Optimal strategies in high risk investments

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Abstract

A decision-maker observes sequentially a given permutation of n uniquely rankable options. He has to invest capital into these opportunities at the moment when they appear. At each step only relative ranks are known. At the end the true rank of the option, at which the investment has been made, is known. [Bruss and Ferguson(2002)] have considered such problems under the assumption that an investment on the very best opportunity yields a lucrative, possibly time-dependent, rate of return. Uninvested capital keeps its risk-free value. Wrong investments lose their value.

In this paper we partially extend results by [Bruss and Ferguson(2002)]. We confine our study to linear utility but a wider range of payoffs is taken into account. Two cases are considered. The first-type payoff gives a positive rate of return if the investment is made on the best or the second best option. The second-type payoff pays when the investment is at the second best option. We motivate these payoff choices. A few examples are explicitly solved.

1   Introduction

In their paper [Bruss and Ferguson(2002)] have considered the investment models based on the best-only payoff of the classical *secretary problem*. Although a version of the secretary problem (the beauty contest problem, the dowry problem or the marriage problem) was first solved by [Cayley(1875)], it was not until four decades ago there had been sudden resurgence of interest in this problem. Since the articles by [Gardner(1960a), Gardner(1960b)] the secretary problem has been extended

and generalized in many different directions. Excellent reviews of the development of this colorful problem and its extensions have been given by [Rose(1982b)], [Freeman(1983)], [Samuels(1991)] and [Ferguson(1989)].

[Bruss and Ferguson(2002)] introduce the following model. The choice of secretary corresponds now to an investment: At the beginning of the investment process the fortune of investor is x_0. Observing the rankable opportunities sequentially the decision maker is able to invest any amount b_1, $0 \leq b_1 \leq x_0$, in the opportunity. After such investment he is leaving fortune $x_1 = x_0 - b_1$ for future investments. If, after all n opportunities have been observed, this opportunity is the best one, then the return on our investment is $y_1 = \beta_1 b_1$, where $\beta_1 \geq 1$ is a known rate of return available at stage 1, otherwise the investment is lost.

Similarly, at stage $k = 2, 3, \ldots, n$, if the k-th opportunity is relatively best and our remaining (uninvested) fortune is x_{k-1}, we may invest any amount b_k, $0 \leq b_k \leq x_{k-1}$, in the kth opportunity and the return on the investment will be $y_k = \beta_k b_k$ if the k-th opportunity is best overall and 0 otherwise, where $\beta_k \geq 1$. Our problem is to choose a sequence of investments to maximize the final expected value of our total fortune from invested and uninvested capital. No interest accumulates on uninvested capital or on lost capital. The uninvested capital contribution makes here an essential difference. If this part is neglected, the unified approach of [Bruss(1984)], and the general setting of the Odds–Theorem (see [Bruss(2000)]) give both results which are, in some ways, stronger.

Our work is also related with that of [Rasmussen and Pliska(1976)], where a discount penalty α, $0 < \alpha < 1$, for each additional observation is taken. It is an investment problem in which the one unit can be invested at moment k, $1 \leq k \leq N$, and the return will be α^k if we invest to the best overall opportunity. Otherwise we get nothing. This approach models the time pressure rather than the division of our fortune between investment opportunities. We finally mention the work of [Assaf et al.(2000)], where the objective is avoiding bankruptcy.

The mathematical models related to the best choice problem also stimulate the research in the psychology (see [Corbin(1980)] and [Miller and Todd(1998)]). They observed that it is not rare that the aim in search for the best in practice can mean to look for non-extremal option. It will be formulated as the problem of choosing the option which has the rank belonging to a fixed set A. In the mathematical literature there are papers devoted to extensions of the classical *secretary problem* of such kind. In special cases, when $A = \{1, 2, \ldots, s\}$, the statement of the optimal strategy for $s = 2$ has been given by [Gilbert and Mosteller(1966)]. [Dynkin and Yushkevich(1969)] outline a proof. The paper by [Quine and Law(1996)] has been devoted for the case $s = 3$. For $s \geq 3$ authors such as [Gusein-Zade(1966)] and [Frank and Samuels(1980)] provide asymptotic results for the optimal strategy. In all these papers the character of set A is such that it contains all ranks from 1 to some s. A more complicated case, when the sequence of ranks in A has 'holes', has been considered by [Rose(1982a)], [Mori(1988)], [Szajowski(1982)] and [Suchwałko and Szajowski(2002)]. These results are motivation for the extensions of the investment model considered by [Bruss and Ferguson(2002)]. Assuming that the investment to the non-extremal options is also profitable we admit some kind of hedging of the high risk investment.

High risk investments are related to real investments, where wrong investments

are lost. If the only profitable option is the best overall one then the only opportunity of rational investment is a relatively best opportunity. This is the case considered by [Bruss and Ferguson(2002)]. However, in real life, the best opportunity can be spread between few top possibilities. If targets with positive return are those with rank belonging to some set A (assume that the best option has the rank 1), then the only possibility of rational investment are opportunities with relative rank from $R_A = \{1, \ldots, \max(A)\}$. At the moment of investment the relative rank of opportunity are usually different from the absolute rank. The prospective value of the option will change from moment of investment to the end of investment period when the final is known. Taking this into account one can consider items with relative ranks at investment moment not belonging to the set A as "hedging". The details will be specified later.

In this paper two types of payoffs are adopted. In section 2 the successful investments are those to the best and the second best option. Section 3 is devoted to the case when success is to choose of the second best option. The form of the optimal strategy in the finite horizon case and the asymptotic solution are given.

2 Investment on the best or the second best

Let us denote the relative ranks by $Y_1, \ldots, Y_n$ and the absolute ranks by $X_1, \ldots, X_n$. The investment on the opportunity at moment k gives the return $\alpha_{(X_k, Y_k), k} \mathbb{I}_A(X_k)$, where $A = \{1, 2\}$, $\mathbb{I}_A(\cdot)$ is the characteristic function of the set A and $\alpha_{(i,r),k}$ is the rate of return when the unit has been invested at moment k into the relative r-th option which is absolutely the i-th best. The initial capital is x_0 and the investment horizon is $n > 0$. At the first opportunity, at moment $k = 1$, we can invest a part of capital b_1, where $0 \leq b_1 \leq x_0$. The rest of money, $x_1 = x_0 - b_1$, can be used for investment at following opportunities. This option has the relative rank 1. If at the end of investment period this option is absolutely first then the return is $y_1 = \alpha_1 b_1$. If it happens that this option will be the absolutely second, then the return is $z_1 = \beta_1 b_1$, where $\alpha_1 = \alpha_{(1,1),1}$ and $\beta_1 = \alpha_{(2,1),1}$ are given. In other cases the investment in the first opportunity is lost.

Similarly, at stage $k = 2, 3, \ldots, n$, if the k-th opportunity is relatively best and our uninvested money is x_{k-1}, we may invest any amount b_k, $0 \leq b_k \leq x_{k-1}$ to this opportunity. The return on this investment will be $y_k = \alpha_k b_k$, $\alpha_k = \alpha_{(1,1),k}$, if at the end it turn out that it is the best one, and $z_k = \gamma_k y_k$, $\gamma_k = \alpha_{(2,1),k}$, if at the end it turn out that it is the absolutely second best one (i.e. at stages $i = k+1, \ldots, n$ there is only one better option than that at stage k), where y_k is the value of the previous investment at the last relatively best. If the k-th opportunity is relatively second, and at the end of investment period, it is the absolutely second then the return will be $z_k = \beta_k b_k$, $\beta_k = \alpha_{(2,2),k}$. In other cases we lose the invested money.

At the stage n, if the n-th opportunity is the relatively best, then it is certain to be best overall. In this case we must invest all our remaining fortune. The return will be $y_n = \alpha_n x_{n-1}$. If the n-th opportunity is the relatively second, it is the second best overall and we have to invest all our remaining money. The return are then $z_n = \beta_n x_{n-1}$. The aim of the decision-maker is to maximize the expected utility of sequential investments at the end of the considered period.

2.1 The recursive equations

Let us adopt the utility function $u(\cdot)$ and let us denote by $V_k(x, y, z)$ the expected utility of our assets (x, y, z) at stage k, where x is the money still available for investment, y is the value of the investment at the relatively best option and z is the value of the investment at the relatively second best. Based on the Bellman principle and the properties of the relative ranks we formulate the recursive relations

$$V_n(x, y, z) = \frac{n-2}{n} u(x + y + z) + \frac{1}{n} u(y + \beta_n x) + \frac{1}{n} u(\alpha_n x + \gamma_n y), \qquad (1)$$

$$
\begin{aligned}
V_k(x, y, z) &= \frac{k-2}{k} V_{k+1}(x, y, z) \\
&+ \frac{1}{k} \left(\max_{0 \le b \le x} V_{k+1}(x - b, \alpha_k b, \gamma_k y) + \max_{0 \le b \le x} V_{k+1}(x - b, y, \beta_k b) \right),
\end{aligned}
\qquad (2)
$$

for $k = 2, ..., n - 1$, and

$$V_1(x, y, z) = \max_{0 \le b \le x} V_2(x - b, \alpha_1 b, 0). \qquad (3)$$

For linear utility function it is possible to get $V_k(x, y, z)$ in an explicit form.

Theorem 2.1. *Let $u(x) = x$. For $k = 3, \dots, n$ we have*

$$V_k(x, y, z) = \frac{(k-2)(k-1)}{n(n-1)} [x + y + z + a_k y + b_k x]. \qquad (4)$$

For $k = 1, 2$

$$
\begin{aligned}
V_2(x, y, z) &= \frac{1}{n(n-1)} [y(1 + \gamma_2 + a_3) & (5) \\
&\quad + x[\max\{1 + b_3, \alpha_2(1 + a_3)\} + \max\{1 + b_3, \beta_2\}]] \\
V_1(x, y, z) &= \frac{x}{n(n-1)} \max\{\alpha_1(1 + \gamma_2 + a_3), & (6) \\
&\quad \max\{1 + b_3, \alpha_2(1 + a_3)\} + \max\{1 + b_3, \beta_2\}\},
\end{aligned}
$$

where $a_n = \frac{1 + \gamma_n}{n - 2}$, $b_n = \frac{\alpha_n + \beta_n}{n - 2}$ and

$$a_k = a_{k+1} + \frac{1}{k-2}(1 + \gamma_k + a_{k+1}), \qquad (7)$$

$$b_k = b_{k+1} + \frac{1}{k-2}[\max\{1 + b_{k+1}, \alpha_k(1 + a_{k+1})\} + \max\{1 + b_{k+1}, \beta_k\}]. \qquad (8)$$

Let $k = 1$. The optimal investment strategy is to allocate all money to the first coming option when

$$\alpha_1(1 + \gamma_2 + a_3) \ge \max\{1 + b_3, \alpha_2(1 + a_3)\} + \max\{1 + b_3, \beta_2\},$$

otherwise for $k \ge 2$

- *all money are invested to the first relative best opportunity when $\alpha_k(1 + a_{k+1}) > 1 + b_{k+1}$;*

- *all money are invested to the first relatively second opportunity when $\beta_k > 1 + b_{k+1}$.*

Proof. For $k = n$

$$
\begin{aligned}
V_n(x, y, z) &= \frac{n-2}{n}(x + y + z + \frac{1 + \gamma_n}{n-2}y + \frac{\alpha_n + \beta_n}{n-2}x) \\
&= \frac{n-2}{n}(x + y + z + a_n y + b_n x)
\end{aligned}
$$

Let us assume that

$$
V_{k+1}(x, y, z) = \frac{k(k-1)}{n(n-1)}((1 + b_{k+1})x + (1 + a_{k+1})y + z)
$$

then we have

$$
\begin{aligned}
V_k(x, y, z) &= \frac{k-2}{k}V_{k+1}(x, y, z) \\
&\quad + \frac{1}{k}\left(\max_{0 \le b \le x} V_{k+1}(x - b, y, \beta_k b) + \max_{0 \le b \le x} V_{k+1}(x - b, \alpha_k b, \gamma_k y)\right) \\
&= \frac{k-2}{k}V_{k+1}(x, y, z) + \frac{1}{k}\left(\max\{V_{k+1}(x, 0, \gamma_k y), V_{k+1}(0, \alpha_k x, \gamma_k y)\}\right. \\
&\quad + \max\{V_{k+1}(x, y, 0), V_{k+1}(0, y, \beta_k x)\}) \\
&= \frac{(k-1)(k-2)}{n(n-1)}((1 + b_{k+1})x + (1 + a_{k+1})y + z) \\
&\quad + \frac{1}{n-2}\left(\max\{1 + b_{k+1}, (1 + a_{k+1})\alpha_k\}x + (1 + \gamma_k + a_{k+1})y\}\right) \\
&= \frac{(k-1)(k-2)}{n(n-1)}(x + y + z + a_k y + b_k x)
\end{aligned}
$$

It gives the form of $V_k(x, y, z)$ for $k = 3, 4, \ldots, n$. From (2) and (4) we have

$$
\begin{aligned}
V_2(x, y, z) &= \frac{1}{2}\left(\max_{0 \le b \le x} V_3(x - b, \alpha_2 b, \gamma_2 y) + \max_{0 \le b \le x} V_3(x - b, y, \beta_2 b)\right) \\
&= \frac{1}{2}\left(\max\{V_3(x, 0, \gamma_2 y), V_3(0, \alpha_2 x, \gamma_2 y)\}\right. \\
&\quad + \max\{V_3(x, y, 0), V_3(0, y, \beta_2 x)\}) \\
&= \frac{1}{n(n-1)}\left(y(1 + \gamma_2 + a_3)\right. \\
&\quad + x[\max\{1 + b_3, \alpha_2(1 + a_3)\} + \max\{1 + b_3, \beta_2\}]) .
\end{aligned}
$$

The form of $V_2(x, y, z)$ and (3) follow that $V_1(x, y, z) = \max\{V_2(x, 0, 0) + V_2(0, \alpha_1 x, 0)\}$. Hence $V_1(x, y, z)$ has the form (6). ∎

2.2 Continuous approximation of discrete time investment problem

Similarly as in [Bruss and Ferguson(2002)] we investigate now the asymptotic behaviour of the total return under an optimal strategy.

Definition 2.2. *Let $\vec{\alpha}_{(i,r),n} = (\alpha_{(i,r),1}, \alpha_{(i,r),2}, \ldots, \alpha_{(i,r),n})$ for $i \in A = \{1, 2, \ldots, s\}$, $i \geq r$ and $n \in \mathbb{N}$ be the given vector of the rate of returns and let $\alpha_{(i,r)}(t)$ be the real value functions defined on $[0,1]$ such that $\lim\limits_{\substack{n \to \infty \\ \frac{k(n)}{n} \to t}} \alpha_{(i,r),k(n)} = \alpha_{(i,r)}(t)$. For the initial capital x_0 denote $\rho_n(\vec{\alpha}_n)$ the optimal return of investments for the rate of return defined by $\vec{\alpha}_n$. If the limits $\lim\limits_{\substack{n \to \infty \\ \frac{k(n)}{n} \to t}} \rho_n(\vec{\alpha}_n) = \rho(\vec{\alpha}(t))$ and $\rho = \lim_{n \to \infty} \rho(\vec{\alpha}(t))$ exist the value of ρ is called the optimal asymptotic return (the limit expected fortune). The limit of corresponding strategies are called the asymptotic optimal strategy.*

We now allow that the parameters α_k, β_k, γ_k and the coefficients a_k and b_k depend on n. Let us assume there are continuous functions: $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ on $(0, 1]$, such that $\alpha_k = \alpha(\frac{k}{n})$, $\beta_k = \beta(\frac{k}{n})$, $\gamma_k = \gamma(\frac{k}{n})$. The equations (7) and (8) can be written in the form

$$\frac{a_{k+1,n} - a_{k,n}}{\frac{1}{n}} = -\frac{n}{k-2}[1 + \gamma(\frac{k}{n}) + a_{k,n}] \tag{9}$$

$$\frac{b_{k+1,n} - b_{k,n}}{\frac{1}{n}} = -\frac{n}{k-2}\left[\max\{1 + b_{k+1}, \alpha(\frac{k}{n})(1 + a_{k+1})\} \right. \tag{10}$$
$$\left. + \max\{1 + b_{k+1}, \beta(\frac{k}{n})\}\right]$$

These show that the sequences a_k and b_k are monotone decreasing. Let $f_n(t)$ and $g_n(t)$ interpolate on $[0, 1]$ points $(\frac{k}{n}, a_{k,n})$ and $(\frac{k}{n}, b_{k,n})$, respectively. Both functions are monotone on $[0, 1]$. Using these functions we get from (9) and (10)

$$\frac{f_n(\frac{k+1}{n}) - f_n(\frac{k}{n})}{\frac{1}{n}} = -\frac{n}{k-2}[1 + \gamma(\frac{k}{n}) + f_n(\frac{k}{n})] \tag{11}$$

$$\frac{g_n(\frac{k+1}{n}) - g_n(\frac{k}{n})}{\frac{1}{n}} = -\frac{n}{k-2}[\max\{1 + g_n(\frac{k+1}{n}), \alpha(\frac{k}{n})\left(1 + f_n(\frac{k+1}{n})\right)\} \tag{12}$$
$$+ \max\{1 + g_n(\frac{k+1}{n}), \beta(\frac{k}{n})\}].$$

Theorem 2.3. *Let us assume the function $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ defined on $[0, 1]$ are continuous and let $\alpha(t) > 1$, $\beta(t) > 1$ on $[0, 1]$.*

(i) As n tends to ∞, $f_n(t) \to f(t)$, $g_n(t) \to g(t)$, where $f(t)$ and $g(t)$ satisfy the set of differential equations

$$f'(t) = -\frac{1}{t}(1 + \gamma(t) + f(t)) \tag{13}$$

$$g'(t) = -\frac{1}{t}[\max\{1 + g(t), \alpha(t)(1 + f(t))\} + \max\{1 + g(t), \beta(t)\}] \tag{14}$$

on $(0, 1]$ with boundary conditions $f(1) = 0$ and $g(1) = 0$.

(ii) The limiting optimal investment policy is to invest the whole money in the first relatively best option at moment t when $\alpha(t)(1 + f(t)) \geq 1 + g(t)$ and in the first relatively second occurring at a time t for which $\beta(t) \geq 1 + g(t)$.

(iii) The asymptotic optimal expected fortune at moment t is

$$V_t(x, y, z) = \lim_{\substack{n \to \infty \\ \frac{k(n)}{n} \to t}} V_{k(n)}(x, y, z) = t^2(x + y + z + f(t)y + g(t)x). \qquad (15)$$

If $t_0 = \min\{0 < t \leq 1 : 1 + g(t) \leq \beta(t) \text{ or } 1 + g(t) \leq \alpha(t)(1 + f(t))\} > 0$ then the optimal asymptotic return $\rho = \lim_{t \to 0^+} t^2(f(t)y + g(t)x)$.

Proof. This proof, using results of [Henrici(1962)] on Euler-Cauchy approximations and of [Bruss(1988)] on limiting record frequencies, is along the same lines as the proof of Theorem 2 of [Bruss and Ferguson(2002)], and can therefore be omitted.

∎

Remark 2.4. *The solution of the equation (13) on $(0, 1]$ is*

$$f(t) = \frac{1 - t + \int_t^1 \gamma(s)d\,s}{t}. \qquad (16)$$

2.3 The constant rate of return

Consider the case of the constant rate of return assuming that $\alpha(t) = \alpha \geq 1$, $\beta(t) = \gamma(t) = \beta \geq 1$ and $\alpha \geq \beta$. For $t \in (0, 1]$, for every α and β we have $\alpha(\frac{1+\beta}{t} - \beta) \geq \beta$. The equations (13) and (14) for $0 < t \leq 1$ have the form:

$$f'(t) = -\frac{1}{t}(1 + \beta + f(t)) \qquad (17)$$

$$g'(t) = \begin{cases} -\frac{2}{t}(1 + g(t)) & \text{for } 1 + g(t) \geq \alpha(-\beta + \frac{1+\beta}{t}), \\ -\frac{1}{t}(\frac{\alpha(1+\beta)}{t} - \alpha\beta + 1 + g(t)) & \text{for } \beta \leq 1 + g(t) < \alpha(-\beta + \frac{1+\beta}{t}), \\ -\frac{1}{t}(\frac{\alpha(1+\beta)}{t} - \alpha\beta + \beta) & \text{for } 1 + g(t) < \beta \end{cases} \qquad (18)$$

with boundary condition $f(1) = 0$ and $g(1) = 0$. The solution of this set of equations has the form: $f(t) = \frac{(1+\beta)(1-t)}{t}$,

$$g(t) = \begin{cases} -\beta(1 - \alpha)\ln(t) + \frac{\alpha(1+\beta)}{t}(1 - t) & \text{for } t_2 < t \leq 1, \\ -\alpha(1 + \beta)\ln(t) + \frac{t_2(1-\alpha)\beta + \alpha(1+\beta)\ln(t_2)}{t} + \alpha\beta - 1 & \text{for } t_1 < t \leq t_2, \\ \frac{\alpha((1+\beta)t_1 - \beta t_1^2)}{t^2} - 1 & \text{for } 0 < t \leq t_1, \end{cases}$$

where t_2 is the solution in $(0, 1]$ of the equation $1 + g(t) = \beta$ and t_1 is the solution of the equation $1 + g(t) = \alpha(1 + f(t))$ in $(0, t_2]$. In this case the asymptotic optimal fortune is equal $\rho = \alpha((1 + \beta)t_1 - \beta t_1^2)x$, where $x = x_0$ is the initial fortune.

Corollary 2.5. *The asymptotically optimal strategy for the problem of investment, when the positive rate of return is given by the best or the second best option, is as follows. We do not invest any capital into an option which appear from 0 to t_1. On the interval $[t_1, 1]$ we invest all our capital at the first relatively best option and on the interval $[t_2, 1]$ also at the relatively second best option, which appears first.*

3   Investment on the second best

Let us consider the investment problem at the sequence of different option when the positive return is possible on the absolutely second best opportunity and the initial capital is x_0. This may seem strange but can indeed be a good objective, if–as e.g. for targeted investments– a strong competitor is known to hunt for similar opportunities. We observe the relative ranks of options as in the problem solved by [Bruss and Ferguson(2002)] and in Section 2. In this case the investment opportunity at moment k gives the return $\alpha_{(X_k,Y_k),k}\mathbb{I}_A(X_k)$, where $A = \{2\}$. The positive return is possible when we invest on the relatively best or the second best opportunity. Let us assume that at some moment k, having still whole capital $x_{k-1} = x_0$ not engaged, we observe the relatively best option and we invest b_k, $0 \leq b_k \leq x_{k-1}$ in that opportunity, leaving fortune $x_k = x_{k-1} - b_k$ for future investments. If, at the end of investment period, this opportunity appears to be the second best overall, it means that at some moment i, $k < i < n$, the next relatively best option appeared. In this case the return on our investment at previous relatively best will work in fact. To formalize this we adopt the following convention: having the best option at moment i and the invested money b_k at moment k our stage of investment would be $z_i = \alpha_{(2,1),i}y_i$, where $y_k = \alpha_{(1,1),k}b_k$ and we can invest b_i, $0 \leq b_i < x_{i-1}$ at the current best. If it happens that the option at moment i will be the second best, then the return is $z_i = \alpha_{(2,2),i}b_i$. At the last stage n, if the option is the second best we have to invest all capital available to get $z_n = \alpha_{(2,2),n}x_{n-1}$. Taking into account the payoff structure we can parametrize the rate of return as follows: $\alpha_{(1,1),k} = 1$, $\alpha_{(2,1),k} = \gamma_k$ and $\alpha_{(2,2),k} = \beta_k$.

Our aim is to maximize the return from the investment process.

3.1   The recursive determination of the value function

Let $V_k(x, y, z)$ be the expected return under the optimal investment policy when at stage k, $1 \leq k \leq n$, the capital which is not invested is equal x. The capital invested at the current relatively best is y and at the current second best is z. If at stage n the best option appears we do not invest at this option but the investment to previous the best will pay $\gamma_n y$. If at stage n the second best appears the total capital is allocated to this option with return $\beta_n x$. We have

$$V_n(x, y, z) = \frac{n-2}{n}u(x+z) + \frac{1}{n}u(\beta_n x) + \frac{1}{n}u(\gamma_n y) \tag{19}$$

Similar consideration as in construction of the formulae (2) leads to equations

$$\begin{aligned} V_k(x, y, z) &= \frac{k-2}{k}V_{k+1}(x, y, z) + \frac{1}{k}\max_{0 \leq b \leq x} V_{k+1}(x - b, b, \gamma_k y) &(20)\\ &+ \frac{1}{k}\max_{0 \leq b \leq x} V_{k+1}(x - b, y, \beta_k b) \text{ for } k = 2, \ldots, n-1, \\ V_1(x, y, z) &= \max_{0 \leq b \leq x} V_2(x - b, b, 0). &(21) \end{aligned}$$

The set of equations (19)-(21) for the linear utility function can be solved and the result similar to the theorem 2.1 is formulated.

Theorem 3.1. *Let the utility function be* $u(x) = x$. *The expected optimal return functions have the form*

$$V_k(x, y, z) = \frac{(k-2)(k-1)}{n(n-1)}[x + z + a_k y + b_k x] \text{ for } k = 3, \ldots, n,$$

$$V_2(x, y, z) = \frac{1}{n(n-1)}[y(\gamma_2 + a_3) + x[\max\{1 + b_3, a_3\} + \max\{1 + b_3, \beta_2\}]]$$

$$V_1(x, y, z) = \frac{x}{n(n-1)} \max\{\gamma_2 + a_3, \max\{1 + b_3, (1 + a_3)\} + \max\{1 + b_3, \beta_2\}\}$$

where $a_n = \frac{\gamma_n}{n-2}$ *and* $a_k = a_{k+1} + \frac{1}{k-2}(\gamma_k + a_{k+1})$ *for* $k = 3, \ldots, n-1$, $b_n = \frac{\beta_n}{n-2}$ *and* $b_k = b_{k+1} + \frac{1}{k-2}[\max\{1 + b_{k+1}, a_{k+1}\} + \max\{1 + b_{k+1}, \beta_k\}]$ *for* $k = 3, \ldots, n-1$.
The optimal investment strategy at moment $k = 1$ *is to allocate all the money to the first option, if*

$$\gamma_2 + a_3 \geq \max\{1 + b_3, a_3\} + \max\{1 + b_3, \beta_2\},$$

otherwise for $k \geq 2$

- *all money are invested to the first relative best opportunity when* $a_{k+1} > 1 + b_{k+1}$;

- *all money are invested to the first relatively second opportunity when* $\beta_k > 1 + b_{k+1}$.

Proof. For $k = n$

$$V_n(x, y, z) = \frac{n-2}{n}(x + z + \frac{\gamma_n}{n-2}y + \frac{\beta_n}{n-2}x) = \frac{n-2}{n}(x + z + a_n y + b_n x).$$

At stage n we invest when the second best appears. Let us assume that $V_{k+1}(x, y, z) = \frac{k(k-1)}{n(n-1)}(x + z + a_{k+1}y + b_{k+1}x)$, then we have

$$
\begin{aligned}
V_k(x, y, z) &= \frac{k-2}{k}V_{k+1}(x, y, z) + \frac{1}{k}\left(\max_{0 \leq b \leq x} V_{k+1}(x - b, b, \gamma_k y) + \max_{0 \leq b \leq x} V_{k+1}(x - b, y, \beta_k b)\right) \\
&= \frac{k-2}{k}V_{k+1}(x, y, z) + \frac{1}{k}(\max\{V_{k+1}(x, 0, \gamma_k y), V_{k+1}(0, x, \gamma_k y)\} \\
&\quad + \max\{V_{k+1}(x, y, 0), V_{k+1}(0, y, \beta_k x)\}) \\
&= \frac{(k-1)(k-2)}{n(n-1)}\left[x + z + y(a_{k+1} + \frac{1}{k-2}(\gamma_k + a_{k+1})\right. \quad (22) \\
&\quad \left. + x\left(b_{k+1} + \frac{1}{k-2}(\max\{1 + b_{k+1}, a_{k+1}\} + \max\{1 + b_{k+1}, \beta_k\})\right)\right] \quad (23) \\
&= \frac{(k-1)(k-2)}{n(n-1)}(x + y + z + a_k y + b_k x).
\end{aligned}
$$

It gives the form of $V_k(x, y, z)$ for $k = 3, \ldots, n$ and the optimal investment strategy follows from (22)-(23). For $k = 2$, by backward induction, we have

$$
\begin{aligned}
V_2(x, y, z) &= \frac{1}{2}\left(\max_{0 \leq b \leq x} V_3(x - b, b, \gamma_2 y) + \max_{0 \leq b \leq x} V_3(x - b, y, \beta_2 b)\right) \\
&= \frac{1}{n(n-1)}((\gamma_2 + a_3)y + (\max\{a_3, 1 + b_3\} + \max\{\beta_2, 1 + b_3\})x).
\end{aligned}
$$

Finally, the form of the expected return at $k = 1$ follows the optimal investment strategy at this stage.

$$
\begin{aligned}
V_1(x, y, z) &= \max_{0 \le b \le x} V_2(x - b, b, 0) \\
&= \frac{1}{n(n-1)} \max\{\gamma_2 + a_3, \max\{a_3, 1 + b_3\} + \max\{\beta_2, 1 + b_3\}\}
\end{aligned}
$$

■

3.2 Asymptotic solution of the investment in the second best option

In this section, similarly as in Section 2.2, the asymptotic behaviour of the total return and the optimal strategies are investigated. Let us allow that the parameters β_k, γ_k and the coefficients a_k and b_k depend on n. Let us assume there are continuous functions: $\beta(t)$ and $\gamma(t)$ on $(0, 1]$, such that $\beta_k = \beta(\frac{k}{n})$, $\gamma_k = \gamma(\frac{k}{n})$. Let $f_n(t)$ and $g_n(t)$ interpolate on $[0, 1]$ points $(\frac{k}{n}, a_{k,n})$ and $(\frac{k}{n}, b_{k,n})$, respectively. Based on the recurrence relation for a_k and b_k formulated in Theorem 3.1 we get the differential equations for coefficients of the asymptotic solution. The details of the proof are omitted.

Theorem 3.2. *Let us assume the function $\beta(t)$ and $\gamma(t)$ defined on $[0, 1]$ to be continuous and let $\beta(t) > 1$ and $\gamma(t) > 1$ on $[0, 1]$.*

(i) *As n tends to ∞, $f_n(t) \to f(t)$, $g_n(t) \to g(t)$, where $f(t)$ and $g(t)$ satisfy the set of differential equations*

$$
f'(t) = -\frac{1}{t}(\gamma(t) + f(t)) \tag{24}
$$

$$
g'(t) = -\frac{1}{t}[\max\{1 + g(t), f(t)\} + \max\{1 + g(t), \beta(t)\}] \tag{25}
$$

on $(0, 1]$ with boundary conditions $f(1) = 0$ and $g(1) = 0$.

(ii) *The limiting optimal investment policy is to invest all money in the first relatively best option at time t if $f(t) \ge 1 + g(t)$ and in the first relatively second occurring at a time t for which $\beta(t) \ge 1 + g(t)$.*

(iii) *The asymptotic optimal expected fortune at moment t is*

$$
V_t(x, y, z) = \lim_{\substack{n \to \infty \\ \frac{k(n)}{n} \to t}} V_{k(n)}(x, y, z) = t^2(x + y + z + f(t)y + g(t)x). \tag{26}
$$

If $t_0 = \min\{0 < t \le 1 : 1 + g(t) \le \beta(t) \text{ or } 1 + g(t) \le f(t)\} > 0$ then the optimal asymptotic return $\rho = \lim_{t \to 0^+} t^2(f(t)y + g(t)x)$.

3.3 Constant rate of return

Consider the case of a constant rate of return assuming that $\beta(t) = \beta \geq 1$ and $\gamma(t) = \gamma \geq 1$. The equations (24) for $0 < t \leq 1$ has the form:

$$f'(t) = -\frac{1}{t}(\gamma + f(t)) \tag{27}$$

with boundary condition $f(1) = 0$. The solution of this equation in $(0, 1]$ is the function $f(t) = \gamma\frac{1-t}{t}$. In some left hand side neighbourhood of 1 the equation (25) takes form $g'(t) = -\frac{1}{t}(1 + \beta + g(t))$ with boundary condition $g(1) = 0$. It has the solution $g_1(t) = (1+\beta)(\frac{1}{t} - 1)$. If $\beta \geq \gamma$, then $1 + g(t) \geq f(t)$ in $(0, 1]$. The inequality $1 + g(t) < \beta$ is fulfilled for $t \in (\frac{1+\beta}{2\beta}, 1]$. We have

$$g'(t) = \begin{cases} -\frac{1+\beta+g(t)}{t} & \text{if } t \in (\frac{1+\beta}{2\beta}, 1], \\ -\frac{2(1+g(t))}{t} & \text{if } t \in (0, \frac{1+\beta}{2\beta}]. \end{cases} \tag{28}$$

The function

$$g(t) = \begin{cases} \frac{(1+\beta)(1-t)}{t} & \text{if } t \in (\frac{1+\beta}{2\beta}, 1], \\ \frac{(1+\beta)^2}{4\beta t^2} - 1 & \text{if } t \in (0, \frac{1+\beta}{2\beta}]. \end{cases} \tag{29}$$

is the solution of (28) with boundary condition $g(1) = 0$ when $\beta \geq \gamma$. Hence in this case the optimal investment strategy is to put all money at the first relatively second opportunity after the moment $\frac{1+\beta}{2\beta}$. The optimal limit expected fortune is $\rho = \frac{(1+\beta)^2}{4\beta}x$. This solution is also valid for $\beta \leq \gamma < \frac{(1+\beta)^2}{\beta}$.

When $\gamma > \frac{(1+\beta)^2}{\beta}$ then the solution of (25) has form

$$g(t) = \begin{cases} \frac{(1+\beta)(1-t)}{t} & \text{if } t \in (\frac{1+\beta}{2\beta}, 1], \\ \frac{(1+\beta)^2}{4\beta t^2} - 1 & \text{if } t \in (t_1, \frac{1+\beta}{2\beta}], \\ \frac{\gamma}{t}\ln\frac{t_1}{t} + (\gamma - 1) + \frac{\gamma(1-2t_1)}{t} & \text{if } t \in (t_0, t_1], \\ \frac{\gamma t_0(1-t_0)}{t^2} - 1 & \text{if } t \in (0, t_0], \end{cases} \tag{30}$$

where $t_1 = \frac{1 + \sqrt{1 - \frac{(1+\beta)^2}{\beta\gamma}}}{2}$ and $t_0 = \min\{0 < t < t_1 : \frac{\gamma}{t}\ln\frac{t_1}{t} + (1 - \gamma)\frac{t_1}{t} < 0\}$. The construction of the solution follows the consideration of optimal strategies in turn at the intervals $(\frac{1+\beta}{2\beta}, 1]$, $(t_1, \frac{1+\beta}{2\beta}]$, $(t_0, t_1]$ and for $t \leq t_0$. The optimal limit expected fortune is $\rho = \gamma t_0(1 - t_0)x$.

Corollary 3.3. *The asymptotically optimal strategy for the problem of investment when the positive rate of return is from the second best option is as follows. If $\gamma \leq \frac{(1+\beta)^2}{\beta}$ then we do not invest any capital until time $\frac{1+\beta}{2\beta}$. On the interval $(\frac{1+\beta}{2\beta}, 1]$ we invest all our capital at the first relatively second best option, which appears first.*

If $\gamma > \frac{(1+\beta)^2}{\beta}$ then we do not invest any capital at option which appear from 0 to t_0. On the interval $(t_0, t_1]$ we invest all our capital at the first relatively best option and on the interval $(\frac{1+\beta}{2\beta}, 1]$ we invest all our capital at the first relatively second best option.

4   Final remarks

It is difficult to obtain solutions in analytical form. From the theorems presented in this paper it is possible to find solution by numerical methods.

We may also consider more general utility functions. Typical utility function have been suggested by [Bruss and Ferguson(2002)] such as $u_\alpha(x) = \frac{x^\alpha - 1}{\alpha}$, for $\alpha \neq 1$ and $u_0(x) = \log(x)$ for $\alpha = 0$. The solution of the investment problem for such functions will be studied of a separate paper.

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