Almost Derivations on C^*-Ternary Rings

Mohammad Sal Moslehian*

Abstract

We establish the generalized Hyers–Ulam–Rassias stability of derivations in C^*-ternary rings associated to the Cauchy functional equation. We also show that any so-called almost derivation on a C^*-ternary ring is a true derivation.

1 Introduction and Preliminaries

A *C^*-ternary ring* is a Banach space $\mathcal{A}$ equipped with a ternary product $(x, y, z) \mapsto [x\ y\ z]$ of $\mathcal{A} \times \mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$ which is linear in the outer variables, conjugate linear in the middle variable, and associative in the sense that $[x\ y\ [z\ t\ s]] = [x\ [t\ z\ y]\ s] = [[x\ y\ z]\ t\ s]$, and satisfies $\|[x\ y\ z]\| \leq \|x\|\ \|y\|\ \|z\|$ and $\|[x\ x\ x]\| = \|x\|^3$; cf. [24]. For instance, any ternary ring of operators, namely any closed subspace of the space $B(\mathfrak{H}, \mathfrak{K})$ of bounded linear operators between Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ which is closed under the ternary product $[x\ y\ z] := xy^*z$ is a C*-ternary ring.

If a C^*-ternary ring $(\mathcal{A}, [\ ])$ has an identity, i.e. an element e such that $x = [x\ e\ e] = [e\ e\ x]$ for all $x \in \mathcal{A}$, then it is routine to verify that $\mathcal{A}$ endowed with $x \odot y := [x\ e\ y]$ and $x^* := [e\ x\ e]$ is a unital C^*-algebra. The most important thing is the C^*-condition. To see this, note that

$$\begin{aligned} \|x \odot x^* \odot x\| &= \|[x\ e\ x^*] \odot x\| = \|[[x\ e\ x^*]\ e\ x]\| = \|[[x\ e\ [e\ x\ e]]\ e\ x]\| \\ &= \|[[[x\ e\ e]\ x\ e]\ e\ x]\| = \|[[x\ x\ e]\ e\ x]\| = \|[x\ [e\ e\ x]\ x]\| \\ &= \|[x\ x\ x]\| \\ &= \|x\|^3, \end{aligned}$$

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whence

$$\begin{aligned} \|x \odot x^*\|^3 &= \|(x \odot x^*) \odot (x \odot x^*)^* \odot (x \odot x^*)\| = \|(x \odot x^* \odot x) \odot (x \odot x^* \odot x)^*\| \\ &\leq \|(x \odot x^* \odot x)\| \, \|(x \odot x^* \odot x)^*\| \leq \|x\|^3 \, \|x^*\|^3 = \|x\|^6, \end{aligned}$$

by applying $\|x\| = \|x^*\|$ which is followed from $\|x\|^3 = \|[x\ x^*\ x]\| \leq \|x\|\,\|x^*\|\,\|x\|$.

Conversely, if $(A, \odot)$ is a (unital) C^*-algebra, then $[x\ y\ z] := x \odot y^* \odot z$ makes $\mathcal{A}$ into a C^*-ternary ring (with the unit e such that $x \odot y = [x\ e\ y]$).

A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a *derivation* if $\delta([x\ y\ z]) = [\delta(x)\ y\ z] + [x\ \delta(y)\ z] + [x\ y\ \delta(z)]$ for all $x, y, z \in \mathcal{A}$. This notion is a generalization of derivation on a Hilbert C^*-module; cf. [11].

We say a functional equation $(\mathcal{E})$ is *stable* if any function g satisfying the equation $(\mathcal{E})$ "approximately" is near to a true solution of $(\mathcal{E})$. The equation $(\mathcal{E})$ is called *superstable* if every approximate solution of $(\mathcal{E})$ is an exact solution (see [3] for another notion of superstability namely *superstability modulo the bounded functions*)

The stability problem of functional equations originated from a question of Ulam [23], posed in 1940, concerning the stability of group homomorphisms:

Let $(\mathcal{G}_1, *)$ be a group and let $(\mathcal{G}_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : \mathcal{G}_1 \to \mathcal{G}_2$ satisfies the inequality $d(h(x * y), h(x) \diamond h(y)) < \delta$ for all $x, y \in \mathcal{G}_1$, then there is a homomorphism $H : \mathcal{G}_1 \to \mathcal{G}_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in \mathcal{G}_1$?

In the next year, Hyers [6] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1978, Rassias [21] extended the theorem of Hyers by considering the unbounded Cauchy difference $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$, $(\varepsilon > 0, p \in [0, 1))$. The result of Th. M. Rassias has provided a lot of influence in the development of what we now call *Hyers–Ulam–Rassias stability* of functional equations. In 1994, a generalization of Rassias' theorem, the so-called generalized Hyers–Ulam–Rassias stability, was obtained by Găvruta [5]. During the last decades several stability problems of functional equations have been investigated in the spirit of Hyers–Ulam–Rassias. See [4, 7, 8, 22] for more detailed information on stability of functional equations. Some results on stability of mappings on other ternary structures may be found in [1, 16].

Recently, the stability of various types of derivations has been extensively investigated by some mathematicians; see [10, 12, 13, 14, 15, 17, 18, 19]. In this paper, using some strategies from [2, 17], we establish the generalized Hyers–Ulam–Rassias stability of derivations associated to the Cauchy equations. Because of the interrelation between unital C^*-algebras and unital C^*-ternary rings our approach may be applied to study of stability of derivations in unital C^*-algebras; see [17]. Introducing the notion of almost derivation on a C^*-ternary ring and using some ideas from [12] we prove that every almost derivation is a true derivation.

Throughout this paper, $\mathcal{A}$ denotes a C^*-ternary ring.

2 Generalized Hyers–Ulam–Rassias Stability

In this section, we are going to establish the generalized Hyers–Ulam–Rassias stability of derivations in C^*-ternary rings associated with the Cauchy functional equation. See [1] for a fixed point approach in the framework of Hilbert C^*-modules.

Theorem 2.1. *Suppose $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^5 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, u, v, w) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y, 2^n u, 2^n v, 2^n w) < \infty,$$

and

$$\begin{aligned} \|f(\mu x + \mu y + [u\ v\ w]) - \mu f(x) - \mu f(y) - [f(u)\ v\ w] - [u\ f(v)\ w] - [u\ v\ f(w)]\| \\ \leq \varphi(x, y, u, v, w), \end{aligned} \tag{2.1}$$

for all $\mu \in T^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, u, v, w \in \mathcal{A}$. Then there exists a unique derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \delta(x)\| \leq \widetilde{\varphi}(x, x, 0, 0, 0),$$

for all $x \in \mathcal{A}$.

Proof. Set $u = v = w = 0, \mu = 1, y = x$ in (2.1) to get

$$\|f(2x) - 2f(x)\| \leq \varphi(x, x, 0, 0, 0)$$

for all $x \in \mathcal{A}$. Using the induction, one can show that

$$\|2^{-n} f(2^n x) - f(x)\| \leq \frac{1}{2}\sum_{k=0}^{n-1} 2^{-k}\varphi(2^k x, 2^k x, 0, 0, 0), \tag{2.2}$$

for all $x \in \mathcal{A}$ and for all positive integers n, and

$$\|2^{-n} f(2^n x) - 2^{-m} f(2^m x)\| \leq \frac{1}{2}\sum_{k=m}^{n-1} 2^{-k}\varphi(2^k x, 2^k x, 0, 0, 0),$$

for all $x \in \mathcal{A}$ and for all non-negative integers m, n with $m < n$. Hence $\{2^{-n} f(2^n x)\}$ is a Cauchy sequence in $\mathcal{A}$. Due to the completeness of $\mathcal{A}$ we conclude that this sequence is convergent. Set

$$\delta(x) = \lim_{n\to\infty} 2^{-n} f(2^n x), \qquad x \in \mathcal{A}.$$

If $n \to \infty$ in inequality (2.2), we obtain

$$\|f(x) - \delta(x)\| \leq \widetilde{\varphi}(x, x, 0, 0, 0),$$

for all $x \in \mathcal{A}$.

Putting $u = v = w = 0, y = 2^{n-1}x$ and replacing x by $2^{n-1}x$ in (2.1) we obtain

$$\|f(2^n \mu x) - 2\mu f(2^{n-1}x)\| \leq \varphi(2^{n-1}x, 2^{n-1}x, 0, 0, 0),$$

for all $x \in \mathcal{A}, \mu \in \mathbb{T}^1$. Then

$$\begin{aligned} \|\mu f(2^n x) - 2\mu f(2^{n-1}x)\| &\leq |\mu| \cdot \|f(2^n x) - 2f(2^{n-1}x)\| \\ &\leq \varphi(2^{n-1}x, 2^{n-1}x, 0, 0, 0), \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. So

$$\begin{aligned}\|2^{-n}f(2^n\mu x) - 2^{-n}\mu f(2^n x)\| &\leq 2^{-n}\|f(2^n\mu x) - 2\mu f(2^{n-1}x)\| \\ &\quad +2^{-n}\|2\mu f(2^{n-1}x) - \mu f(2^n x)\| \\ &\leq 2^{-n+1}\varphi(2^{n-1}x, 2^{n-1}x, 0, 0, 0),\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Since the right hand side tends to zero as $n \to \infty$, we have

$$\delta(\mu x) = \lim_{n\to\infty} \frac{f(2^n\mu x)}{2^n} = \lim_{n\to\infty} \frac{\mu f(2^n x)}{2^n} = \mu\delta(x),$$

for all $\mu \in \mathbb{T}^1$ and all $x \in \mathcal{A}$. Obviously, $\delta(0x) = 0 = 0\delta(x)$.

Next, let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and let M be a natural number greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = 1/3$. By Theorem 1 of [9], there exist three numbers $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. By the additivity of δ we get $\delta(\frac{1}{3}x) = \frac{1}{3}\delta(x)$ for all $x \in \mathcal{A}$. Therefore,

$$\begin{aligned}\delta(\lambda x) &= \delta(\frac{M}{3}\cdot 3\cdot\frac{\lambda}{M}x) = M\delta(\frac{1}{3}\cdot 3\cdot\frac{\lambda}{M}x) = \frac{M}{3}\delta(3\cdot\frac{\lambda}{M}x) \\ &= \frac{M}{3}\delta(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(\delta(\mu_1 x) + \delta(\mu_2 x) + \delta(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)\delta(x) = \frac{M}{3}\cdot 3\cdot\frac{\lambda}{M} \\ &= \lambda\delta(x),\end{aligned}$$

for all $x \in \mathcal{A}$. So that δ is $\mathbb{C}$-linear.

Set $x = y = 0$ and replace u, v, w by $2^n u, 2^n v, 2^n w$, respectively, in (2.1). Then

$$\begin{aligned}&\tfrac{1}{2^{3n}}\|f(2^{3n}[u\ v\ w]) - [f(2^n u)\ 2^n v\ 2^n w] - [2^n u\ f(2^n v)\ 2^n w] - [2^n u\ 2^n v\ f(2^n w)]\| \\ &\qquad \leq \tfrac{1}{2^{3n}}\varphi(0, 0, 2^n u, 2^n v, 2^n w),\end{aligned}$$

for all $u, v, w \in \mathcal{A}$. It follows from the continuity of the mapping $\mathcal{A}\times\mathcal{A}\times\mathcal{A} \to \mathcal{A}$ given by $(x, y, z) \mapsto [x\ y\ z]$ that

$$\begin{aligned}\delta([u\ v\ w]) &= \lim_{n\to\infty}\frac{f(2^{3n}[u\ v\ w])}{2^{3n}} \\ &= \lim_{n\to\infty}[\frac{f(2^n u)}{2^n}\ v\ w] + [u\ \frac{f(2^n v)}{2^n}\ w] + [u\ v\ \frac{f(2^n w)}{2^n}] \\ &= [\delta(u)\ v\ w] + [u\ \delta(v)\ w] + [u\ v\ \delta(w)],\end{aligned}$$

for all $u, v, w \in \mathcal{A}$. Thus δ is a derivation satisfying the required inequality. ■

Theorem 2.2. *Suppose that $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ and there exists a function $\varphi : \mathcal{A}^5 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y, u, v, w) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y, 2^n u, 2^n v, 2^n w) < \infty,$$

$$\begin{aligned}&\|f(\mu x + \mu y + [u\ v\ w]) - \mu f(x) - \mu f(y) - [f(u)\ v\ w] - [u\ f(v)\ w] - [u\ v\ f(w)]\| \\ &\qquad \leq \varphi(x, y, u, v, w), \qquad (2.3)\end{aligned}$$

for $\mu = 1, \mathbf{i}$ and all $x, y, u, v, w \in \mathcal{A}$. If for each fixed $x \in \mathcal{A}$ the function $t \mapsto f(tx)$ is continuous on $\mathbb{R}$, then there exists a unique derivation $\delta : \mathcal{A} \to \mathcal{A}$ such that

$$\|f(x) - \delta(x)\| \leq \widetilde{\varphi}(x, x, 0, 0, 0),$$

for all $x \in \mathcal{A}$.

Proof. By the same arguing as in the proof of Theorem 2.1 we can appropriately approximate f by a unique additive mapping $\delta : \mathcal{A} \to \mathcal{A}$ given by $\delta(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$, $(x \in \mathcal{A})$.

By the same reasoning as in the proof of the main theorem of [21], the mapping δ is $\mathbb{R}$-linear.

Assuming $y = u = v = w = 0$ and $\mu = \mathbf{i}$, it follows from (2.3) that

$$\|f(\mathbf{i}x) - \mathbf{i}f(x)\| \le \varphi(x, 0, 0, 0, 0),$$

for all $x \in \mathcal{A}$. Hence

$$2^{-n}\|f(2^n\mathbf{i}x) - \mathbf{i}f(2^n x)\| \le 2^{-n}\varphi(2^n x, 0, 0, 0, 0),$$

for all $x \in \mathcal{A}$. The right hand side tends to zero as $n \to \infty$, hence

$$\delta(\mathbf{i}x) = \lim_{n\to\infty} \frac{f(2^n\mathbf{i}x)}{2^n} = \lim_{n\to\infty} \frac{\mathbf{i}f(2^n x)}{2^n} = \mathbf{i}\delta(x),$$

for all $x \in \mathcal{A}$. For every $\lambda \in \mathbb{C}, \lambda = s + \mathbf{i}t$ in which $s, t \in \mathbb{R}$ we have

$$\delta(\lambda x) = \delta(sx + \mathbf{i}tx) = s\delta(x) + t\delta(\mathbf{i}x) = s\delta(X) + \mathbf{i}t\delta(x) = (s + \mathbf{i}t)\delta(x) = \lambda\delta(x),$$

for all $x \in \mathcal{A}$. Thus δ is $\mathbb{C}$-linear. The rest of the proof is similar to the last part of the proof of Theorem 2.1. ■

3 Superstability

In this section, we aim to prove the superstability of derivations on C^*-ternary rings. We start our work with following result in which we give some sufficient conditions in order an approximate derivation to be an exact one.

Proposition 3.1. *Let $r > 1$, and let $\delta : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying $\delta(rx) = r\delta(x)$ for all $x \in \mathcal{A}$ and let there exist a function $\varphi : \mathcal{A}^5 \to [0, \infty)$ such that*

$$\lim_{n\to\infty} r^{-n}\varphi(r^n x, r^n y, r^n u, r^n v, r^n w) = 0,$$

and

$$\begin{aligned}\|\delta(\lambda x + \lambda y + [u\ v\ w]) - \lambda\delta(x) - \lambda\delta(y) - [\delta(u)\ v\ w] - [u\ \delta(v)\ w] - [u\ v\ \delta(w)]\| \\ \le \varphi(x, y, u, v, w),\end{aligned} \tag{3.1}$$

for all $\lambda \in \mathbb{C}$ and all $x, y, u, v, w \in \mathcal{A}$. Then δ is a derivation.

Proof. $\delta(0) = 0$, since $\delta(0) = r\delta(0)$. Set $x = y = 0$ in (3.1). Then

$$\begin{aligned}&\|\delta([u\ v\ w]) - [\delta(u)\ v\ w] - [u\ \delta(v)\ w] - [u\ v\ \delta(w)]\| \\ =\ & \frac{1}{r^{3n}}\|\delta([r^n u\ \ r^n v\ \ r^n w]) - [\delta(r^n u)\ \ r^n v\ \ r^n w] \\ & -[r^n u\ \ \delta(r^n v)\ \ r^n w] - [r^n u\ \ r^n v\ \ \delta(r^n w)]\| \\ \le\ & \frac{1}{r^{3n}}\varphi(r^n u, r^n v, r^n w) \\ \le\ & \frac{1}{r^n}\varphi(r^n u, r^n v, r^n w),\end{aligned}$$

for all $u, v, w \in \mathcal{A}$. The right hand side tends to zero as $n \to \infty$. So that $\delta([u\ v\ w]) = [\delta(u)\ v\ w] + [u\ \delta(v)\ w] + [u\ v\ \delta(w)]$ for all $u, v, w \in \mathcal{A}$.

Similarly, one can shows that $\delta(\lambda x + y) = \lambda\delta(x) + \delta(y)$ for all $x, y \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$. ■

Now we introduce an appropriate definition of almost derivation regarding to Rassias's inequality (see the introduction).

Definition 3.2. Given numbers $\varepsilon > 0$ and $0 \leq p < 1$, a mapping $f : \mathcal{A} \to \mathcal{A}$ is called an (ε, p)-*almost derivation* if $f(0) = 0$ and

$$\begin{aligned}&\|f(\mu x + \mu y + [u\ v\ w]) - \mu f(x) - \mu f(y) - [f(u)\ v\ w] - [u\ f(v)\ w] - [u\ v\ f(w)]\|\\&\qquad\leq \varepsilon(\|x\|^p + \|y\|^p + \|u\|^p + \|v\|^p + \|w\|^p),\end{aligned}$$

for all $x, y, u, v, w \in \mathcal{A}$ and all $\mu \in \mathbb{T}^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

The following theorem is our main result.

Theorem 3.3. *Let $f : \mathcal{A} \to \mathcal{A}$ be an (ε, p)-almost derivation. Then f is a derivation.*

Proof. Put $\varphi(x, y, u, v, w) = \varepsilon(\|x\|^p + \|y\|^p + \|u\|^p + \|v\|^p + \|w\|^p)$ in Theorem 2.1. Then we get a derivation δ defined by $\delta(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ such that

$$\|\delta(x) - f(x)\| \leq \frac{\varepsilon\|x\|^p}{1 - 2^{p-1}},$$

for all $x \in \mathcal{A}$. We have

$$\begin{aligned}
&\|2^n([u\ \ v\ \ f(2^m w)] - [u\ \ v\ \ 2^m f(w)])\|\\
\leq\ &\|f([2^n u\ \ v\ \ 2^m w]) - [f(2^n u)\ \ v\ \ 2^m w] - [2^n u\ \ f(v)\ \ 2^m w] - [2^n u\ \ v\ \ f(2^m w)]\|\\
&+\|f([2^n u\ \ v\ \ 2^m w]) - [f(2^n u)\ \ v\ \ 2^m w] - [2^n u\ \ f(v)\ \ 2^m w] - [2^n u\ \ v\ \ 2^m f(w)]\|\\
\leq\ &\varepsilon(\|2^n u\|^p + \|v\|^p + \|2^m w\|^p)\\
&+\|f([2^n u\ \ v\ \ 2^m w]) - [f(2^n u)\ \ v\ \ 2^m w] - [2^n u\ \ f(v)\ \ 2^m w] - [2^n u\ \ v\ \ 2^m f(w)]\|\\
\leq\ &\varepsilon(\|2^n u\|^p + \|v\|^p + \|2^m w\|^p) + \|f([2^n u\ \ v\ \ 2^m w]) - \delta([2^n u\ \ v\ \ 2^m w])\|\\
&+\|\delta([2^n u\ \ v\ \ 2^m w]) - [f(2^n u)\ \ v\ \ 2^m w] - [2^n u\ \ f(v)\ \ 2^m w] - [2^n u\ \ v\ \ 2^m f(w)]\|\\
\leq\ &\varepsilon(\|2^n u\|^p + \|v\|^p + \|2^m w\|^p) + \frac{\varepsilon}{1 - 2^{p-1}}\|[2^n u\ \ v\ \ 2^m w]\|^p\\
&+2^m\|\delta([2^n u\ \ v\ \ w]) - [f(2^n u)\ \ v\ \ w] - [2^n u\ \ f(v)\ \ w] - [2^n u\ \ v\ \ f(w)]\|\\
\leq\ &\varepsilon(\|2^n u\|^p + \|v\|^p + \|2^m w\|^p) + \frac{\varepsilon}{1 - 2^{p-1}}\|[2^n u\ \ v\ \ 2^m w]\|^p\\
&+2^m\|f([2^n u\ \ v\ \ w]) - \delta([2^n u\ \ v\ \ w])\|\\
&+2^m\|f([2^n u\ \ v\ \ w]) - [f(2^n u)\ \ v\ \ w] - [2^n u\ \ f(v)\ \ w] - [2^n u\ \ v\ \ f(w)]\|\\
\leq\ &\varepsilon(\|2^n u\|^p + \|v\|^p + \|2^m w\|^p) + \frac{\varepsilon}{1 - 2^{p-1}}\|[2^n u\ \ v\ \ 2^m w]\|^p\\
&+\frac{2^m\varepsilon}{1 - 2^{p-1}}\|[2^n u\ \ v\ \ 2^m w]\|^p + 2^m\varepsilon(\|2^n u\|^p + \|v\|^p + \|w\|^p),
\end{aligned}$$

for all nonnegative integers m, n and all $u, v, w \in \mathcal{A}$. Fix m, divide the both sides of the last inequality by 2^n and let n tend to ∞ to obtain

$$\|[u\ \ v\ \ f(2^m w)] - [u\ \ v\ \ 2^m f(w)]\| \leq 0,$$

for all m and all $u, v, w \in \mathcal{A}$. Therefore $\|[u\ \ v\ \ (\frac{f(2^m w)}{2^m} - f(w))]\| = 0$ for all m and all $u, v, w \in \mathcal{A}$. Letting m to ∞ we get $\|[u\ \ v\ \ (\delta(w) - f(w))]\| = 0$ for all $u, v, w \in \mathcal{A}$. Putting $u = v = \delta(w) - f(w)$ we obtain

$$\|\delta(w) - f(w)\|^3 = \|[(\delta(w) - f(w))\ \ (\delta(w) - f(w))\ \ (\delta(w) - f(w))]\| = 0,$$

and so $\delta(w) = f(w)$ for all $w \in \mathcal{A}$. ∎

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Department of Mathematics
Ferdowsi University
P. O. Box 1159
Mashhad 91775
Iran
email:moslehian@ferdowsi.um.ac.ir