. The monotonicity property for the solution to the PDE requires an assumption that the PDE defined on Ω restricted to an arbitrary subdomain of Ω possesses the same structure. In particular it implies the maximum principle for its solution also on the subdomain. In such case we have $\max_{\partial\Omega_1}|u| = \max_{\Omega_1}|u| \leq \max_{\Omega_2}|u| = \max_{\partial\Omega_2}|u|$, provided that $\Omega_1 \subseteq \Omega_2 \subseteq \Omega$. Therefore the monotonicity property follows.

This is not our case. The structure of our equations is not invariant under the restriction to the subdomain. For that reason our principles admit oscillating functions as well.

Many papers deal also with various principles admitting oscillating functions (as e.g. [1, 4, 6, 7, 10, 14, 15, 25]). For example Duhoux [6, 7, 8] considers an eigenvalue Sturm-Liouville problem

$$-(r(t)u')' + p(t)u = \lambda m(t)u + h(t),$$

with boundary conditions of Dirichlet type on a bounded interval $[a, b]$ and uses properties of the Green function, spectral theory and topological degree theory to obtain maximum principles that are subordinated to the weight.

The paper [10] deals with the antimaximum principle for the problem

$$-\Delta_p u = \mu g(x)|u|^{p-2}u + h(x)$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p–Laplacian, $1 < p < \infty$, $\mu \in \mathbf{R}$, with some special g and h.

Quasilinear problems

$$-\operatorname{div}(A(x,u)\nabla u) + u = h \quad \text{in} \quad \Omega$$

with nonhomogeneous Neuman boundary conditions were studied in [1]. Both results [1] and [10] generalize an approach initiated in [4] (which dealt with the Laplace operator and Dirichlet boundary conditions) and were extended in [14] to linear elliptic operators. The paper [25] deals with elliptic systems.

Our techniques are inspired by an old theory by Szegö, who dealt with maximum principles for orthogonal polynomials (see [21, 22]). As was shown in [15] his ideas well apply to the study of maximum principles for solutions of a certain class of linear equations of the second order. For example hypergeometric equations in many cases can be treated by such an approach. We extend the results originated in [15] to fully nonlinear equations like (1). In particular some of the quasilinear equations can be also treated by this theory.

Our approach to equation (1) follows as a special case from the result dealing with a more general class of equations considered in Section 2. Therefore we believe that also some of other fully nonlinear equations can be treated by the method proposed here. We, however, restrict ourselves to the illustration of the technique from Section 2 within the class of equations starting from linear and ending up on equations like (1). Finally, in Chapter 4 we construct some nonlinear PDEs on the ball B, whose radial solutions achieve their extremas either in the center of B or on its boundary. Our examples deal with quasilinear equations involving the p–Laplacian. We also show some applications to the Sturm-Liouville problem and to the existence theory.

Let us mention that in the cases discussed here the function $\tau \mapsto A(\tau, \lambda_0, \lambda_1)$ in equation (1) achieves 0 on an interval $[\alpha, \beta]$. For example it vanishes at both its endpoints like in equation (2). Therefore our approach seems to be convenient to work with singular equations. This justifies our title.

The techniques of this paper can be extended in many directions. For example one can deal with positive radial solutions to PDEs, or obtain the weighted variants of the maximum principles. Some extensions to study oscilatory properties of the solutions are also possible, see e.g. [2, 3, 9] and their references for the related approaches. Therefore we hope that the ideas influenced by an old Szegö's theory and presented in this paper will contribute to the theory of maximum principles.

2 Maximum modulus principles. The general approach

2.1 Notation and basic definitions

Notation. In the sequel we assume that $\alpha, \beta \in \mathbf{R}$ and $\alpha < \beta$. We use the standard notation for Sobolev spaces $W^{m,p}((\alpha, \beta))$ and $W^{m,p}_{loc}((\alpha, \beta))$. By ∇f we denote the distributional gradient of f. The k–th distributional derivative of a one-variable function is denoted by $f^{(k)}$. For $k, N \in \mathbf{N}$, $k \leq N$, by k–Caratheodory function we will mean an arbitrary function defined on a subset of $\mathbf{R}^N$, which is measurable with respect to its first k variables and continuous with respect to the remaining ones.

We will deal with the following maximum modulus principles in the class of scalar functions defined on an interval.

Definition 2.1 (Maximum modulus principle)**.** Assume that $\alpha, \beta \in \mathbf{R}$ are given numbers, $\alpha < \beta$, and let M be a subset of $C^0([\alpha, \beta])$. We will say that M fulfills a maximum principle on $[\alpha, \beta]$ if for each $u \in M$ we have

$$\sup_{x\in[\alpha,\beta]}|u(x)| = \max\{|u(\alpha)|, |u(\beta)|\}.$$

Definition 2.2 (Left- and right-hand side maximum modulus principles)**.** Assume that $\alpha, \beta \in \mathbf{R}$ are given numbers, $\alpha < \beta$, and let M be a subset of $C^0([\alpha, \beta])$. We will say that M fulfills a left (right) hand side maximum principle on $[\alpha,\beta]$ if for each $u\in M$ we have

$$\sup_{x\in[\alpha,\beta]}|u(x)| = |u(\alpha)| \quad (|u(\beta)|).$$

The following two conditions will play a special role in our approach.

Definition 2.3 (Gradient condition)**.** We say that a 1–Caratheodory function $f:(\alpha,\beta)\times\mathbf{R}^3\to\mathbf{R}$ satisfies the *gradient condition* if there exists a function $\Psi\in C^1([\alpha,\beta]\times\mathbf{R}^2)$ such that

$$f(\tau,\lambda_0,\lambda_1,\lambda_2)\le\langle\nabla\Psi(\tau,\lambda_0,\lambda_1),(1,\lambda_1,\lambda_2)\rangle, \tag{3}$$

for almost every $\tau\in(\alpha,\beta)$ and every $(\lambda_0,\lambda_1,\lambda_2)\in\mathbf{R}^3$.

Definition 2.4 (Majorization property)**.** Let $L:(\alpha,\beta)\times\mathbf{R}^3\to\mathbf{R}$ be a 1-Caratheodory function, $G:(\alpha,\beta)\times\mathbf{R}^4\to\mathbf{R}$ be a 2–Caratheodory function and $\phi:\mathbf{R}\to\mathbf{R}$ be a continuous function. We will say L has *majorization property* with the pair (ϕ,G), if for almost every $\tau\in(\alpha,\beta)$ and every $\lambda=(\lambda_0,\lambda_1,\lambda_2)\in\mathbf{R}^3$

$$\phi(\lambda_0)\lambda_1\le G(\tau,\lambda,L(\tau,\lambda)). \tag{4}$$

2.2 Maximum modulus principles for Sobolev functions

We will deal with the following sets of conditions.

(Mo)

1. $G:(\alpha,\beta)\times\mathbf{R}^4\to\mathbf{R}$ is a 1–Caratheodory function and $\Psi\in C^1([\alpha,\beta]\times\mathbf{R}^2)$
2. $G(\cdot,0)$ satisfies gradient condition with the function Ψ.
3. $\phi:\mathbf{R}\to\mathbf{R}$ is an odd continuous function and $\tau\phi(\tau)>0$ for $\tau\ne0$.

(LMo):

1. $\Psi(\alpha,\lambda)\ge0$ and $\Psi(\beta,\lambda)\le0$ for every $\lambda\in\mathbf{R}^2$,
2. $\Psi(\tau,\lambda_0,0)\le0$ for every $(\tau,\lambda_0)\in(\alpha,\beta)\times(\mathbf{R}\setminus\{0\})$.

(RMo):

1. $\Psi(\alpha,\lambda)\ge0$ and $\Psi(\beta,\lambda)\le0$ for every $\lambda\in\mathbf{R}^2$,
2. $\Psi(\tau,\lambda_0,0)\ge0$ for every $(\tau,\lambda_0)\in(\alpha,\beta)\times(\mathbf{R}\setminus\{0\})$.

(MoI):

1. $\Psi(\alpha,\lambda)\ge0$ for every $\lambda\in\mathbf{R}^2$,
2. $\Psi(\tau,\lambda_0,0)\le0$ for every $(\tau,\lambda_0)\in(\alpha,\beta)\times(\mathbf{R}\setminus\{0\})$.

(Mor):

1. $\Psi(\beta, \lambda) \le 0$ for every $\lambda \in \mathbf{R}^2$,
2. $\Psi(\tau, \lambda_0, 0) \ge 0$ for every $(\tau, \lambda_0) \in (\alpha, \beta) \times (\mathbf{R} \setminus \{0\})$.

Our first result reads as follows.

Theorem 2.1. *Suppose that* ϕ, G *and* Ψ *satisfy* **(Mo)**, $L : (\alpha, \beta) \times \mathbf{R}^3 \to \mathbf{R}$ *is a* 1*-Caratheodory function, and let* M *be a set of solutions to the O.D.E:*

$$L(\tau, u(\tau), u^{'}(\tau), u^{''}(\tau)) = 0 \text{ a.e. in } (\alpha, \beta), \quad u \in W^{2,1}((\alpha, \beta)). \tag{5}$$

Then we have:

i) *If* L *has majorization property with the pair* (ϕ, G) *and the condition* **(LMo)** *is satisfied then* M *fulfills the left hand side maximum principle on* $[\alpha, \beta]$.

ii) *If* L *has majorization property with the pair* $(-\phi, G)$ *and the condition* **(RMo)** *is satisfied then* M *fulfills the right hand side maximum principle on* $[\alpha, \beta]$.

iii) *If* L *has majorization property with the pair* (ϕ, G) *and the conditions* **(Mol)** *is satisfied or if* L *has majorization property with the pair* $(-\phi, G)$ *and the conditions* **(Mor)** *are satisfied then* M *fulfills the maximum principle on* $[\alpha, \beta]$.

Proof. We give the proof of part i) only. The proofs of all remaining cases follow the same line. For the proof of part iii) we need to verify that if $u \in M$ and $|u|$ has a local maximum at $x \in (\alpha, \beta)$ then $|u(x)| \le |u(\alpha)|$ or $|u(x)| \le |u(\beta)|$ and apply the same arguments as in the previous parts.

Let $\Phi(x) = \int_0^x \phi(\tau) d\tau$, $\mathcal{M}$ be the set of all all points $x \in (\alpha, \beta)$ where $|u(x)| \neq 0$ and $|u|$ (so also u^2) attains its local maximum at x, and let $\mathcal{A} = \mathcal{M} \cup \{\beta\}$. Then Φ is strictly increasing on $[0, \infty)$. Therefore it suffices to show that for every $x \in \mathcal{A}$ we have: $A(\alpha, x) := \Phi(|u(x)|) - \Phi(|u(\alpha)|) \le 0$.

As $|u| \in W^{1,1}((\alpha, \beta))$ and $\Phi \in C^1$, using Nikodym ACL Characterization Theorem (see e.g. Theorem 1, Section 1.1.3 in [17]), we easily verify that $\Phi(|u|) \in W^{1,1}((\alpha, \beta))$, in particular

$$A(\alpha, x) = \int_\alpha^x \frac{d}{d\tau} \Phi(|u(\tau)|) d\tau = \int_\alpha^x \Phi^{'}(|u(\tau)|) \mathrm{sgn} u(\tau) u^{'}(\tau) d\tau.$$

Note that $\Phi^{'}(|\lambda|)\mathrm{sgn}\lambda = \phi(\lambda)$. Moreover, according to (4) and (5) we have for almost every $\tau \in (\alpha, \beta)$

$$\phi(u(\tau))u^{'}(\tau) \le G(\tau, u(\tau), u^{'}(\tau), u^{''}(\tau), 0)$$

and $G(\cdot, 0)$ satisfies the gradient condition. Therefore

$$\begin{aligned} A(\alpha, x) &\le \int_\alpha^x G(\tau, u(\tau), u^{'}(\tau), u^{''}(\tau), 0) d\tau \le \\ &\le \int_\alpha^x \langle \nabla \Psi(\tau, u(\tau), u^{'}(\tau)), \frac{d}{d\tau}(\tau, u(\tau), u^{'}(\tau)) \rangle d\tau, \end{aligned}$$

and using again Nikodym ACL Characterization Theorem again we observe that $\Psi(x,u(x),u^{'}(x)) \in W^{1,1}((\alpha,\beta))$ and the function under the sign of an integral equals $\frac{d}{d\tau}\Big(\Psi(\tau,u(\tau),u^{'}(\tau))\Big)$. This gives

$$A(\alpha,x) \leq \Psi(\tau,u(\tau),u^{'}(\tau))|_{\alpha}^{x}, \tag{6}$$

and the last term is nonnegative according to our assumptions. This ends the proof of the theorem. ■

Remark 2.1. As by the Sobolev Embedding Theorem $W^{2,1}((\alpha,\beta)) \subseteq C^1([\alpha,\beta])$, we observe that $G(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau),0)$ is measurable for every $u \in W^{2,1}((\alpha,\beta))$ and set M of solutions to (5) is contained in $C([\alpha,\beta])$.

2.3 Maximum modulus principles for classical solutions

Our goal now is to study the situation when the set M of solutions to the equation $L(\tau,u(\tau),u^{'}(\tau),u^{''}(\tau)) = 0$ consists of more regular functions than Sobolev's ones. Namely, we are interested in the case when additionally $M \subseteq C^2((\alpha,\beta))$. In such a situation we obtain another variant of the maximum principle.

For this purpose we introduce the following two sets:

$$\begin{aligned} \mathcal{S} &:= \{(\tau,\lambda_0,0,\lambda_2) \in (\alpha,\beta)\times(\mathbf{R}\setminus\{0\})\times\{0\}\times\mathbf{R}:\ \lambda_0\lambda_2 \leq 0 \text{ and } L(\tau,\lambda_0,0,\lambda_2)=0\},\\ \mathcal{S}^{'} &:= \{(\tau,\lambda_0,0) : \exists\lambda_2 \in \mathbf{R} : (\tau,\lambda_0,0,\lambda_2) \in \mathcal{S}\}, \end{aligned} \tag{7}$$

and we will deal with the following sets of conditions:

(Mo)

1. $G : (\alpha,\beta)\times\mathbf{R}^4 \to \mathbf{R}$ is a 1–Caratheodory function, $\Psi \in C^1([\alpha,\beta]\times\mathbf{R}^2)$ and $G(\cdot,0)$ satisfies the gradient condition with the function Ψ,
2. $\phi : \mathbf{R} \to \mathbf{R}$ is an odd continuous function and $\tau\phi(\tau) > 0$ for $\tau \neq 0$.

(LMo1):

1. $\Psi(\alpha,\lambda) \geq 0$ and $\Psi(\beta,\lambda) \leq 0$ for every $\lambda \in \mathbf{R}^2$,
2. if $\mathcal{S} \neq \emptyset$ then $\Psi(s) \leq 0$ for every $s \in \mathcal{S}^{'}$.

(RMo1):

1. $\Psi(\beta,\lambda) \leq 0$ and $\Psi(\alpha,\lambda) \geq 0$ for every $\lambda \in \mathbf{R}^2$,
2. if $\mathcal{S} \neq \emptyset$ then $\Psi(s) \geq 0$ for every $s \in \mathcal{S}^{'}$.

(Mol1):

1. $\Psi(\alpha,\lambda) \geq 0$ for every $\lambda \in \mathbf{R}^2$,
2. $\mathcal{S} \neq \emptyset$ and $\Psi(s) \leq 0$ for every $s \in \mathcal{S}^{'}$.

(Mor1):

1. $\Psi(\beta, \lambda) \leq 0$ for every $\lambda \in \mathbf{R}^2$,
2. $\mathcal{S} \neq \emptyset$ and $\Psi(s) \geq 0$ for every $s \in \mathcal{S}'$.

The result stated below is the interplay between the classical maximum principle for a one–variable function and the maximum principle presented in Theorem 2.1.

Theorem 2.2. *Suppose that $L : (\alpha, \beta) \times \mathbf{R}^3 \to \mathbf{R}$ is a continuous function, and M is the set of solutions to the O.D.E:*

$$L(\tau, u(\tau), u'(\tau), u''(\tau)) = 0 \text{ a.e. in } (\alpha, \beta), \quad u \in W^{2,1}((\alpha, \beta)) \cap C^2((\alpha, \beta)). \tag{8}$$

Then we have:

i) *If L has majorization property with the pair (ϕ, G) and conditions* **(Mo)** *and* **(LMo1)** *are satisfied then M fulfills the left hand side maximum principle on $[\alpha, \beta]$.*

ii) *If L has majorization property with the pair $(-\phi, G)$ and conditions* **(Mo)** *and* **(RMo1)** *are satisfied then M fulfills the right hand side maximum principle on $[\alpha, \beta]$.*

iii) *If one of the conditions is satisfied:*

- $\mathcal{S} = \emptyset$
- *L has majorization property with the pair (ϕ, G) and conditions* **(Mo)** *and* **(Mol1)** *is satisfied*
- *L has majorization property with the pair $(-\phi, G)$ and conditions* **(Mo)** *and* **(Mor1)** *are satisfied*

then M fulfills the maximum principle on $[\alpha, \beta]$.

Proof. We use the same techniques as previously. The only difference is that if L is continuous, u is the solution to (8) and $\mathcal{M}$ is the set of all points in (α, β), where $|u|$ achieves its local nonzero maximum then for every $x \in \mathcal{M}$ we have $(x, u(x), u'(x), u''(x)) \in \mathcal{S}$. ■

2.4 Preliminary remarks

Remark 2.2. Theorem 2.1 deals with more general assumptions than Theorem 2.2. Indeed, in Theorem 2.2 one assumes that L is continuous and that the solution u of the equation $L(x, u(x), u'(x), u''(x)) = 0$ is of class $C^2((\alpha, \beta))$. In such case the equation $L(\tau, \lambda_0, \lambda_1, \lambda_2) = 0$ is well defined at every point $(\tau, \lambda_0, \lambda_1, \lambda_2) \in (\alpha, \beta) \times \mathbf{R}^3$, and also the sign of the expression $u(\tau)u''(\tau)$ is well prescribed everywhere in (α, β). In particular the set $\mathcal{S}$ is well defined, and also the condition $(\tau, u(\tau), u'(\tau), u''(\tau)) \in \mathcal{S}$. This was not the case in Theorem 2.1. Therefore for purposes of Theorem 2.1 we had to verify the sign of the expression of the right hand side of (6) at β and on the set of all critical points of u inside (α, β). For purposes of Theorem 2.2 this set could be reduced to an essentially smaller one.

Remark 2.3. Theorem 2.2 links Theorem 2.1 and the classical maximum principles. Indeed, part iii) of Theorem 2.2 in the case when $\mathcal{S} = \emptyset$ is a variant of the classical one–dimensional maximum principle: the structure of the equation does not allow local extremas, and so they must be achieved at the boundary points of an interval. Note also that even in the case when one can use the classical maximum modulus principles to deduce that extremas of the solution to the equation are achieved at the boundary points of an interval, our statements allow to deduce at which boundary point of an interval this extremum is achieved.

Remark 2.4. The statement of part iii) in Theorem 2.2 in the case when $\mathcal{S} = \emptyset$ works also under weaker assumptions: $M \subseteq C([\alpha,\beta]) \cap C^{(2)}((\alpha,\beta))$. For proofs of the remaining cases we have used the assumption: $u \in W^{2,1}((\alpha,\beta)) \cap C^{(2)}((\alpha,\beta))$.

Remark 2.5. Contrary to the classical maximum principles, our principles cannot be inherited by subintervals. This means that if M fulfills the maximum principle on (α,β) then it does not imply that M fulfills a maximum principle on its arbitrary subinterval, as it is true for classical maximum principles. This is because our maximum principles admit the solutions of the equations to achieve their local extremas inside the domain (α,β). In particular our solutions can be non-monotonic as it happens in many known situations (see e.g. [16]). However, it is obvious that if M fulfills the left-hand side maximum principle on (α,β) then it also fulfills the left-hand side maximum principle on its arbitrary left subinterval: (α,t), for any $t \in (\alpha,\beta)$. Similar observation holds for the right–hand side maximum principles.

Remark 2.6. The assumptions required for our maximum principles involve conditions at the boundary points of an interval. For the classical maximum principles no assumptions involving the boundary of the domain are required.

3 Applications. Singular O.D.Es

Our goal now is to present equations, to which our techniques can be applied. In this Section we show that our maximum principles are very convenient to deal with singular O.D.Es. Our discussion will be supported by examples of linear, quasilinear and fully nonlinear equations.

3.1 Linear equations

Our first equations deal with the linear case. Let us introduce the following set of conditions:

(Lin) $a, c \in C^1([\alpha,\beta])$ and $c(\tau) \neq 0$ for every $\tau \in [\alpha,\beta]$, $b \in L^1((\alpha,\beta))$;

(LLin)

1. $\frac{a}{c}(\alpha) \leq 0$ and $\frac{a}{c}(\beta) \geq 0$,
2. for every $x \in (\alpha,\beta)$ we have: $(a^{'}(x) - 2b(x))c(x) \leq a(x)c^{'}(x)$;

(RLin)

1. $\frac{a}{c}(\alpha) \geq 0$ and $\frac{a}{c}(\beta) \leq 0$,
2. for every $x \in (\alpha, \beta)$ we have: $(a^{'}(x) - 2b(x))c(x) \geq a(x)c^{'}(x)$;

(Linl)

1. $\frac{a}{c}(\alpha) \leq 0$,
2. for every $x \in (\alpha, \beta)$ we have: $(a^{'}(x) - 2b(x))c(x) \leq a(x)c^{'}(x)$;

(Linr)

1. $\frac{a}{c}(\beta) \leq 0$,
2. for every $x \in (\alpha, \beta)$ we have: $(a^{'}(x) - 2b(x))c(x) \geq a(x)c^{'}(x)$.

Proposition 3.1 (Linear equation)**.** *Let M be the set of solutions to the O.D.E:*

$$a(\tau)u^{''}(\tau) + b(\tau)u^{'}(\tau) + c(\tau)u(\tau) = 0, \text{ for a.e. } x \in (\alpha, \beta), \; u \in W^{2,1}((\alpha, \beta)). \quad (9)$$

Then we have

i) *If the conditions* **(Lin)** *and* **(LLin)** *are satisfied then M fulfills the left hand side maximum principle on $[\alpha, \beta]$.*

ii) *If the conditions* **(Lin)** *and* **(RLin)** *are satisfied then M fulfills the right hand side maximum principle on $[\alpha, \beta]$.*

iii) *If the conditions* **(Lin)** *and either* **(Linl)** *or* **(Linr)** *are satisfied then M fulfills the maximum principle on $[\alpha, \beta]$.*

Proof. We give a detailed proof of part i) and sketch the proof of part ii) only, as the proof of the remaining part and missing details follow the same line.

i): Let $u \in M$ and

$$L(\tau, \lambda_0, \lambda_1, \lambda_2) = a(\tau)\lambda_2 + b(\tau)\lambda_1 + c(\tau)\lambda_0. \quad (10)$$

Then u satisfies the equation: $L(\tau, u(\tau), u^{'}(\tau), u^{''}(\tau)) = 0$ a.e., and for every $\tau \in (\alpha, \beta)$ and every $\lambda_0, \lambda_1, \lambda_2 \in \mathbf{R}$ we have

$$\lambda_0\lambda_1 = -\frac{a}{c}(\tau)\lambda_1\lambda_2 - \frac{b}{c}(\tau)\lambda_1^2 + \frac{L(\tau, \lambda_0, \lambda_1, \lambda_2)\lambda_1}{c(\tau)} := G(\tau, \lambda_0, \lambda_1, \lambda_2, L(\tau, \lambda_0, \lambda_1, \lambda_2)). \quad (11)$$

Now the result follows from part i) of Theorem 2.1 if we take $\phi(\lambda_0) = \lambda_0$ and verify that the function

$$g(\tau, \lambda_0, \lambda_1, \lambda_2) = -\frac{a}{c}(\tau)\lambda_1\lambda_2 - \frac{b}{c}(\tau)\lambda_1^2$$

satisfies the gradient condition with the function $\Psi(\tau,\lambda_0,\lambda_1) = -\frac{a}{2c}(\tau)\lambda_1^2$, which reads as:

$$g(\tau,\lambda_0,\lambda_1,\lambda_2) = \langle(-\frac{b}{c}(\tau)\lambda_1^2, 0, -\frac{a}{c}(\tau)\lambda_1), (1,\lambda_1,\lambda_2)\rangle \leq \langle\nabla\Psi(\tau,\lambda_0,\lambda_1), \langle 1,\lambda_1,\lambda_2\rangle\rangle.$$

ii): Let L be given by (10). Then instead of (11) used in the proof of part i) now we use the equation:

$$(-\lambda_0)\lambda_1 = \frac{a}{c}(\tau)\lambda_1\lambda_2 + \frac{b}{c}(\tau)\lambda_1^2 - \frac{L(\tau,\lambda_0,\lambda_1,\lambda_2)\lambda_1}{c(\tau)} := G(\tau,\lambda_0,\lambda_1,\lambda_2,L(\tau,\lambda_0,\lambda_1,\lambda_2)),$$

and check that for $\lambda = (\lambda_1,\lambda_1,\lambda_2)$ the function $g(\tau,\lambda) := G(\tau,\lambda,0)$ satisfies gradient condition with the function $\Psi(\tau,\lambda_0,\lambda_1) = \frac{a}{2c}(\tau)\lambda_1^2$. Now the result follows from part ii) of Theorem 2.1. ∎

Remark 3.1. In the proof of Proposition 3.1 we have used Theorem 2.1 and did not apply Theorem 2.2. The reason is that we did not have any extra regularity assumptions on solutions to (9) as we do not assume that $M \subseteq C^2((\alpha,\beta))$.

Example 3.1. The Legendre polynomial

$$W_n(\tau) = C_n((1-\tau)^n(1+\tau)^n)^{(n)} = W_n(-\tau)$$

satisfies hypergeometric equation of Gauss:

$$(1-\tau^2)W_n^{(2)}(\tau) - 2\tau W_n'(\tau) + (n+1)nW_n(\tau) = 0, \text{ for } \tau \in (-1,1)$$

and fulfills the assumptions of Theorem 3.1. Hence $\max_{x\in[-1,1]}|W_n(x)| = |W_n(1)| = |W_n(-1)|$. This result is known in the theory of orthogonal polynomials (see [21], Theorem 7.4.1, and [15] for some other variants of this theorem).

Remark 3.2. Maximum principles for linear equations (9), their applications to the theory of special functions, to the existence of solutions of (9), and also relations with Szegö's theory were studied in [15], where more general forms of Proposition 3.1 were obtained.

Remark 3.3. An adaptation of Theorem 2.2, parts i), ii) and iii) for $\mathcal{S} \neq \emptyset$, where $\mathcal{S}$ is defined in (7) for the equation (9) does not bring a new result, as we only impose additional restrictions: $b \in C((\alpha,\beta))$, $M \subseteq C^2((\alpha,\beta))$, so we can adopt the same proof as that for Proposition 3.1. The situation changes when we deal with the case $\mathcal{S} = \emptyset$ in part iii). An analysis when such situation can appear together with Remark 2.4 leads to the following variant of the weak maximum principle:

Theorem 3.1 (Weak maximum principle)**.** *Assume that $\alpha,\beta \in \mathbf{R}$ are given numbers such that $\alpha < \beta$. Let $a,b,c \in C((\alpha,\beta))$ be given functions such that $a \geq 0$, $c < 0$ in (α,β). Suppose that u is a solution of the equation*

$$a(x)u''(x)+b(x)u'(x)+c(x)u(x) = 0, \text{ for } x \in (\alpha,\beta), u \in C^2((\alpha,\beta))\cap C([\alpha,\beta]). \quad (12)$$

Then $\max_{\tau\in[\alpha,\beta]}|u(\tau)| = \max\{|u(\alpha)|,|u(\beta)|\}$.

Proof. We verify that the situation $a\lambda_2 + c\lambda_0 = 0$ for $\lambda_0 \neq 0$ and $\lambda_2\lambda_0 \leq 0$ is impossible. □

More sophisticated arguments (see e.g. [11], Sections 3.1, 3.2 and 3.3 in [13], [18]) show that the assumption: $a \geq 0, c < 0$ can be changed to: $a > 0, c \leq 0$ in (α,β). The variant of the above theorem with such an assumption is a case of the classical weak maximum principle.

3.2 Quasilinear equations

For our purposes we will deal with the following sets of conditions.

(Qlin)

1. $\Psi \in C^1([\alpha,\beta] \times \mathbf{R}^2)$,
2. $A : (\alpha,\beta) \times \mathbf{R}^2 \to \mathbf{R}$ is continuous and $A(\tau,\lambda_0,\lambda_1) = -\dfrac{\partial \Psi}{\partial \lambda_1}(\tau,\lambda_0,\lambda_1)$,
3. $B : (\alpha,\beta) \times \mathbf{R}^2 \to \mathbf{R}$ is a 1-Caratheodory function,
4. $\phi : \mathbf{R} \to \mathbf{R}$ is an odd continuous function such that $\tau\phi(\tau) > 0$ for $\tau \neq 0$;

(LQlin)

1. $\Psi(\alpha,\cdot) \geq 0$ and $\Psi(\beta,\cdot) \leq 0$,
2. $\Psi(\tau,\lambda_0,0) \leq 0$ for every $\tau \in (\alpha,\beta), \lambda_0 \in \mathbf{R} \setminus \{0\}$,
3. $-B(\tau,\lambda_0,\lambda_1) \leq -\phi(\lambda_0)\lambda_1 + \dfrac{\partial \Psi}{\partial \tau}(\tau,\lambda_0,\lambda_1) + \dfrac{\partial \Psi}{\partial \lambda_0}(\tau,\lambda_0,\lambda_1)\lambda_1$;

(RQlin)

1. $\Psi(\alpha,\cdot) \geq 0$ and $\Psi(\beta,\cdot) \leq 0$,
2. $\Psi(\tau,\lambda_0,0) \geq 0$ for every $\tau \in (\alpha,\beta), \lambda_0 \in \mathbf{R} \setminus \{0\}$,
3. $B(\tau,\lambda_0,\lambda_1) \leq \phi(\lambda_0)\lambda_1 + \dfrac{\partial \Psi}{\partial \tau}(\tau,\lambda_0,\lambda_1) + \dfrac{\partial \Psi}{\partial \lambda_0}(\tau,\lambda_0,\lambda_1)\lambda_1$;

(Qlinl)

1. $\Psi(\alpha,\cdot) \geq 0$,
2. $\Psi(\tau,\lambda_0,0) \leq 0$ for every $\tau \in (\alpha,\beta), \lambda_0 \in \mathbf{R} \setminus \{0\}$,
3. $-B(\tau,\lambda_0,\lambda_1) \leq -\phi(\lambda_0)\lambda_1 + \dfrac{\partial \Psi}{\partial \tau}(\tau,\lambda_0,\lambda_1) + \dfrac{\partial \Psi}{\partial \lambda_0}(\tau,\lambda_0,\lambda_1)\lambda_1$;

(Qlinr)

1. $\Psi(\beta,\cdot) \leq 0$,
2. $\Psi(\tau,\lambda_0,0) \geq 0$ for every $\tau \in (\alpha,\beta), \lambda_0 \in \mathbf{R} \setminus \{0\}$,
3. $B(\tau,\lambda_0,\lambda_1) \leq \phi(\lambda_0)\lambda_1 + \dfrac{\partial \Psi}{\partial \tau}(\tau,\lambda_0,\lambda_1) + \dfrac{\partial \Psi}{\partial \lambda_0}(\tau,\lambda_0,\lambda_1)\lambda_1$.

Our next proposition deals with the quasilinear case.

Proposition 3.2 (Quasilinear equation)**.** *Let M be the set of solutions to the O.D.E:*

$$A(\tau,u(\tau),u'(\tau))u''(\tau) + B(\tau,u(\tau),u'(\tau)) = 0, \text{ for a.e } \tau \in (\alpha,\beta), u \in W^{2,1}((\alpha,\beta)). \tag{13}$$

Then we have

i) *If the conditions* **(Qlin)** *and* **(LQlin)** *are satisfied then M fulfills the left hand side maximum principle on $[\alpha, \beta]$.*

ii) *If the conditions* **(Qlin)** *and* **(RQlin)** *are satisfied then M fulfills the right hand side maximum principle on $[\alpha, \beta]$.*

iii) *If the conditions* **(Qlin)** *and either* **(Qlinl)** *or* **(Qlinr)** *are satisfied then M fulfills the maximum principle on $[\alpha, \beta]$.*

Proof. We give the detailed proof of part i) only and sketch the proof of part ii). The remaining proofs follow the same line and are left to the reader.

i): Let $u \in M$, $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ and

$$L(\tau, \lambda) = A(\tau, \lambda_0, \lambda_1)\lambda_2 + B(\tau, \lambda_0, \lambda_1). \tag{14}$$

Then $L(\tau, u(\tau), u'(\tau), u''(\tau)) = 0$ for almost every $\tau \in (\alpha, \beta)$, and

$$\phi(\lambda_0)\lambda_1 = -B(\tau, \lambda_0, \lambda_1) + \phi(\lambda_0)\lambda_1 - A(\tau, \lambda_0, \lambda_1)\lambda_2 + L(\tau, \lambda) := G(\tau, \lambda, L(\tau, \lambda)). \tag{15}$$

We set $g(\tau, \lambda) := G(\tau, \lambda, 0)$ and verify that

$$g(\tau, \lambda) = -B(\tau, \lambda_0, \lambda_1) + \phi(\lambda_0)\lambda_1 - A(\tau, \lambda_0, \lambda_1)\lambda_2 \leq \langle \nabla\Psi(\tau, \lambda_0, \lambda_1), (1, \lambda_1, \lambda_2)\rangle,$$

in particular the function g satisfies the gradient condition. Now the result follows from part i) of Theorem 2.1.

ii): We follow the same line as in the proof of part i), but instead of (15) we use the equation

$$-\phi(\lambda_0)\lambda_1 = B(\tau, \lambda_0, \lambda_1) - \phi(\lambda_0)\lambda_1 + A(\tau, \lambda_0, \lambda_1)\lambda_2 - L(\tau, \lambda) := G(\tau, \lambda, L(\tau, \lambda)).$$

■

Remark 3.4. Proposition 3.1 can be deduced directly from Proposition 3.2 when we consider

$$A(\tau, \lambda_0, \lambda_1) = \frac{a}{c}(\tau)\lambda_1, \quad B(\tau, \lambda_0, \lambda_1) = \frac{b}{c}(\tau)\lambda_1^2 + \lambda_0\lambda_1,$$

and note that the solution to (9) satisfies a weaker equation:

$$\frac{a}{c}(\tau)u'(\tau)u''(\tau) + \frac{b}{c}(\tau)(u'(\tau))^2 + u(\tau)u'(\tau) = 0 \text{ a.e. in } (\alpha, \beta).$$

For example in the proof of part i) of Proposition 3.1 we take $\Psi(\tau) = -\frac{a}{2c}(\tau)\lambda_1^2$, $\phi(\lambda_0) = \lambda_0$ and verify the assumptions **(Qlin)** and **(LQlin)**.

For the sake of completeness we present below the counterpart of Theorem 2.2 adopted to the quasilinear equation. We deal with the following set of conditions.

(Qlin1)

1. $\Psi \in C^1([\alpha, \beta] \times \mathbf{R}^2)$,
2. $A, B : (\alpha, \beta) \times \mathbf{R}^2 \to \mathbf{R}$ are continuous and $A(\tau, \lambda_0, \lambda_1) = -\dfrac{\partial\Psi}{\partial\lambda_1}(\tau, \lambda_0, \lambda_1)$,
3. $\phi : \mathbf{R} \to \mathbf{R}$ is an odd continuous function such that $\tau\phi(\tau) > 0$ for $\tau \neq 0$;

(LQlin1)

1. $\Psi(\alpha,\cdot)\geq 0$ and $\Psi(\beta,\cdot)\leq 0$,
2. if $\mathcal{S}\neq\emptyset$ then $\Psi(s)\leq 0$ for every $s\in\mathcal{S}'$,
3. $-B(\tau,\lambda_0,\lambda_1)\leq -\phi(\lambda_0)\lambda_1+\dfrac{\partial\Psi}{\partial\tau}(\tau,\lambda_0,\lambda_1)+\dfrac{\partial\Psi}{\partial\lambda_0}(\tau,\lambda_0,\lambda_1)\lambda_1$;

(RQlin1)

1. $\Psi(\alpha,\cdot)\geq 0$ and $\Psi(\beta,\cdot)\leq 0$,
2. if $\mathcal{S}\neq\emptyset$ then $\Psi(s)\geq 0$ for every $s\in\mathcal{S}'$,
3. $B(\tau,\lambda_0,\lambda_1)\leq \phi(\lambda_0)\lambda_1+\dfrac{\partial\Psi}{\partial\tau}(\tau,\lambda_0,\lambda_1)+\dfrac{\partial\Psi}{\partial\lambda_0}(\tau,\lambda_0,\lambda_1)\lambda_1$;

(Qlinl1)

1. $\Psi(\alpha,\cdot)\geq 0$,
2. $\mathcal{S}\neq\emptyset$ and $\Psi(s)\leq 0$ for every $s\in\mathcal{S}'$,
3. $-B(\tau,\lambda_0,\lambda_1)\leq -\phi(\lambda_0)\lambda_1+\dfrac{\partial\Psi}{\partial\tau}(\tau,\lambda_0,\lambda_1)+\dfrac{\partial\Psi}{\partial\lambda_0}(\tau,\lambda_0,\lambda_1)\lambda_1$;

(Qlinr1)

1. $\Psi(\beta,\cdot)\leq 0$,
2. $\mathcal{S}\neq\emptyset$ and $\Psi(s)\geq 0$ for every $s\in\mathcal{S}'$,
3. $B(\tau,\lambda_0,\lambda_1)\leq \phi(\lambda_0)\lambda_1+\dfrac{\partial\Psi}{\partial\tau}(\tau,\lambda_0,\lambda_1)+\dfrac{\partial\Psi}{\partial\lambda_0}(\tau,\lambda_0,\lambda_1)\lambda_1$,

where $\mathcal{S}$ and $\mathcal{S}'$ are given by

$$\begin{aligned}\mathcal{S} &:= \{(\tau,\lambda_0,0,\lambda_2)\in(\alpha,\beta)\times(\mathbf{R}\setminus\{0\})\times\{0\}\times\mathbf{R}:\ \lambda_0\lambda_2\leq 0 \text{ and}\\ &\quad\ A(\tau,\lambda_0,0)\lambda_2+B(\tau,\lambda_0,0)=0\},\\ \mathcal{S}' &:= \{(\tau,\lambda_0,0):\exists_{\lambda_2\in\mathbf{R}}:(\tau,\lambda_0,0,\lambda_2)\in\mathcal{S}\}.\end{aligned}$$

Our result reads as follows. Its easy proof is left to the reader.

Proposition 3.3. *Let M be the set of solutions to the O.D.E:*

$$A(\tau,u(\tau),u'(\tau))u''(\tau)+B(\tau,u(\tau),u'(\tau))=0,\ \text{for a.e } \tau\in(\alpha,\beta),\qquad u\in W^{2,1}((\alpha,\beta))\cap C^2((\alpha,\beta)).\tag{16}$$

Then we have:

i) *If the conditions* **(Qlin1)** *and* **(LQlin1)** *are satisfied then M fulfills the left hand side maximum principle on $[\alpha,\beta]$.*

ii) *If the conditions* **(Qlin1)** *and* **(RQlin1)** *are satisfied then M fulfills the right hand side maximum principle on $[\alpha,\beta]$.*

iii) *If one of the conditions is satisfied:*

- $\mathcal{S} = \emptyset$
- *the conditions* **(Qlin1)** *and* **(Qlinl1)** *are satisfied*
- *the conditions* **(Qlin1)** *and* **(Qlinr1)** *are satisfied*

then M fulfills the maximum principle on $[\alpha, \beta]$.

3.3 Fully nonlinear equations

Now it is only a matter of routine to generalize the results of Proposition 3.2 to the fully nonlinear case. The generalization reads as follows.

Proposition 3.4 (Fully nonlinear equation). *Let $M \subseteq W^{2,1}((\alpha, \beta))$ be the set of solutions to the O.D.E:*

$$A(\tau, u(\tau), u^{'}(\tau))u^{''}(\tau)+B(\tau, u(\tau), u^{'}(\tau))+f(\tau, u(\tau), u^{'}(\tau), u^{''}(\tau)) = 0, \text{ a.e. in } (\alpha, \beta). \tag{17}$$

Then we have

i) *If the conditions* **(Qlin)** *and* **(LQlin)** *are satisfied and $f \geq 0$ is an arbitrary 1-Caratheodory function then M fulfills the left hand side maximum principle on $[\alpha, \beta]$.*

ii) *If the conditions* **(Qlin)** *and* **(RQlin)** *are satisfied and $f \leq 0$ is an arbitrary 1-Caratheodory function then M fulfills the right hand side maximum principle on $[\alpha, \beta]$.*

iii) *If the conditions* **(Qlin)** *and either (*(**Qlinl**) *and $f \geq 0$) or (*(**Qlinr**) *and $f \leq 0$) are satisfied and f is an arbitrary 1-Caratheodory function then M fulfills the maximum principle on $[\alpha, \beta]$.*

Proof. We use the same techniques as in the proof of Proposition 3.2 for

$$L(\tau, \lambda_0, \lambda_1, \lambda_2) = A(\tau, \lambda_0, \lambda_1)\lambda_2 + B(\tau, \lambda_0, \lambda_1) + f(\tau, \lambda_0, \lambda_1, \lambda_2).$$

□

The counterpart of Proposition 3.3 reads as follows. Its easy proof is left to the reader.

Proposition 3.5. *Let M be the set of solutions to the O.D.E:*

$$A(\tau, u(\tau), u^{'}(\tau))u^{''}(\tau) + B(\tau, u(\tau), u^{'}(\tau)) + f(\tau, u(\tau), u^{'}(\tau), u^{''}(\tau)) = 0, \tag{18}$$

for a.e $\tau \in (\alpha, \beta)$, $u \in W^{2,1}((\alpha, \beta)) \cap C^2((\alpha, \beta))$. Then we have:

i) *If the conditions* **(Qlin1)** *and* **(LQlin1)** *are satisfied and $f \geq 0$ is an arbitrary continuous function then M fulfills the left hand side maximum principle on $[\alpha, \beta]$.*

ii) *If the conditions* **(Qlin1)** *and* **(RQlin1)** *are satisfied and* $f \leq 0$ *is an arbitrary continuous function then* M *fulfills the right hand side maximum principle on* $[\alpha, \beta]$.

iii) *If one of the conditions is satisfied:*

- $\mathcal{S} = \emptyset$
- *the conditions* **(Qlin1)** *and* **(Qlinl1)** *are satisfied and* $f \geq 0$ *is an arbitrary continuous function*
- *the conditions* **(Qlin1)** *and* **(Qlinr1)** *are satisfied and and* $f \leq 0$ *is an arbitrary continuous function*

then M *fulfills the maximum principle on* $[\alpha, \beta]$.

4 The PDEs. Equations involving the p-Laplacian

Our goal now is to illustrate our approach by showing the PDEs, which are invariant under rotations, and to which our techniques can be applied. We will specialize in inequalities involving the p–Laplacian. Obviously, one can construct many examples illustrating our previous results and we are not claiming that our choice is complete. For some other related results dealing with the radial solutions of equations involving the p–Laplacian we refer for example to [5, 12, 20, 23, 24, 26] and to their references.

Our first example shows that radial solutions to some quasilinear equations involving the p-Laplacian are upperbounded by their value at 0. This property is described in the Proposition 4.1 stated below.

Proposition 4.1. *Let* B *be an arbitrary ball in* $\mathbf{R}^n$ *with radius* r *and center at* 0 *and* $w \in W^{2,1}(B)$ *be a radial solution to the equation*

$$-|x|^\alpha \left(\text{div}|\nabla w(x)|^{p-2}\nabla w(x)\right) = \phi(w(x)), \text{ a.e. in } B \tag{19}$$

where $p \geq 2$, $\frac{(n-1)p}{p-1} \geq \alpha \geq 1$ *and* ϕ *is an arbitrary odd function such that* $\tau\phi(\tau) > 0$ *for* $\tau \neq 0$. *Then* $\sup_{x \in B}|w(x)| = |w(0)|$.

Proof. As $w(x) = u(|x|)$, where $u \in W^{2,1}(0, r)$ is a one–variable function, u solves the $O.D.E$:

$$(p-1)\tau^\alpha|u^{'}(\tau)|^{p-2}u^{''}(\tau) + (n-1)\tau^{\alpha-1}|u^{'}(\tau)|^{p-2}u^{'}(\tau) + \phi(u(\tau)) = 0, \tag{20}$$

and also its weaker variant:

$$(p-1)\tau^\alpha|u^{'}(\tau)|^{p-2}u^{'}(\tau)u^{''}(\tau) + (n-1)\tau^{\alpha-1}|u^{'}(\tau)|^p + \phi(u(\tau))u^{'}(\tau) = 0. \tag{21}$$

This is a quasilinear equation of the form:

$$A(\tau, u(\tau), u^{'}(\tau))u^{''}(\tau) + B(\tau, u(\tau), u^{'}(\tau)) = 0$$

where

$$A(\tau, \lambda_0, \lambda_1) = (p-1)\tau^\alpha|\lambda_1|^{p-2}\lambda_1, \qquad B(\tau, \lambda_0, \lambda_1) = (n-1)\tau^{\alpha-1}|\lambda_1|^p + \phi(\lambda_0)\lambda_1.$$

Now it suffices to apply part i) of Proposition 3.2 with $\Psi(\tau, \lambda_0, \lambda_1) = -(1-\frac{1}{p})\tau^\alpha|\lambda_1|^p$ and ϕ as in the statement of the Proposition. ■

As an immediate corollary we obtain the following application to the Sturm-Liouville problem involving the p-Laplacian.

Corollary 4.1. *Let B be an arbitrary ball in $\mathbf{R}^n$ with radius r and center at 0 and $w \in W^{2,1}(B)$ be a solution to the eigenvalue problem*

$$-|x|^\alpha \left(\mathrm{div}|\nabla w(x)|^{p-2}\nabla w(x)\right) = \lambda w(x), \text{ a.e. in } B \tag{22}$$

with an arbitrary positive λ, $p \geq 2$ and $\frac{(n-1)p}{p-1} \geq \alpha \geq 1$. Then $\sup_{x\in B}|w(x)| = |w(0)|$.

One could expect that the radial solutions of the equation $-|x|^\alpha(1-|x|^\beta)\Delta_p w = \phi(w)$, $x \in B(1)$, with some positive parameters α and β achieve their extremas at the boundary points of $B(1)$. The result stated below shows that it is not possible if α is the same as in Proposition 4.1 and $\beta \geq 0$ is taken arbitrary.

Proposition 4.2. *Let B be the unit ball in $\mathbf{R}^n$ with center at 0 and $w \in W^{2,1}(B)$ be a radial solution to the equation*

$$-|x|^\alpha(1-|x|)^\beta \left(\mathrm{div}|\nabla w(x)|^{p-2}\nabla w(x)\right) = \phi(w(x)), \text{ a.e. in } B \tag{23}$$

where $p \geq 2$, $\frac{(n-1)p}{p-1} \geq \alpha \geq 1$, $\beta \geq 0$ and ϕ is an arbitrary odd continuous function such that $\tau\phi(\tau) > 0$ for $\tau \neq 0$. Then $\sup_{x\in B}|w(x)| = |w(0)|$.

Proof. Let $w(x) = u(|x|)$, $u \in W^{2,1}(0,1)$. Then u solves the *O.D.E*:

$$(p-1)\tau^\alpha(1-\tau)^\beta|u'(\tau)|^{p-2}u''(\tau) + (n-1)\tau^{\alpha-1}(1-\tau)^\beta|u'(\tau)|^{p-2}u'(\tau) + \phi(u(\tau)) = 0, \tag{24}$$

and also its weaker variant:

$$(p-1)\tau^\alpha(1-\tau)^\beta|u'(\tau)|^{p-2}u'(\tau)u''(\tau) + (n-1)\tau^{\alpha-1}(1-\tau)^\beta|u'(\tau)|^p + \phi(u(\tau))u'(\tau) = 0, \tag{25}$$

which is a quasilinear equation of the form:

$$A(\tau, u(\tau), u'(\tau))u''(\tau) + B(\tau, u(\tau), u'(\tau)) = 0$$

where

$$A(\tau,\lambda_0,\lambda_1) = (p-1)\tau^\alpha(1-\tau)^\beta|\lambda_1|^{p-2}\lambda_1, \;\; B(\tau,\lambda_0,\lambda_1) = (n-1)\tau^{\alpha-1}(1-\tau)^\beta|\lambda_1|^p + \phi(\lambda_0)\lambda_1.$$

We apply part i) of Proposition 3.2 with $\Psi(\tau,\lambda_0,\lambda_1) = -(1-\frac{1}{p})\tau^\alpha(1-\tau)^\beta|\lambda_1|^p$ and ϕ as in the statement of the Proposition. It suffices to check that $-B \leq -\phi\lambda_1 + \frac{\partial\Psi}{\partial\tau} + \frac{\partial\Psi}{\partial\lambda_0}\lambda_1$. This is equivalent to the inequality:

$$-(n-1)\tau^{\alpha-1}(1-\tau)^\beta \leq -(1-\frac{1}{p})\frac{d}{d\tau}\left(\tau^\alpha(1-\tau)^\beta\right), \quad \tau \in (0,1),$$

which reduces to

$$(n-1) - (1-\frac{1}{p})\alpha \geq \left((n-1) - (1-\frac{1}{p})(\alpha+\beta)\right)\tau, \quad \tau \in (0,1).$$

Within the given range of parameters it is always satisfied. ■

As a corollary we immediately obtain the following result.

Corollary 4.2. *Let B be the unit ball in $\mathbf{R}^n$ with center at 0 and $w \in W^{2,1}(B)$ be a solution to the eigenvalue problem*

$$-|x|^\alpha(1-|x|)^\beta \left(\mathrm{div}|\nabla w(x)|^{p-2}\nabla w(x)\right) = \lambda w(x), \text{ a.e. in } B \tag{26}$$

with a positive λ and $p \geq 2$, $\frac{(n-1)p}{p-1} \geq \alpha \geq 1$, $\beta \geq 0$. Then $\sup_{x\in B}|w(x)| = |w(0)|$.

Our goal now is to deal with equations like $|x|^\alpha(1-|x|)^\beta \Delta_p w(x) = g(|x|, w(x), \langle \nabla w(x), \frac{x}{|x|}\rangle)$, $x \in B(1)$, the disturbed variants of equation (19) and show that in this case we can expect that their radial solutions achieve their extremas at $\partial B(1)$. Our next result reads as follows.

Proposition 4.3. *Let B be the unit ball in $\mathbf{R}^n$ with center at 0 and $w \in W^{2,1}(B)$ be a radial solution to the equation*

$$-|x|^\alpha(1-|x|)^\beta \left(\mathrm{div}|\nabla w(x)|^{p-2}\nabla w(x)\right) = \phi(w(x)) + h(|x|, \langle \nabla w(x), \frac{x}{|x|}\rangle) \text{ a.e. in } B, \tag{27}$$

where

1. ϕ is an arbitrary odd function such that $\tau\phi(\tau) > 0$ for $\tau \neq 0$,

2. $h(\tau, \lambda_1) = -r(\tau)\tau^{\alpha-1}(1-\tau)^{\beta-1}|\lambda_1|^{p-2}\lambda_1$, and $r(\tau)$ is an arbitrary C^1 function such that

$$C_1 + C_2\tau \leq r(\tau) \quad \text{for every} \quad \tau \in (0,1),$$

with $C_1 = n-1+(1-\frac{1}{p})\alpha$, $C_2 = -(1-\frac{1}{p})(\alpha+\beta) - n + 1$, $\alpha, \beta > 0$.

Then $\sup_{x\in B}|w(x)| = \sup_{x\in\partial B}|w(x)|$.

Proof. Let $w(x) = u(|x|)$, Then $u \in W^{2,1}(0,1)$ and u solves the *O.D.E* satisfied on $(0,1)$:

$$\begin{aligned}(p-1)\tau^\alpha(1-\tau)^\beta|u'(\tau)|^{p-2}u''(\tau) &+ (n-1)\tau^{\alpha-1}(1-\tau)^\beta|u'(\tau)|^{p-2}u'(\tau) + \\ &+ \phi(u(\tau)) + h(\tau, u'(\tau)) = 0,\end{aligned}$$

and also its weaker variant

$$\begin{aligned}(p-1)\tau^\alpha(1-\tau)^\beta|u'(\tau)|^{p-2}u'(\tau)u''(\tau) &+ (n-1)\tau^{\alpha-1}(1-\tau)^\beta|u'(\tau)|^p + \\ &+ \phi(u(\tau))u'(\tau) + h(\tau, u'(\tau))u'(\tau) = 0.\end{aligned}$$

This is a quasilinear equation of the form

$$A(\tau, u(\tau), u'(\tau)) + B(\tau, u(\tau), u'(\tau)) = 0,$$

where

$$\begin{aligned}A(\tau, \lambda_0, \lambda_1) &= (p-1)\tau^\alpha(1-\tau)^\beta|\lambda_1|^{p-2}\lambda_1 \\ B(\tau, \lambda_0, \lambda_1) &= (n-1)\tau^{\alpha-1}(1-\tau)^\beta|\lambda_1|^p + \phi(\lambda_0)\lambda_1 + h(\tau, \lambda_1)\lambda_1.\end{aligned}$$

Using part ii) of Proposition 3.2 with $\Psi(\tau,\lambda_0,\lambda_1) = -(1-\frac{1}{p})\tau^\alpha(1-\tau)^\beta|\lambda_1|^p$ and ϕ as in the statement of the Proposition, it suffices to check that $B \leq \phi\lambda_1 + \frac{\partial\Psi}{\partial\tau} + \frac{\partial\Psi}{\partial\lambda_0}\lambda_1$. This will be done after we verify that

$$(n-1)\tau^{\alpha-1}(1-\tau)^\beta|\lambda_1|^p + h(\tau,\lambda_1)\lambda_1 \leq -(1-\frac{1}{p})\frac{d}{d\tau}\left(\tau^\alpha(1-\tau)^\beta\right)|\lambda_1|^p. \tag{28}$$

As

$$\frac{d}{d\tau}\left(\tau^\alpha(1-\tau)^\beta\right) = \tau^{\alpha-1}(1-\tau)^{\beta-1}\left(\alpha-(\alpha+\beta)\tau\right),$$

the inequality (28) reduces to the verification that

$$(n-1)+(1-\frac{1}{p})\alpha+\tau(1-n-(1-\frac{1}{p})(\alpha+\beta)) \leq r(\tau),$$

which is satisfied under our assumptions. ∎

Our results can be applied to the existence theory. As a corollary from Proposition 4.3 we obtain for instance the following result.

Corollary 4.3. *Let B be the unit ball in $\mathbf{R}^n$ with center at 0 and ϕ and h be the same as in Proposition 4.3. Then the problem*

$$-|x|^\alpha(1-|x|)^\beta\left(\mathrm{div}|\nabla w(x)|^{p-2}\nabla w(x)\right) = \phi(w(x)) + h(|x|,\langle\nabla w(x),\frac{x}{|x|}\rangle) \text{ a.e. in } B,$$
$$u(0)=u_0\neq 0, \qquad u\equiv 0 \text{ on } \partial B$$

admits no radial solutions in the class $W^{2,1}(B)$.

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