Projective Planes with a Doubly Transitive Projective Subplane

Alessandro Montinaro

Abstract

Projective planes Π of order up to q^3 with a collineation group G acting 2-transitively on a subplane of order q are investigated.

1 Introduction

A classical problem in finite geometry is the investigation of a projective plane Π of order n admitting a collineation group G which acts 2-transitively on the points of a subplane Π_0 of Π. In 1959 Ostrom and Wagner [20] show that Π is Desarguesian and $PSL(3, n) \leq G$ when $\Pi_0 = \Pi$. Several years later, in 1976, Lüneburg [17] proves that either Π is a Desarguesian plane or a Generalized Hughes plane when Π_0 is a Baer subplane of Π. In 1985, Dempwolff [5] proves that any projective plane Π of order n with a collineation group $G \cong PSL(3, \sqrt[3]{n})$ contains a Desarguesian subplane Π_0 of order $\sqrt[3]{n}$ on which G acts faithfully in its natural permutation representation. Furthermore, in that paper, Dempwolff emphasizes the difficulty to obtain a characterization of Π, even though he gives a complete description of the G-orbits on the points and on the lines of Π. He also shows that examples occur in the Desarguesian planes and in the Hering-Figueroa planes [7], [11].

The aim of this paper is to show that any projective plane Π of order n with a collineation group G acting 2-transitively on the points of a subplane Π_0 of Π order q, with $n \leq q^3$, has actually order $n = q$, or q^2 or q^3. Moreover, the structure of G is determined.

Received by the editors December 2005 - In revised form in March 2006.
Communicated by J. Thas.
2000 *Mathematics Subject Classification* : 51E15, 20B25.
Key words and phrases : Projective plane, Collineation group, Orbit.

2 Preliminaries

We shall use standard notation. For what concerns finite groups the reader is referred to [8] and [13]. The necessary background about finite projective planes may be found in [12].

Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a finite projective plane of order n. If H is a collineation group of Π and $P \in \mathcal{P}$ ($l \in \mathcal{L}$), we denote by $H(P)$ (by $H(l)$) the subgroup of H consisting of perspectivities with the centre P (the axis l). Also, $H(P,l) = H(P) \cap H(l)$. Furthermore, we denote by $H(P,P)$ (by $H(l,l)$) the subgroup of H consisting of elations with the centre P (the axis l).

Now, we give some numerical results which will be useful in the following.

Lemma 1. *Let p^m be a prime power such that p is odd and $p^m \equiv 1 \mod 3$, and let n be an integer such that $n \leq p^{3m}$. Then the Diophantine equation*

$$(n - p^m)(n - p^{2m}) = \frac{2}{3}p^{2m}(p^{3m} - 1)(p^m + 1) \tag{1}$$

has no positive solutions.

Proof. Set $n = p^h\alpha$ with $(p, \alpha) = 1$ and $h \geq 0$. Note that $p^{2m} \parallel (n - p^m)(n - p^{2m})$, since p is odd and $p^m \equiv 1 \mod 3$. Thus $1 \leq h \leq m$ and hence $(n - p^m)(n - p^{2m}) = p^{2h}(\alpha - p^{m-h})(\alpha - p^{2m-h})$. Then $h = m$ and $n = p^m\alpha$, again by the fact that $p^{2m} \parallel (n - p^m)(n - p^{2m})$. Then (1) becomes

$$(\alpha - 1)(\alpha - p^m) = \frac{2}{3}(p^{2m} + p^m + 1)(p^{2m} - 1). \tag{2}$$

Note that $(\alpha - 1, \alpha - p^m) \mid p^m - 1$. Furthermore $(p^m - 1, p^{2m} + p^m + 1) = 3$, since $p^m \equiv 1 \mod 3$. Actually, $p^{2m} + p^m + 1 \equiv 3 \mod 9$ by [10], Lemma 3.9, and hence $(p^m - 1, \frac{p^{2m}+p^m+1}{3}) = 1$. Thus, either $\frac{p^{2m}+p^m+1}{3} \mid \alpha - 1$ or $\frac{p^{2m}+p^m+1}{3} \mid \alpha - p^m$. Assume that $\frac{p^{2m}+p^m+1}{3} \mid \alpha - 1$. Then $\alpha = k\frac{p^{2m}+p^m+1}{3} + 1$ for some integer $k \geq 1$. Note that $k < 3$, since $n \leq p^{3m}$. Then (2) becomes

$$k(k\frac{p^{2m} + p^m + 1}{3} + 1 - p^m) = 2(p^{2m} - 1).$$

At this point it is easily checked that the previous equality has no positive integer solutions for $k \in \{1, 2\}$. Hence, we may assume that $\frac{p^{2m}+p^m+1}{3} \mid \alpha - p^m$. Then $\alpha = s\frac{p^{2m}+p^m+1}{3} + p^m$ for some integer $s \geq 1$. Also $s < 3$, since $n \leq p^{3m}$. Then (2) becomes

$$s(s\frac{p^{2m} + p^m + 1}{3} + p^m - 1) = 2(p^{2m} - 1).$$

Now, elementary calculations show that the previous equality has no positive integer solutions for $s \in \{1, 2\}$. This completes the proof. ■

Lemma 2. *Let p be a prime and $\lambda \in \{1, 2, 3\}$. If $n = p^{3m} - \beta$ with $0 \leq \beta < p^{3m}$, then the positive solutions of the Diophantine equation*

$$(n - p^m)(n - p^{2m}) = \frac{\lambda}{3}p^{3m}(p^m - 1)^2(p^m + 1) \tag{3}$$

are $(n, \beta, \lambda) = (p^m, 0, 3)$, $(6, 2, 1)$ and $(105, 20, 2)$.

Proof. If $\lambda = 3$, then $(n, \beta, \lambda) = (p^m, 0, 3)$ is clearly the unique integer solution. Hence, we may assume that $\lambda \in \{1, 2\}$. By substituting $n = p^{3m} - \beta$ in (3), we have

$$3\beta^2 - 3p^m(p^m - 1)(2p^m + 1)\beta + (3 - \lambda)p^{3m}(p^m - 1)^2(p^m + 1) = 0. \quad (4)$$

Since β must be a positive integer, then the discriminant

$$\Delta = p^{2m}(p^m - 1)^2 \left(9 + 12\lambda p^m(p^m + 1)\right)$$

must be a square. In particular $9 + 12\lambda p^m(p^m + 1)$ must be a square. Thus either $p = 3$ or $p \equiv 2 \mod 3$, since $3 \mid 9+12\lambda p^m(p^m+1)$. Therefore $y^2 = 1+4\frac{\lambda}{3}p^m(p^m+1)$ for some positive integer y, with $p = 3$ or $p \equiv 2 \mod 3$. Hence

$$\left(\frac{y-1}{2}\right) \cdot \left(\frac{y+1}{2}\right) = \frac{\lambda}{3}p^m(p^m + 1) \quad (5)$$

with $\left(\frac{y-1}{2}, \frac{y+1}{2}\right) = 1$ and $p = 3$ or $p \equiv 2 \mod 3$. Assume that $p \equiv 2 \mod 3$. If $(p, \lambda) = 1$, then either $p^m \mid \frac{y-1}{2}$ or $p^m \mid \frac{y+1}{2}$. Assume that $p^m \mid \frac{y-1}{2}$. Then $\frac{y-1}{2} = j\, p^m$ and $\frac{y+1}{2} = jp^m + 1$ for some integer $j \geq 1$. Now, by substituting these values in (5) and dividing by p^m, we obtain $j(jp^m + 1) = \frac{\lambda}{3}(p^m + 1)$. This is impossible, since $j \geq 1 > \frac{\lambda}{3}$ as $\lambda \in \{1, 2\}$. Assume that $p^m \mid \frac{y+1}{2}$. Then $\frac{y+1}{2} = p^m j$ and $\frac{y-1}{2} = jp^m - 1$, and again by substituting these values in (5) and dividing by p^m, we obtain $j(jp^m - 1) = \frac{\lambda}{3}(p^m + 1)$ with unique solutions $(j, \lambda, p^m) = (1, 1, 2)$ and $(1, 2, 5)$. By substituting the values found for (j, λ, p^m) in (4), we obtain $\beta = 2$ and $\beta = 20$, respectively, since $0 \leq \beta < p^{3m}$. Thus $n = 6$ and 105, respectively. Assume that $(p, \lambda) > 1$. Then $p = \lambda = 2$. In particular m is odd, since $2^m \equiv 2 \mod 3$. Then either $2^{m+1} \mid \frac{y-1}{2}$ or $2^{m+1} \mid \frac{y+1}{2}$. Assume that $2^{m+1} \mid \frac{y-1}{2}$. Then $\frac{y-1}{2} = s2^{m+1}$ and $\frac{y+1}{2} = s2^{m+1} + 1$ for some positive integer s. Now, by substituting these values in (5) and dividing by 2^{m+1}, we have $s(s2^{m+1} + 1) = \frac{1}{3}(2^m + 1)$. A contradiction. Hence, $2^{m+1} \mid \frac{y+1}{2}$. Then $\frac{y+1}{2} = t2^{m+1}$ and $\frac{y+1}{2} = t2^{m+1} - 1$ for some positive integer t. Now, by substituting these values in (5) and dividing by 2^{m+1}, we have $t(t2^{m+1} - 1) = \frac{1}{3}(2^m + 1)$. A contradiction.

Assume that $p = 3$. Then either $3^{m-1} \mid \frac{y-1}{2}$ or $3^{m-1} \mid \frac{y+1}{2}$. Assume that $3^{m-1} \mid \frac{y-1}{2}$. Then $k(k3^{m-1} + 1) = \lambda(3^m + 1)$ for some positive integer k. A contradiction, since $\lambda \in \{1, 2\}$. So, $3^{m-1} \mid \frac{y+1}{2}$. Arguing as above, we obtain $h(h3^{m-1} - 1) = \lambda(3^m + 1)$ for some positive integer h. A contradiction, since $\lambda \in \{1, 2\}$. Hence the assertion. ∎

3 The background

In this section we introduce the background for the problem investigated and we state the group-theoretical theorems on which relies the proof of the result exposed in this paper.

Lemma 3. *Let* Π *be a finite projective plane with a collineation group* G *which fixes a projective subplane of* Π *and induces a doubly transitive group on the points of* Π_0*. Then* Π_0 *is Desarguesian and the group induced by* G *on* Π_0 *contains a subgroup isomorphic to* $PSL(3, q)$*.*

Proof. Ostrom-Wagner [20]. ∎

As it is well known $o(\Pi) \geq q^2$ when Π_0 is a proper subplane of Π. Thus $q^2 \leq o(\Pi) \leq q^3$ under our assumption. The only known cases are when $o(\Pi) = q^2$ or $o(\Pi) = q^3$. The following result characterizes the case when $o(\Pi) = q^2$.

Lemma 4. *Let Π be a finite projective plane of order q^2 with a collineation group G that fixes a projective Baer subplane Π_0 and induces a doubly transitive group on Π_0. Then one of the following occurs:*

1. *Π is a Desarguesian or a generalized Hughes plane and G contains a subgroup isomorphic to $PSL(3, q)$;*
2. *Π is the generalized Hughes plane over the exceptional nearfield of order 7^2 and G contains a subgroup isomorphic to $SL(3, 7)$.*

Proof. Lüneburg [17]. ■

The next result deals with the case $o(\Pi) = q^3$. Let $\mathcal{M}$ be the set of points of $\Pi - \Pi_0$ which lie in a secant to Π_0, and let $\mathcal{A}$ be the set of point of $\Pi - \Pi_0$ which do not lie in any secant to Π_0.

Lemma 5. *Let Π be a finite projective plane of order q^3 with a collineation group $G \cong PSL(3, q)$. Then Π has a projective subplane Π_0 of order q which is invariant under G and G acts faithfully on Π_0. Moreover, the following occur:*

1. *G is transitive on the points and lines of Π_0, $\mathcal{M}$ and $\mathcal{A}$.*
2. *Let (M, m) be a flag in $\mathcal{M}$. Then $|G_M| = |G_m| = q^2(q-1)/j$, with $j = (3, q-1)$. Moreover, G_M (G_m) has a normal elementary abelian subgroup A (B) of order q^2 and G_M (G_m) is the semidirect product of A (B) with $G_{M,m}$. The group $G_{M,m}$ is cyclic and G_M (G_m) is a Frobenius group.*
3. *If P is a point (or a line) in $\mathcal{A}$, then G_P is cyclic of order $(q^2+q+1)/j$. Any non-trivial element in G_P fixes exactly a triangle in Π which lies in $\mathcal{A}$.*
4. *G is flag-transitive on $\mathcal{M}$, and G is flag-transitive on $\mathcal{A} \times \mathcal{M}$ and on $\mathcal{M} \times \mathcal{A}$ if $j = 1$.*

Proof. Dempwolff [5], Theorems A and B. ■

Unlike the Lemma 4, this result does not seem to determine the plane Π, even though examples occur in the Desarguesian or the Hering-Figueroa planes (e.g. see [7] and [11]). Indeed, Dempwolff remarked in his paper that the group $G_{M,m}$ under (3) of the previous Lemma is too far from being a group of homologies of a Desarguesian or a Hering-Figueroa plane.

Now, we expose some results on the subgroups of $PSL(3, q)$ which will be used extensively in the following.

Lemma 6. *Let M be a maximal subgroup of $PSL(3, 2^h)$. Then M is isomorphic to one of the following groups:*

1. $A : PSL(2, 2^h)$*, where A is elementary abelian of order q^2;*
2. $B.S_3$ *, where B is diagonal group of order $\frac{(2^h-1)^2}{j}$ and $j = (3, 2^h - 1)$;*
3. $Z_{\frac{2^{2h}+2^h+1}{j}}.Z_3$*, where $j = (3, 2^h - 1)$;*
4. $PSL(3, 2^m)$*, where $h = tm$ and t is prime;*
5. *A group containing $PSL(3, 2^m)$ as normal subgroup of index 3, where $h = 3m$ and m is even;*
6. $PSU(3, 2^m)$*, where $h = 2m$;*
7. *A group containing $PSU(3, 2^m)$ as normal subgroup of index 3, where $h = 6m$ and m is odd;*

Proof. Hartley [9]. ■

Lemma 7. *Let M be a non-trivial subgroup of $PSL(3, p^h)$. If M has no non-trivial normal elementary abelian subgroups, then M is isomorphic to one of the following groups:*

1. $PSL(3, p^m)$*, where $m \mid h$;*
2. $PSU(3, p^m)$*, where $2m \mid m$;*
3. *A group containing $PSL(3, p^m)$ as normal subgroup of index 3, when $p^m \equiv 1 \mod 3$ and $3m \mid h$;*
4. *A group containing $PSU(3, p^m)$ as normal subgroup of index 3, when $p^m \equiv 2 \mod 3$ and $6m \mid h$;*
5. $PSL(2, p^m)$ *or* $PGL(2, p^m)$*, where $m \mid h$ and $p^m \neq 3$;*
6. $PSL(2, 5)$ *when $p^h \equiv \pm 1 \mod 10$;*
7. $PSL(2, 7)$ *when $p^{3h} \equiv 1 \mod 7$;*
8. A_6 *or A_7, or a group containing A_6 with index 2, with $p = 5$ and h even;*
9. A_6*, when $p^h \equiv 1 \mod 30$ or $p^h \equiv 19 \mod 30$.*

Moreover, $PSL(3, p^h)$ has exactly one subgroup G of each type mentioned above up to conjugacy in $GL(3, p^h)/Z(SL(3, p^h))$.

Proof. Bloom [3], Theorem 1.1. ■

Lemma 8. *Let G be a subgroup of $PSL(3,p^h)$ not satisfying the hypothesis of Lemma 7. Then the following occurs:*

1. *G has a cyclic normal subgroup H such that $[G:H] \leq 3$ and $(|H|, p) = 1$;*
2. *G has diagonal normal subgroup R such that $G/R \leq S_3$;*
3. *The inverse image G^* of G in $SL(3,p^h)$ has a normal elementary abelian p-subgroup F such that $G^*/F \leq GL(2,p^h)$. The case $F = \langle 1 \rangle$ is also included;*
4. *$p^h \equiv 1 \mod 9$ and G has a normal subgroup T, abelian of type $(3,3)$, with $G/T \leq SL(2,3)$. All subgroups of $SL(2,3)$ do occur in this context;*
5. *$p^h \equiv 1 \mod 3$, $p^h \not\equiv 1 \mod 9$ and G has a normal subgroup Y, abelian of type $(3,3)$, with $G/Y \leq Q_8$. All subgroups of Q_8 do occur in this context.*

Proof. Bloom [3], Theorem 7.1. and Theorem 3.4. ∎

4 The faithful action

Throughout this section we assume that G acts faithfully on Π_0. Hence, we may assume that G is minimal and $G \cong PSL(3,q)$. The proof relies on combinatorics and a detailed knowledge of the structure of the group $PSL(3,q)$. A preliminary step is to prove that the involutions in G are perspectivities. We show this fact for q even in Lemma 9, using the Cauchy-Frobenius Lemma, and for q is odd in Lemma 10 and in Proposition 11, using the list of subgroup of $PSL(2,q)$. Finally in Theorem 12, we show that if $o(\Pi) \leq q^3$ then Π must have order q, q^2 or q^3. Here the list of the subgroups of $PSL(3,q)$ is extensively used.

Lemma 9. *Let Π be a finite projective plane of order n and let $G \cong PSL(3,q)$ be a collineation group of Π with a point-orbit $\Pi_0 \cong PG(2,q)$. If $n \leq q^3$ and q is even, then each involution in G is a perspectivity of Π.*

Proof. Assume that q is even. Let r be a secant of Π_0 and let T be the elementary abelian 2-group of order q^2 inducing an elation group of axis r on Π_0. Then either T is a Baer collineation group or $T = T(r,r)$ on Π, since all the elements in T lie in a unique conjugate class under G. Assume that T is a Baer collineation group of Π. Then each non trivial element in T fixes exactly $\sqrt{n}+1$ lines of $[P]$, where P is any point of $r \cap \Pi_0$. Then $q^2 \mid [n+1+(q^2-1)(\sqrt{n}+1)]$ and hence $q^2 \mid n-\sqrt{n}$. Either $q^2 \mid \sqrt{n}$ or $q^2 \mid \sqrt{n}-1$, since q is a prime power. So $q^2 \leq \sqrt{n}$ in any case. A contradiction, since $n \leq q^3$ by our assumption. Hence $T = T(r,r)$ on Π, and the assertion follows by the fact that there exists a unique conjugate class of involutions in $PSL(3,q)$. ∎

Lemma 10. *Let Π be a finite projective plane of order n and let $G \cong PSL(3,q)$ be a collineation group of Π with a point-orbit $\Pi_0 \cong PG(2,q)$. If $n \leq q^3$, q is odd and $q \notin \{5,7,9,11,19\}$, then each involution in G is a perspectivity of Π.*

Proof. Assume that q is odd and $q \notin \{5, 7, 9, 11, 19\}$. Assume also that each involution in G is a Baer collineation of Π. Denote by α the involution in G represented by the matrix $diag(-1, -1, 1)$. Then α induces a (C, l)-homology on Π_0. Let H be the group consisting of the matrices $diag(A, 1)$ with $A \in SL(2, q)$. Then $H \leq C_G(\alpha)$ and H acts on $Fix(\alpha)$ inducing $\bar{H} \cong PSL(2, q)$. In particular $\bar{H}$ fixes l and acts on $l \cap \Pi_0$ in its natural 2-transitive permutation representation of degree $q + 1$. Set $\mathcal{C} = l \cap Fix(\alpha) - \Pi_0$. Then $|\mathcal{C}| > 0$ by [19], Corollary 5.2.(ii), since $\bar{H}$ contains Baer collineations and $q \notin \{5, 9\}$ by our assumption. Then $|\mathcal{C}| > 0$ and $n > q^2$. Furthermore $\bar{H}$ acts on $\mathcal{C}$.

(A) There exists $X \in \mathcal{C}$ such that $\left|X^{\bar{H}}\right| > 1$.

Assume that $\bar{H}$ fixes $\mathcal{C}$ pointwise. Assume also that $q \equiv 3 \mod 4$. Note that the stabilizer in $\bar{H}$ of a point on $l \cap \Pi_0$ has odd order, since it is isomorphic to $E_q.Z_{\frac{q-1}{2}}$. Thus the points on $l \cap Fix(\alpha)$ fixed by any involution in $\bar{H}$ are exactly those fixed lying in $\mathcal{C}$. Set $k = |C|$. Clearly $0 < k < \sqrt{n} + 1$. Assume that $k \geq 3$. Thus each involution in $\bar{H}$ is a Baer collineation of $Fix(\alpha)$ and hence $k = \sqrt[4]{n} + 1$, since $k < \sqrt{n} + 1$. Let $Y \in l \cap \Pi_0$. Then $\bar{H}_Y \cong E_q.Z_{\frac{q-1}{2}}$ fixes Y and the k points of $\mathcal{C}$. Recall that α induces a (C, l)-homology on Π_0. Therefore $\bar{H}_Y$ fixes C and the lines joining the $k + 1$ points of $\mathcal{C} \cup \{Y\}$ with C. In particular $\bar{H}_Y$ cannot contain planar elements, since it fixes $\sqrt[4]{n} + 2$ points on $l \cap Fix(\alpha)$. Hence $\bar{H}_Y$ must be semiregular on $AC \cap Fix(\alpha) - \{A, C\}$ for any $A \in \mathcal{C} \cup \{Y\}$. So, $\frac{q(q-1)}{2} \mid \sqrt{n} - 1$. Then $(q, n) = (3, 16)$, since $q^2 < n \leq q^3$ and $q \equiv 3 \mod 4$ by our assumption. Then $\bar{H} \cong PSL(2, 3)$ and $Fix(\alpha) \cong PG(2, 4)$. A contradiction, since $\bar{H}$ fixes $l \cap Fix(\alpha)$ and a point on it. Hence $0 < k \leq 2$. Assume that $k = 1$. Then $\mathcal{C} = \{R\}$, where R is the unique point of $l \cap Fix(\alpha)$ fixed by $\bar{H}$. Then all involutions in $\bar{H}$ must be elations with same axis $s = RC$ and the same centre R, since $\bar{H}$ fixes R, C and l. Thus $\bar{H} = \bar{H}(R, s)$, since the involutions in $\bar{H}$ generate $\bar{H}$. A contradiction, since $\bar{H}_{B_1, B_2} \cong Z_{\frac{q-1}{2}}$ with $B_1, B_2 \in l \cap \Pi_0$ and $B_1 \neq B_2$. Thus $k = 2$. Then $\mathcal{C} = \{P, Q\}$, where P and Q are the unique points of $l \cap Fix(\alpha)$ fixed by $\bar{H}$. Then there are no triangular configurations for commuting homologies in $\bar{H}$, otherwise one of them would have the axis coinciding with $l \cap Fix(\alpha)$, while $\bar{H} \cong PSL(2, q)$ acts not trivially on $l \cap Fix(\alpha)$. Thus all involutions in $\bar{H}$ must have the same center and the same axis, since $\bar{H}$ fixes the triangle $\{C, P, Q\}$ pointwise. Thus either $\bar{H} = \bar{H}(P, QC)$ or $\bar{H} = \bar{H}(Q, PC)$. A contradiction by the same argument as above. Hence $q \equiv 1 \mod 4$. Then $k > 0$ as $k = |\mathcal{C}|$ and $\mathcal{C} \neq \emptyset$. Then $\bar{H}_{Y_1, Y_2} \cong Z_{\frac{q-1}{2}}$ has even order, for any $Y_1, Y_2 \in l \cap \Pi_0$ such that $Y_1 \neq Y_2$. Therefore the points on $l \cap Fix(\alpha)$ fixed by any involution $\bar{\gamma}$ in $\bar{H}_{Y_1, Y_2}$ are exactly those fixed lying in $\mathcal{C}$ plus the points Y_1, Y_2. Then $Fix(\bar{\gamma})$ is a Baer subplane of $Fix(\alpha)$ and $\sqrt[4]{n} + 1 = k + 2$, since $k > 0$. If $k \leq 2$, then $\sqrt{n} \leq 9$. Then either $q > \sqrt{n}$, since $q \equiv 1 \mod 4$ and $q \notin \{5, 9\}$. A contradiction by [19], Theorem 1.1. Hence $k \geq 3$. Let $\bar{E}$ be any Klein subgroup of $\bar{H}$ containing $\bar{\gamma}$. Then the points of $l \cap Fix(\alpha)$ fixed by $\bar{E}$ are exactly those fixed lying in $\mathcal{C}$, since the stabilizer in $\bar{H}$ of a point on $l \cap \Pi_0$ is isomorphic to the Frobenius group $E_q.Z_{\frac{q-1}{2}}$. Thus $Fix(\bar{E})$ is a Baer subplane of $Fix(\bar{\gamma})$ and hence $\sqrt[8]{n} + 1 = k$, since $k \geq 3$. Then $\sqrt[4]{n} = \sqrt[8]{n} + 2$, since $\sqrt[4]{n} + 1 = k + 2$. This yields $\sqrt{n} = 16$, $q = 13$ and $k = 3$. A contradiction by [14], since $\bar{H}$ contains a Frobenius group of order 39 with a planar 13-element.

(B) Either $\bar{H}_X = \bar{H}$ or $\bar{H}_X \cong E_q.Z_{\frac{q-1}{\theta}}$, $1 < \theta < \frac{q-1}{2}$, θ even or $\bar{H}_X \cong$

$PGL(2, \sqrt{q})$ **for** q **square.**

By **(A)** there exists $X \in C$ such that $\left|X^{\bar{H}}\right| > 1$. Then

$$\sqrt{n} + 1 \geq \left|X^{\bar{H}}\right| + |l \cap \Pi_0|, \tag{6}$$

since $X^{\bar{H}} \cup (l \cap \Pi_0) \subset l \cap Fix(\alpha)$. By managing (6), we have

$$\left|\bar{H}_X\right| \geq \frac{(\sqrt{q}+1)(q+1)}{2}, \tag{7}$$

since $|l \cap \Pi_0| = q + 1$, $\left|X^{\bar{H}}\right| = \frac{q(q^2-1)}{2\left|\bar{H}_X\right|}$ and $n \leq q^3$. Now, we filter the list of the proper subgroups of $\bar{H}$ given in [13], Haupsatz II.8.27, with respect to (7):

(i). $\bar{H}_X \leq D_{q\pm1}$. Then $2(q+1) \geq 2\left|\bar{H}_X\right| \geq (\sqrt{q}+1)(q+1)$ in any case. A contradiction.

(ii). $\bar{H}_X \cong PSL(2, p^m)$ with $q = p^{mt}$ and $t \geq 2$. Then $p^m(p^{2m}-1) \geq (p^{tm/2}+1)(p^{tm}+1)$ by substituting in (7). So $p^m(p^{2m}-1) \geq (p^m+1)(p^{2m}+1)$, since $t \geq 2$. A contradiction.

(iii). $\bar{H}_X \cong PGL(2, p^m)$ with $q = p^{2mt}$ and $t \geq 1$. Then $2p^m(p^{2m}-1) \geq (p^{tm}+1)(p^{2tm}+1)$ by substituting in (7). Thus $t = 1$ and q is a square. Assume there are at least two $\bar{H}$-orbits on $\mathcal{C}$ with point-stabilizer isomorphic to $PGL(2, \sqrt{q})$. Then $\sqrt{n}+1 \geq 2\left|X^{\bar{H}}\right| + |l \cap \Pi_0|$. This yields $\left|\bar{H}_X\right| \geq (\sqrt{q}+1)(q+1)$ arguing as above. A contradiction, since $\bar{H}_X \cong PGL(2, \sqrt{q})$.

(iv). The cases $A_4 \leq \bar{H}_X \leq S_4$ or $\bar{H}_X \cong A_5$ cannot occur, since $q \notin \{5, 7, 9, 11, 19\}$.

(v). $\bar{H}_X \leq E_{p^m}.Z_{\frac{p^m-1}{2}}$. Then $\bar{H}_X \cong E_{p^f}.K$, with $f > 1$ and $\langle 1 \rangle < K \leq Z_{\frac{p^m-1}{2}}$ by (7). Then $|K| \mid p^m - 1$. Furthermore $|K| \mid p^f - 1$, since H_X is a Frobenius group, as $f > 1$ and $K \neq \langle 1 \rangle$. Then $|K| \mid p^e - 1$, where $p^e - 1 = (p^m - 1, p^f - 1)$ and $e = (m, f)$. Set $m = ae$ and $f = be$, then $a \geq b \geq 1$, and $\bar{H}_X \cong E_{p^{be}}.Z_{\frac{p^e-1}{\theta}}$ with $\theta \geq 1$. By (7), we have

$$p^{be}\frac{p^e-1}{\theta} \geq \frac{(\sqrt{q}+1)(q+1)}{2} \geq \frac{(p^{ae/2}+1)(p^{ae}+1)}{2}.$$

This yields $a = b$. That is $m = f = e$ and hence $\bar{H}_X \cong E_q.Z_{\frac{q-1}{\theta}}$, with θ even.

(C) q **is a square and** $\bar{H}$ **has exactly one orbit on** $\mathcal{C}$ **with the stabilizer of a point isomorphic to** $PGL(2, \sqrt{q})$**.**

Assume that $q \equiv 3 \mod 4$. As a consequence of the list given above, the stabilizer in $\bar{H}$ of a point on $l \cap Fix(\alpha)$ is either the whole group $\bar{H}$ or it is isomorphic to $E_q.Z_{\frac{q-1}{\theta}}$, θ even. Since $E_q.Z_{\frac{q-1}{\theta}}$ has odd order, then the points of $l \cap Fix(\alpha)$ fixed by any involution in $\bar{H}$ coincide with those fixed by $\bar{H}$. Let h be the number of these points. A similar argument to that used to rule out the case where $\bar{H}$ fixes $\mathcal{C}$ pointwise and $q \equiv 3 \mod 4$ (part **(A)**), with h in the role of k, still works and we may rule out this case. Hence, we may assume that $q \equiv 1 \mod 4$. Assume also that $\bar{H}$ does not contain any stabilizer of a point isomorphic to $PGL(2, \sqrt{q})$. Then either $\bar{H}_X = \bar{H}$ or $\bar{H}_X \cong E_q.Z_{\frac{q-1}{\theta}}$, θ even, by the list of the admissible subgroups of $\bar{H}$ given above. Furthermore, two distinct commuting involutions have no common fixed points on $l \cap Fix(\alpha)$, other than the h ones fixed by the whole $\bar{H}$, since the point-stabilizer in $\bar{H}$ elsewhere on $l \cap Fix(\alpha)$ is isomorphic to $E_q.Z_{\frac{q-1}{\theta}}$, with q odd and θ even. Assume that $h \geq 3$. Then each involution in $\bar{H}$ is a Baer collineation of $Fix(\alpha)$. Note that for any two distinct commuting involutions in $\bar{H}$, each induces a Baer collineation on the subplane fixed by the other one in $Fix(\alpha)$, as $h \geq 3$. Thus $h = \sqrt[8]{n} + 1$.

Let $\bar{U} \leq \bar{H}$ such that $\bar{U} \cong D_{q+1}$. Then $\bar{U}$ contains exactly $\frac{q+1}{2}$ distinct involutions, since $q \equiv 1 \mod 4$. Let $\bar{\sigma}$ and $\bar{\rho}$ be two distinct involutions in $\bar{U}$. If $\bar{\sigma}$ and $\bar{\rho}$ fixes a point O on $l \cap Fix(\alpha)$, then $\left\langle \bar{\sigma}, \bar{\delta} \right\rangle \leq \bar{H}_O \cap \bar{U}$. Thus $\bar{H}_O = \bar{H}$, since $\left| \left\langle \bar{\sigma}, \bar{\delta} \right\rangle \right| > 2$ and $\left| E_q.Z_{\frac{q-1}{\theta}} \cap \bar{U} \right| \leq 2$. Hence $\bar{\sigma}$ and $\bar{\rho}$ have no common fixed points on $l \cap Fix(\alpha)$, other than the $h = \sqrt[8]{n} + 1$ fixed by the whole group $\bar{H}$. Hence each involution in $\bar{U}$ fixes exactly $\sqrt[4]{n} - \sqrt[8]{n}$ points on $l \cap Fix(\alpha)$ which are not fixed by any other involution in $\bar{U}$. Therefore

$$(\sqrt[4]{n} - \sqrt[8]{n})\frac{q+1}{2} \leq \sqrt{n} - \sqrt[8]{n}, \tag{8}$$

since $\bar{U}$ contains exactly $\frac{q+1}{2}$ distinct involutions. By managing (8), we have that $\frac{q-1}{2} \leq \sqrt[4]{n} + \sqrt[8]{n}$. Hence $\frac{q-1}{2} \leq q^{3/4} + q^{3/8}$, since $n \leq q^3$. Thus $q \in \{13, 17, 25, 29, 37, 41\}$, since q is odd, $q \equiv 1 \mod 4$ and $q > 9$. Actually, $(q, n) = (13, 2^8), (5^2, 3^8), (41, 4^8)$, since $\frac{q-1}{2} \leq \sqrt[4]{n} + \sqrt[8]{n}$ with $\sqrt[8]{n}$ integer, $n > q^2$ and $\sqrt[8]{n} \geq 2$. Also the case $(q, n) = (13, 2^8)$ cannot occur by **(A)**, since $\bar{H}$ fixes $\mathcal{C}$ pointwise in this case. Assume that $(q, n) = (5^2, 3^8)$. In this case $l \cap Fix(\alpha)$ consists of either 3 $\bar{H}$-orbits of length 26 and $h = 4$ fixed points by $\bar{H}$, or 1 $\bar{H}$-orbit of length 26, 1 $\bar{H}$-orbit of length 52 and $h = 4$ fixed points by $\bar{H}$. At this point it is a plain to see that any element of order 5 must fix a subplane of order 6 in any case, since $\sqrt{n} + 1 \equiv 2 \mod 5$ and since it fixes exactly 7 points on $l \cap Fix(\alpha)$. A contradiction by [12], Theorem 3.6. Hence $(q, n) = (41, 4^8)$. Let $\bar{S} \cong Z_{41}$. Since $\sqrt{n} + 1 \equiv 11 \mod 41$ and $n \equiv 18 \mod 41$, then $Fix(\bar{S})$ fixes a subplane $Fix(\alpha)$ of order at least 10. Actually, $o(Fix(\bar{S})) = 10$ by [12], Theorem 3.7. Let $\bar{T} \leq N_{\bar{H}}(\bar{S})$ such that $\bar{T} \cong Z_2$. Then $\bar{T}$ acts trivially on $Fix(\bar{S})$ by [12], Theorem 13.18. Hence $Fix(\bar{S}) \subsetneq Fix(\bar{T})$, since $\bar{S} \cong Z_{41}$ and $\bar{T} \cong Z_2$ fix exactly 1 and 2 points on $l \cap \Pi_0$, respectively. Thus $o(Fix(\bar{T}) \geq 10^2$ and $o(Fix(\alpha)) \geq 10^4$ by [12], Theorem 3.7, since $\bar{T}$ acts not trivially on $Fix(\alpha)$. A contradiction, since $o(Fix(\alpha)) = 4^4$. Hence $h \in \{0, 1, 2\}$. Arguing as above it is easily seen that $(\sqrt[4]{n} + 1 - h)\frac{q+1}{2} \leq \sqrt{n} + 1 - h$ with $h \in \{0, 1, 2\}$, since each involution in $\bar{H}$ fixes exactly 2 points on Π_0, other than the h fixed by the whole $\bar{H}$. Elementary calculations show that $(n, q, h) = (6^8, 13, 2)$ or $(n, q, h) = (8^8, 17, 2)$, since $n \leq q^3$, $q \equiv 1 \mod 4$ and $q \notin \{5, 9\}$. In particular $h = 2$ in any admissible case. Thus, let P_1 and P_2 be the unique points of $l \cap Fix(\alpha)$ fixed by $\bar{H}$. Then the stabilizer in $\bar{H}$ of any point on $l \cap Fix(\alpha) - \{P_1, P_2\}$ is a subgroup of $E_q.Z_{\frac{q-1}{2}}$. As a consequence, each $\bar{H}$-orbit on $l \cap Fix(\alpha) - \{P_1, P_2\}$ has length divisible by $q + 1$. So $q + 1 \mid (\sqrt{n} + 1) - h$. A contradiction. At this point the assertion **(C)** follows by the final remark in (iii).

(D) The final contradiction.

Let B be any point of $\mathcal{C}$ such that $\bar{H}_B \cong PGL(2,\sqrt{q})$. By **(B)** and **(C)** we have that either $\bar{H}_X = \bar{H}$ or $\bar{H}_X \cong E_q.Z_{\frac{q-1}{\theta}}$, θ even, for any $X \in \mathcal{C} - B^{\bar{H}}$, and $\bar{H}_X \cong PGL(2,\sqrt{q})$ for any $X \in B^{\bar{H}}$. Note that $\bar{H}$ contains a unique conjugate class of involutions and each of them fixes exactly $\sqrt{q}$ points on $B^{\bar{H}}$ by [19], Table III, lines 9a and 9b. Furthermore, $\bar{H}$ contains two conjugate classes of Klein subgroups. In particular each subgroup in the first conjugate class fixes exactly 1 point on $B^{\bar{H}}$ and each subgroup in the second conjugate class fixes exactly 3 points on $B^{\bar{H}}$ (e.g. see [19], Table III, lines 9a and 9b). Let $\bar{\tau}$ be any involution in $\bar{H}$ and let be $\bar{E}_1$ and $\bar{E}_2$ the representative of the two conjugate classes of Klein subgroups of $\bar{H}$ containing $\bar{\tau}$. For any subgroup $\bar{J}$ of $\bar{H}$, we set $Fix_X(\bar{J})$ the number of points fixed by $\bar{J}$ in the $X^{\bar{H}}$. The following table gives a description of the points fixed by $\bar{\tau}$, $\bar{E}_1$ and $\bar{E}_2$ in each admissible $\bar{H}$-orbit on $l \cap Fix(\alpha)$:

Table I

Type	$\bar{H}_X$	$[\bar{H} : \bar{H}_X]$	$Fix_X(\bar{\tau})$	$Fix_X(\bar{E}_1)$	$Fix_X(\bar{E}_2)$
1	$\bar{H}$	1	1	1	1
2	$E_q.Z_{\frac{q-1}{\theta}}$, $\frac{q-1}{\theta}$ even	$\theta(q+1)/2$	θ	0	0
3	$E_q.Z_{\frac{q-1}{\theta}}$, $\frac{q-1}{\theta}$ odd	$\theta(q+1)/2$	0	0	0
4	$PGL(2,\sqrt{q})$	$\frac{\sqrt{q}(q+1)}{2}$	$\sqrt{q}$	1	3

By the Table I, we have that $Fix(\bar{\tau})$ is a Baer subplane of $Fix(\alpha)$. Recall that h is the number of points fixed by $\bar{H}$ on $l \cap Fix(\alpha)$. Assume that $h \geq 2$. Then $Fix(\bar{E}_1)$ and $Fix(\bar{E}_2)$ are Baer subplanes of $Fix(\bar{\tau})$. Hence the order of $Fix(\bar{E}_1)$ and $Fix(\bar{E}_2)$ is $\sqrt[8]{n}$. By Table I, we have that $\sqrt[8]{n}+1 = h+1$ for $\bar{E}_1$ and $\sqrt[8]{n}+1 = h+3$ for $\bar{E}_2$ at the same time. A contradiction. Hence $h \leq 1$. However, $Fix(\bar{E}_2)$ is still a Baer subplane of $Fix(\bar{\tau})$ as $\sqrt[8]{n}+1 = h+3$. Thus $(h, \sqrt[8]{n}) = (1,3)$ or $(0,2)$, since $h \leq 1$. Note that each non trivial $\bar{H}$-orbit on $l \cap Fix(\alpha)$ has length a multiple of $\frac{(q+1)}{2}$ by Table I. Hence, $\frac{(q+1)}{2} \mid \sqrt{n}+1-h$. Now by substituting in the previous relation the values $(h, \sqrt[8]{n}) = (1,3)$ or $(0,2)$, we have that either $\frac{q+1}{2} \mid 81$ or $\frac{q+1}{2} \mid 17$, respectively. A contradiction in any case, since q is a prime power with even exponent. ■

Proposition 11. *Let* Π *be a finite projective plane of order* n *and let* $G \cong PSL(3,q)$ *be a collineation group of* Π *with a point-orbit* $\Pi_0 \cong PG(2,q)$. *If* $n \leq q^3$, *then each involution in* G *is a perspectivity of* Π.

Proof. Assume that G contains a Baer collineation of Π. Then each involution is a Baer collineation of Π, since G contains a unique conjugate class of involutions by [6]. Then $q \in \{5,7,9,11,19\}$ by Lemma 9 and Lemma 10. Let α, $\bar{H} \cong PSL(2,q)$ and $\mathcal{C}$ be defined as in Lemma 10.

Assume that $q = 5$. Then $6 \leq \sqrt{n} \leq 11$, since $n \leq 5^3$. If $\bar{H} \cong PSL(2,5)$ contains involutions which are Baer collineations of $Fix(\alpha)$, then $\sqrt{n} = 9$. Nevertheless this case cannot occur by [2],Theorem 1, since $l \cap \Pi_0$ is a 2-transitive $\bar{H}$-orbit of length 6 on $l \cap Fix(\alpha)$. Thus each involution in $\bar{H}$ is a perspectivity of $Fix(\alpha)$. As a consequence, $\bar{H}$ does not fix any point of $\mathcal{C}$, since any involution fixes two points on $l \cap \Pi_0$. Hence $\mathcal{C}$ is union of non-trivial $\bar{H}$-orbits. Thus $\sqrt{n} \geq 10$, since the minimal permutation representation of $\bar{H} \cong PSL(2,5)$ is 5. Actually $\sqrt{n} = 10$ cannot occur by [12], Theorem 3.6. Hence $\sqrt{n} = 11$ and $PSL(2,5)$ acts in its natural 2-transitive permutation representation of degree 6 on $\mathcal{C}$. Then $\bar{\gamma}$ must be a Baer collineation of $Fix(\alpha)$, since $\bar{\gamma}$ fixes exactly 4 on $l \cap Fix(\alpha)$. A contradiction.

Assume that $q = 7$. We may also assume that $\bar{H}_Z \cong S_4$ for some point Z in $\mathcal{C}$, otherwise we obtain a contradiction be the same argument of parts **(A)** and **(C)** of Lemma 10. Hence $\sqrt{n} \geq 14$, since $\Pi_0 \cup Z^{\bar{H}} \subseteq l \cap Fix(\alpha)$ and $\left|Z^{\bar{H}}\right| = 7$. Furthermore, it is easily seen that each involution in $\bar{H}$ fixes exactly 3 points on $Y^{\bar{H}}$. Therefore each involution in $\bar{H}$ is a Baer collineation of $Fix(\alpha)$ and hence $\sqrt[4]{n} \geq 2$. On the other hand $\sqrt{n} \leq 18$, as $n \leq 7^3$. All these informations yield $n = 2^8$. So Π has exactly 65793 points. Assume that Π consists of non trivial G-orbits of points. Since each G-orbit is multiple of the index of some maximal subgroup of $G \cong PSL(3,7)$ and since the indices of the maximal subgroups of G are 57, 5586, 26068, 32928 by [4], then there must be a partition of the number 65793 restricted to the numbers 57, 5586, 26068, 32928. A contradiction. Hence G fixes a point $P \in \Pi - \Pi_0$. Since 57 is the unique primitive permutation representation of G which is less than $n + 1 = 2^8 + 1$, and since $n + 1 \equiv 29 \mod 57$, we have that G fixes a subplane of Π of order at least 28. So, $n \geq 28^2$ by [12], Theorem 3.7. A contradiction, since $n \leq 7^3$.

Assume that $q = 9$. Then $10 < \sqrt{n} \leq 27$, since $n \leq 9^3$. Assume that $\bar{H}_Y \cong S_4$ for some point Y in $\mathcal{C}$. Then $\left|Y^{\bar{H}}\right| = 15$. Hence $\sqrt{n} \geq 24$, since $(l \cap \Pi_0) \cup Y^{\bar{H}} \subseteq l \cap Fix(\alpha)$. Clearly each involution in $\bar{H}$ fixes exactly 2 points on $l \cap \Pi_0$ and 3 points on $Y^{\bar{H}}$. So, each involution in $\bar{H}$ induces a Baer collineation on $Fix(\alpha)$ and $\sqrt[4]{n} \geq 4$. Thus $\sqrt{n} = 25$, since $24 < \sqrt{n} \leq 27$. Then $l \cap Fix(\alpha)$ consists of the following $\bar{H}$-orbits: $l \cap \Pi_0$ of length 10, $Y^{\bar{H}}$ of length 15 and a point R fixed by $\bar{H}$. Pick $\bar{\rho} \in \bar{H}$ such that $o(\bar{\rho}) = 4$. Elementary calculations show that $\bar{\rho}$ fixes 2 points on $l \cap \Pi_0$, 1 points on $Y^{\bar{H}}$ and R. Hence $\bar{\rho}$ fixes exactly 4 points on $l \cap Fix(\alpha)$. Furthermore, $\bar{\rho}^2$ fixes 2 points on $l \cap \Pi_0$, 3 points on $Y^{\bar{H}}$ and R. Thus $\bar{\rho}^2$ is a Baer collineation of $Fix(\alpha)$. Moreover, $\bar{\rho}$ induces a Baer collineation on $Fix(\bar{\rho}^2)$, since $\bar{\rho}$ fixes exactly 4 points on $l \cap Fix(\bar{\rho}^2)$ and $Fix(\bar{\rho}) \subsetneq Fix(\bar{\rho}^2)$. A contradiction, since $o(Fix(\bar{\rho}^2)) = 5$. Hence, we may assume that $\bar{H}_Y \cong A_5$ for some point Y in $\mathcal{C}$. Thus $\left|Y^{\bar{H}}\right| = 6$. Hence $\sqrt{n} \geq 15$, since $(l \cap \Pi_0) \cup Y^{\bar{H}} \subseteq l \cap Fix(\alpha)$. Clearly, each involution in $\bar{H}$ fixes exactly 2 points on $l \cap \Pi_0$ and at least 2 points on $Y^{\bar{H}}$. So, each involution in $\bar{H}$ induces a Baer collineation on $Fix(\alpha)$ and $\sqrt[4]{n} \geq 3$. Therefore, either $\sqrt{n} = 16$ or $\sqrt{n} = 25$, since $10 < \sqrt{n} \leq 27$. Assume that $n = 16$. Set $\{F\} = \mathcal{C} - Y^{\bar{H}}$. Let $\bar{S}$ be a Sylow 2-subgroup of $\bar{H}$. Then $S = \left\langle \bar{\varphi}, \bar{\beta} \right\rangle$ with $\bar{\varphi}^4 = 1$, $\bar{\beta}^2 = 1$ and $\bar{\varphi}^{\bar{\beta}} = \bar{\varphi}^{-1}$. Note that $|Fix(\bar{\varphi}) \cap l| = 3$, $|Fix(\bar{\varphi}^2) \cap l| = 5$ and $\left|Fix(\bar{\beta}) \cap l\right| = 5$, since $l = (l \cap \Pi_0) \cup Y^{\bar{H}} \cup \{F\}$, and since $\bar{H} \cong PSL(2,9)$ acts in its 2-transitive permutation representations of degree 10 and 6 on $l \cap \Pi_0$ and on $Y^{\bar{H}}$, respectively. Furthermore, $\left|Fix(\bar{\varphi}^2) \cap Fix(\bar{\beta}) \cap l\right| = 3$. This yields $Fix(\bar{\varphi}^2) \cong Fix(\beta) \cong PG(2,4)$ and $Fix(\bar{\varphi}) \cong PG(2,2)$ with $Fix(\bar{\varphi}) \subset Fix(\bar{\varphi}^2)$. Moreover, $Fix(\bar{\varphi}^2) \cap Fix(\bar{\beta}) \cong PG(2,2)$ and $Fix(\bar{\varphi}) \cap Fix(\bar{\beta})$ consists of 3 collinear points of $Fix(\bar{\varphi}^2)$ including F. Thus $\left|Fix(\bar{\varphi}^2) - (Fix(\bar{\varphi}) \cup Fix(\bar{\beta}) \cup l)\right| = 10$. Let $\bar{U} \leq \bar{H}$ such that $\bar{U} \cong E_9$. Is is easily seen that $Fix(\bar{U})$ fixes exactly 2 points on l, since the $\bar{\gamma} = (123)(456)$ lies in $\bar{U}$ and $\bar{\gamma}$ is f.p.f. on $Y^{\bar{H}}$. Thus $Fix(\bar{U})$ cannot be a subplane of Π. Then there exists a line r of Π such that $Fix(\bar{U}) - l \subset r$. In particular $Fix(\bar{H}) \subset Fix(\bar{U})$ and $\left|Fix(\bar{U}) \cap Fix(\bar{\varphi}) - l\right| \leq 3$. Hence, there are at least 2 points of $\Pi - l$ lying in $Fix(\alpha) - l$, say X_1 and X_2, such that $\bar{H}_{X_1} \cong Z_2$ and $\bar{H}_{X_2} \cong Z_4$, since $Fix(\bar{H}) \subset Fix(\bar{U})$, since $Fix(\bar{U}) - l \subset r$, since $Fix(\bar{\varphi}) \cap Fix(\bar{\beta})$ consists of 3 collinear points of $Fix(\bar{\varphi}^2)$ including F, and since the are no proper subgroups of $\bar{H}$ of order divisible by 20. Then $|\Pi - l| \geq 270$, since $X_1^{\bar{H}} \cup X_2^{\bar{H}} \subset \Pi - l$ with $\left|X_1^G\right| = 180$ and $\left|X_2^{\bar{H}}\right| = 90$. A contradiction, since $\sqrt{n} = 16$. Hence $\sqrt{n} = 25$. It is easily seen that any involution $\bar{\zeta}$ in $\bar{H}$ fixes 2 points on $l \cap \Pi_0$ and 2 points on $Y^{\bar{H}}$. Thus $\bar{\zeta}$ is a Baer collineation of Π and $o(Fix(\bar{\zeta})) = 5$. So $\bar{\zeta}$ must fix exactly 2 points on $\mathcal{C} - Y^{\bar{H}}$ and $\left|\mathcal{C} - Y^{\bar{H}}\right| = 10$. This forces $\bar{H} \cong PSL(2,9)$ to act in its 2-transitive permutation representation of degree 10 on $\mathcal{C} - Y^{\bar{H}}$. Let $\bar{\rho}$ and $\bar{\rho}^2$ be defined as above. Clearly $\bar{\rho}$ and $\bar{\rho}^2$ fixes the same point on $l \cap Fix(\alpha) - Y^{\bar{H}}$, since this set consists of two 2-transitive $\bar{H}$-orbits both of length 10. Nevertheless, $\bar{\rho}$ is f.p.f. on $Y^{\bar{H}}$ while $\bar{\rho}^2$ fixes 2 points on $Y^{\bar{H}}$. So, we may apply the above argument to rule out this case. As in Lemma 10, part **(C)**, h denotes the number of points fixed by $\bar{H}$ in $\mathcal{C}$ and hence on $l \cap Fix(\alpha)$. Now, we assume that $\bar{H}$ fixes at least a point on $l \cap Fix(\alpha)$. Thus $h > 0$. At this point may use the similar argument to that of parts **(A)** and **(C)** of Lemma 10 and we may rule out this case, since there are not $\bar{H}$-orbits on $\mathcal{C}$ with that point-stabilizer isomorphic either to S_4 or to A_5. Hence $h = 0$. Then $\bar{H}_M \cong E_9.Z_4$ for any $M \in \mathcal{C}$. Then each $\bar{H}$-orbit on $\mathcal{C}$ has length 10 and hence $\sqrt{n} + 1 = 10t$, since $|l \cap \Pi_0| = 10$. Then $\sqrt{n} = 19$, since $n \leq 9^3$. Hence $\bar{H}$ acts in its 2-transitive permutation representation of degree 10 on $\mathcal{C}$. Then any involution is a Baer collineation of $Fix(\alpha)$, since it fixes exactly 2 points on $l \cap \Pi_0$ and 2 points on $\mathcal{C}$. A contradiction, since $\sqrt{n} = 19$.

Assume that $q = 11$. We may also assume that $\bar{H}_P \cong A_5$ for some point P in $\mathcal{C}$, otherwise we obtain a contradiction be the same argument of parts **(A)** and **(C)** of Lemma 10. Thus $\left|P^{\bar{H}}\right| = 11$. Hence $\sqrt{n} \geq 22$, since $(l \cap \Pi_0) \cup P^{\bar{H}} \subseteq l \cap Fix(\alpha)$. Furthermore, it is easily seen that each involution in $\bar{H}$ fixes exactly 3 points on $P^{\bar{H}}$. Therefore each involution in $\bar{H}$ is a Baer collineation of $Fix(\alpha)$ and hence $\sqrt[4]{n} \geq 2$. On the other hand, we have that $\sqrt{n} \leq 36$ as $n \leq 11^3$. All these informations yield either $n = 5^4$ or $n = 6^4$. Assume that the former occurs. Hence $\bar{H}$ fixes $\mathcal{C} - P^{\bar{H}}$ pointwise, since $\left|\mathcal{C} - P^{\bar{H}}\right| = 3$. Let $\bar{C} \leq \bar{H}$ such that $\bar{C} \cong Z_{11}$. Then $\bar{C}$ fixes exactly 4 points on $l \cap Fix(\alpha)$, since $\bar{C}$ fixes $\mathcal{C} - P^{\bar{H}}$ pointwise, $|l \cap \Pi_0| = 12$ and $\left|P^{\bar{H}}\right| = 11$. In particular $\bar{C}$ fixes a subplane of Π of order 3. Now, let $\bar{D} \leq N_{\bar{H}}(\bar{C})$ such that $\bar{D} \cong Z_5$. Clearly $\bar{D}$ fixes $l \cap Fix(\bar{C})$ pointwise. Nevertheless $\bar{D}$ cannot be a homology group, since $o(Fix(\bar{C})) = 3$. Thus $\bar{D}$ acts trivially on $Fix(\bar{C})$. Actually $Fix(\bar{C}) \subsetneq Fix(\bar{D})$, since $\bar{C}$ and $\bar{D}$ fix exactly 1 and 2 points on $l \cap \Pi_0$, respectively. So $Fix(\bar{C}) \subsetneq Fix(\bar{D}) \subsetneq Fix(\alpha)$. A contradiction by [12], Theorem 3.7, since $o(Fix(\bar{C})) = 3$ and $o(Fix(\alpha)) = 5^2$. Hence $n = 6^4$. Then any involution of $\bar{H}$ fixes a subplane of order 6. A contradiction by [12], Theorem 3.6.

Assume that $q = 19$. We may also assume that $\bar{H}_Q \cong A_5$ for some point Q in $\mathcal{C}$, otherwise we obtain a contradiction be the same argument of parts **(A)** and **(C)** of Lemma 10. Thus $\left|Q^{\bar{H}}\right| = 11$. Hence $\sqrt{n} \geq 30$, since $(l \cap \Pi_0) \cup Q^{\bar{H}} \subseteq l \cap Fix(\alpha)$. Furthermore, it is easily seen that each involution in $\bar{H}$ fixes exactly 3 points on $Q^{\bar{H}}$. Therefore each involution in $\bar{H}$ is a Baer collineation of $Fix(\alpha)$ and hence $\sqrt[4]{n} \geq 2$. Actually, $\sqrt[4]{n} \geq 3$ since $\sqrt{n} \geq 30$. On the other hand by $\sqrt{n} \leq 82$ as

$n \leq 19^3$. All these informations yield $n = 3^8$. Let $\bar{K} \leq \bar{H}$ such that $\bar{K} \cong Z_{19}$. Then $\bar{K}$ fixes at least 6 points on $l \cap Fix(\alpha)$, since $\sqrt{n} + 1 \equiv 6 \mod 19$. Then $\bar{K}$ fixes a subplane of $Fix(\alpha)$ of order at least 5, since $n \equiv 6 \mod 19$. Actually, $o(Fix(\bar{K})) = 5$ by [12], Theorem 3.7. Now, let $\bar{L} \leq N_{\bar{H}}(\bar{K})$ such that $\bar{L} \cong Z_9$. Clearly $\bar{L}$ fixes $l \cap Fix(\bar{K})$. In particular there exists a subgroup $\bar{L}_0$ of $\bar{L}$, with $\left[\bar{L} : \bar{L}_0\right] \leq 3$, which fixes $l \cap Fix(\bar{K})$ pointwise, since $\bar{L} \nleq P\Gamma L(2,5)$. Nevertheless $\bar{L}_0$ cannot be a homology group, since $o(Fix(\bar{K})) = 5$. Thus $\bar{L}_0$ acts trivially on $Fix(\bar{K})$. Actually $Fix(\bar{K}) \subsetneq Fix(\bar{L}_0)$, since $\bar{K}$ and $\bar{L}$ fix exactly 1 and 2 points on $l \cap \Pi_0$, respectively. So $Fix(\bar{K}) \subsetneq Fix(\bar{L}_0) \subsetneq Fix(\alpha)$. A contradiction by [12], Theorem 3.7, since $o(Fix(\bar{K})) = 5$ and $o(Fix(\alpha)) = 3^4$. ∎

Theorem 12. *Let Π be a finite projective plane of order n and let $G \cong PSL(3,q)$ be a collineation group of Π with a point-orbit $\Pi_0 \cong PG(2,q)$. If $n \leq q^3$, then one of the following occurs:*

1. *$n = q$ and $\Pi = \Pi_0$;*
2. *$n = q^2$, Π is a Desarguesian plane or a Generalized Hughes plane and Π_0 is a Baer subplane of Π;*
3. *$n = q^3$.*

Proof. If $n \leq v$, $v = q^2 + q + 1$, the assertions (1) and (2) follow by [1], Theorem 3.9. Hence, assume that $v < n$. Let $\mathcal{A}$ be the set of points of $\Pi - \Pi_0$, which do not lie on any secant to Π_0. Then $|\mathcal{A}| = (n - p^m)(n - p^{2m})$. Furthermore, G leaves $\mathcal{A}$ invariant. Clearly each involution in G is a perspectivity of Π by Proposition 11, and its center lies in Π_0 and its axis is a secant of Π_0. Hence $|G_X|$ must be odd for each $X \in \mathcal{A}$. If $|G_X| \leq \frac{p^{2m}-1}{j}$, where $j = (3, p^m - 1)$, then $\left|X^G\right| \geq p^{3m}(p^{3m} - 1)$ and hence $p^{3m}(p^{3m} - 1) \leq p^{3m}(p^m - 1)^2(p^m + 1)$, since $X^G \subseteq \mathcal{A}$ and $n \leq p^{3m}$. A contradiction. Hence $|G_X| > \frac{(p^{2m}-1)}{j}$.

Assume that $p = 2$. Then G_X can be recovered by Lemma 6. If $|G_X|$ is a proper divisor of $\frac{3(2^m-1)^2}{j}$, j defined as above, then $\left|X^G\right| \geq 2^{3m}(2^{2m}+2^m+1)(2^m+1)$, and we have again a contradiction. Then $|G_X| \geq \frac{3(2^m-1)^2}{j}$. If $|G_X| = \frac{3(2^m-1)^2}{j}$, then G_X has a normal subgroup R of index 3 such that $Fix(R) \cap \Pi_0$ is a triangle Δ (see Lemma 6). Hence R is planar, since R fixes the quadrangle $\Delta \cup \{X\}$ as $X \in \mathcal{A}$. Again by Lemma 6 there exists an involution β in G normalizing G_X and R. Clearly β fixes a vertex of Δ and its opposite side. Thus $C_\beta \in \Pi_0 - \Delta$. Nevertheless $C_\beta \in Fix(R) - \Pi_0$ since β normalizes R, $Fix(R)$ is a subplane and $Fix(R) \cap \Pi_0 = \Delta$. A contradiction. As a consequence $|G_X| > \frac{3(2^m-1)^2}{j}$. Then $|G_X| = 3\frac{2^{2m}+2^m+1}{j\theta}$ by Lemma 6, where θ is a divisor of $\frac{2^{2m}+2^m+1}{j}$, since G_X has odd order. Therefore

$$|\mathcal{A}| = \lambda_1 \frac{\theta 2^{3m}(2^m - 1)^2(2^m + 1)}{3},$$

with $\lambda_1 \in \{1,2,3\}$ and $\theta \in \{1,3\}$, since $n \leq 2^{3m}$. If $\theta = 1$, then $G_X = N_G(Z_{\frac{2^{2m}+2^m+1}{j}})$. If $\lambda_1 = 3$ then $n = q^3$, but this case cannot occur by Lemma 5. Thus $\lambda_1 \leq 2$ and

$n = 6$ by Lemma 2. This case cannot occur by [12], Theorem 3.6. Hence $\theta = 3$. Then $\lambda_1 = 1$ and $n = 2^{3m}$, since $n \leq 2^{3m}$. Then $G_X = Z_{\frac{2^{2m}+2^m+1}{j}}$ by Lemma 5. Thus the assertion (3).

Assume that p is odd. Note that G_X is none of the groups listed in Lemma 7, since G_X has odd order. Then the possibilities for G_X are listed in Lemma 8. Assume that $p \mid |G_X|$. Then there exists a normal elementary abelian p-subgroup U_2 of G_X such that G_X/U_2 is isomorphic to a subgroup of $PGL(2, p^m)$ again by Lemma 8. Actually, G_X/U_2 is isomorphic to a subgroup of $PSL(2, p^m)$, since G_X has odd order. If $U_2 \neq \langle 1 \rangle$, then U_2 is a group of elations with the same axis r in Π_0, for some secant r of Π_0 by [18]. Actually, $U_2 = U_2(r, r)$ in Π by [12], Theorem 4.25, since all involutions in G are homologies of Π by Proposition 11 and since G contains involutory homologies of Π with axis r. But U_2 fixes X with $X \in \mathcal{A}$. A contradiction. Therefore $U_2 = \langle 1 \rangle$. Then G_X is isomorphic to a subgroup of $PSL(2, p^m)$. Then G_X is isomorphic to a Frobenius subgroup of $E_{p^m}.Z_{\frac{p^m-1}{2}}$ by [13], Haupsatz II.8.27, since $p \mid |G_X|$ and $|G_X|$ is odd. Then $G_X \cong E_{p^h}.L$ with $1 \leq h \leq m$ and $|L| > 1$, since $|G_X| > \frac{(p^{2m}-1)}{j}$. Hence $|L| \mid p^m - 1$. Moreover, $|L| \mid p^h - 1$, since G_X is isomorphic to a Frobenius group. Hence $|L| \mid p^e - 1$ where $e = (m, h)$. Then $L \cong Z_{\frac{p^e-1}{t}}$ with $\frac{p^e-1}{t}$ odd. Therefore $G_X \cong E_{p^h}.Z_{\frac{p^e-1}{t}}$. Then $p^h \frac{p^e-1}{t} \geq \frac{(p^{2m}-1)}{j}$. This yields $h = m$ and $G_X \cong E_{p^m}.Z_{\frac{p^m-1}{t}}$. At this points, since $X^G \subseteq \mathcal{A}$ and $|\mathcal{A}| \leq p^{3m}(p^m - 1)^2(p^m + 1)$, we obtain $q \equiv 3 \mod 4$ and $t = 2$, and $q \equiv 1 \mod 3$ and $j = 3$. That is $q \equiv 7 \mod 12$ and $\left|X^G\right| = \frac{2}{3}p^{2m}(p^{3m} - 1)(p^m + 1)$.

Assume that $p \nmid |G_X|$. By Lemma 8, and relations (5.8) in the proof of Lemma 5.6 of [3], we have that either $G_X \cong E_9$ with $q \equiv 1 \mod 3$ and $q \not\equiv 1 \mod 9$, or $G_X \cong E_9.Z_3$ with $q \equiv 1 \mod 9$, or $G_X \leq N_G(Z_{\frac{p^{2m}+p^m+1}{j}})$, since $|G_X|$ is odd. Actually, $G_X \leq N_G(Z_{\frac{p^{2m}+p^m+1}{j}})$, since $X^G \subseteq \mathcal{A}$ and $|\mathcal{A}| \leq p^{3m}(p^m - 1)^2(p^m + 1)$. Then $\left|X^G\right| = \frac{1}{k}p^{3m}(p^m - 1)^2(p^m + 1)$ with $k = 1$ for $G_X \cong Z_{\frac{p^{2m}+p^m+1}{j}}$, and $k = 3$ for $G_X = N_G(Z_{\frac{p^{2m}+p^m+1}{j}})$.

Set μ_1 and μ_2 the number G-orbits in $\mathcal{A}$ of length $A = \frac{2}{3}p^{2m}(p^{3m} - 1)(p^m + 1)$ and $B = \frac{1}{k}p^{3m}(p^m - 1)^2(p^m + 1)$, respectively. Then $\mu_1 A + \mu_2 B = (n - p^m)(n - p^{2m})$, since $|\mathcal{A}| = (n - p^m)(n - p^{2m})$. Then $\mu_2 \neq 0$ by Lemma 1, since $n \leq q^3$. Thus $\mu_1 = 0$, since $|\mathcal{A}| \leq p^{3m}(p^m - 1)^2(p^m + 1)$. Then $k = 1$, $G_X \cong Z_{\frac{p^{2m}+p^m+1}{j}}$ for any point X in $\mathcal{A}$ by Lemma 2, since $n \leq q^3$ and $n \neq 105$. Then $n = q^3$ according with Lemma 5, since $n \leq q^3$. Thus the assertion (3). ■

5 The unfaithful action

Let N be the kernel of G on Π_0 and set $\bar{G} = G/N$. Throughout this section we assume that $N \neq \langle 1 \rangle$. We may also assume that G is the minimal preimage of $\bar{G} \cong PSL(3, q)$.

Firstly, we prove that N is the Frattini subgroup G in Lemma 13. Hence N is nilpotent. This yields $N = Z(G)$ in Theorem 14 using group-theoretical results. Again, an extensive use of the list of the subgroups of $PSL(3, q)$ leads us to assert that Π is the Generalized Hughes plane over the exceptional nearfield of order 7^2, Π_0 is a Baer subplane of Π and G contains $SL(3,7)$.

Lemma 13. *$N = \Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G.*

Proof. Let S be any Sylow t-subgroup of N. Then $G = N_G(S)N$ by the Frattini's argument. Thus $S \lhd G$ by the minimality of G. Therefore N is nilpotent. Suppose that $N \nleq \Phi(G)$. Then there exists a maximal subgroup M of G such that $G = NM$ by [13], Satz 3.2 (b). Clearly $M < G$ and $\frac{M}{M\cap N} \cong \bar{G}$. A contradiction by the minimality of G. Hence, we may assume that $N \leq \Phi(G)$. Note that G_P is maximal in G for each point $P \in \Pi_0$, since $N \lhd G_P$ and $\bar{G}$ is primitive on Π_0. Hence $\Phi(G) \lhd G_P$ for each point $P \in \Pi_0$. Therefore $N = \Phi(G)$. ∎

Theorem 14. *Let Π be a finite projective plane of order n and let G be a collineation group of Π with a point-orbit $\Pi_0 \cong PG(2,q)$ on which G induces $\bar{G} \cong PSL(3,q)$. If $n \leq q^3$ and $Fix(N) = \Pi_0$, then Π is the Generalized Hughes plane over the exceptional nearfield of order 7^2, Π_0 is a Baer subplane of Π and G contains $SL(3,7)$.*

Proof. The assertion follows by [1] for $n \leq v$, $v = q^2+q+1$. Hence, assume that $n > v$. Let l be a secant to Π_0. Then N acts on $l - \Pi_0$. If $N_X \neq \langle 1 \rangle$ for some $X \in l - \Pi_0$, then $Fix(N) \subsetneq Fix(N_X) \subsetneq \Pi$. A contradiction by [12], Theorem 3.7, since $o(Fix(N)) = q$ and $n \leq q^3$. Thus N is semiregular on $l - (l \cap \Pi_0)$ and hence $|N| \mid n - q$. Furthermore, N must have odd order, since $Fix(N) \cong PG(2,q)$ and $q^2 < n$.

(I) $G \cong SL(3,q)$, $q \equiv 1 \mod 3$. **Furthermore, each involution in G is perspectivity of Π having the center in Π_0 and the axis a secant of Π_0.**

Assume that $N \nleq Z(G)$. Then there exists a Sylow t-subgroup S of N such that $S \nleq Z(G)$, since N is nilpotent. Set $V = S/\Phi(S)$, where $\Phi(S)$ is the Frattini subgroup of S. Clearly G acts on V. Let R be the kernel of the action of G on V. If U is the Sylow u-subgroup of N, where u is a prime, $u \neq t$, then $[S,U] = \langle 1 \rangle$, since N is nilpotent. This yields $N \trianglelefteq R \trianglelefteq G$, since $S' \leq \Phi(S)$, being S a t-group. If $R = G$, then each Sylow r-subgroup of G, with $r \neq t$, centralizes S by [8], Theorem 5.1.4. That is $C_G(S) \nleq N$. Furthermore, $C_G(S) \lhd G$ as $S \lhd G$. Then $N \lhd C_G(S)N \trianglelefteq G$. Hence $G = C_G(S)N$, since $\bar{G}$ is non abelian simple and since $C_G(S) \nleq N$. Actually, $G = C_G(S)$ since $N = \Phi(G)$ by Lemma 13. A contradiction, since $S \nleq Z(G)$. Hence $R < G$. Then $R = N$ as $\bar{G}$ is non abelian simple. Then $\bar{G} \leq \Gamma L(V)$, since V is a vector space over $GF(t)$. Actually $\bar{G} \leq SL(V)$, since $\bar{G}$ is non abelian simple. Then $\bar{G} \leq PSL(V)$, where $V = S/\Phi(S)$ is a vector spaces over $GF(t)$. If $t \neq p$, then $|V| \geq t^{q^2-1}$ by [16], Theorem 5.3.9, for $q \notin \{2,4\}$. In particular, for $t \neq p$ we have that $|V| \geq 2^{q^2-1}$ for any q. If $t = p$, then $|V| \geq q^3$ by [16], Proposition 5.4.13. Hence $|V| \geq q^3$ in any case. Thus $|N| \geq q^3$. A contradiction, since $|N| \mid n - q$ and $n \leq q^3$. Hence, we may assume that $N \leq Z(G)$. Then $G \cong SL(3,q)$ by [15], Theorem 7.7.1, since N has odd order. Furthermore, $N \cong Z_3$ and $q \equiv 1 \mod 3$, since $N \neq \langle 1 \rangle$. Then each involution in G is perspectivity of Π having the center in Π_0 and the axis a secant of Π_0, since the proof of Proposition 11 still works being $N \cong Z_3$.

(II) The final contradiction.

Denote by $\mathcal{A}$ the set of points of $\Pi - \Pi_0$ not lying on any secant of Π_0. Then $|\mathcal{A}| = (n-p^m)(n-p^{2m})$. Furthermore, G leaves $\mathcal{A}$ invariant. Note that each N-orbit on $\mathcal{A}$ is a triangle, since $N \cong Z_3$, N is semiregular on $\mathcal{A}$, and $Fix(N) = \Pi_0$.

Denote by $\mathcal{A}_N$ be the set of N-orbits on $\mathcal{A}$. Then $|\mathcal{A}_N| = \frac{|\mathcal{A}|}{3}$. Pick $\Delta \in \mathcal{A}_N$. Then $G_\Delta = G(\Delta) \times N$, where $G(\Delta)$ is the pointwise-stabilizer of Δ. Furthermore, G_Δ has odd order by **(I)**. Set $\bar{G}_\Delta = G_\Delta/N$. Then $\bar{G}_\Delta \leq \bar{G} \cong PSL(3,q)$ and $G(\Delta) \cong \bar{G}_\Delta$. Hence, the proof of Theorem 12 still works with $\bar{G}_\Delta$ in role of G_X (where $X \in \mathcal{A}$) and $\mathcal{A}_N$ in role of $\mathcal{A}$, since $|\mathcal{A}_N| = \frac{|\mathcal{A}|}{3}$. Thus $n = q^3$ and $\bar{G}_\Delta \leq Z_{\frac{p^{2m}+p^m+1}{3}}.Z_3$. If $\bar{G}_\Delta \cong Z_{\frac{p^{2m}+p^m+1}{3}}.Z_3$, then $G_\Delta = Z_{\frac{p^{2m}+p^m+1}{3}}.Z_3 \times N$. Note that the group Z_3 in the normalizer of $Z_{\frac{p^{2m}+p^m+1}{3}}$ consists of generalized homologies of Π_0 having the centres in Π_0 and the axes which are secants of Π_0 by [5], Proposition 3.4 (i). Actually, the group Z_3 in the normalizer of $Z_{\frac{p^{2m}+p^m+1}{3}}$ consists of generalized homologies of Π having the centres in Π_0 and the axes which are secants of Π_0 by using the proof of Proposition 3.4 (i) of [5], since the involutions in G are perspectivities of Π by **(I)**. Hence $\bar{G}_\Delta \lneq Z_{\frac{p^{2m}+p^m+1}{3}}.Z_3$. Thus $\left|\Delta^{\bar{G}}\right| = y\frac{p^{3m}(p^m-1)^2(p^m+1)}{3}$, with y odd, $y > 1$. So,

$$y\frac{p^{3m}(p^m-1)^2(p^m+1)}{3} \leq \frac{p^{3m}(p^m-1)^2(p^m+1)}{3},$$

since $\Delta^{\bar{G}} \subseteq \mathcal{A}_N$ and $|\mathcal{A}_N| \leq \frac{p^{3m}(p^m-1)^2(p^m+1)}{3}$. A contradiction, since y is odd and $y > 1$. ■

We conclude this paper with the following remark:

Remark. The problem of classifying the projective planes Π of order q^3 with a collineation group isomorphic to $PSL(3,q)$ is still open today. So, it might be interesting and useful to know what would happen if the projective plane Π has order q^3 and the collineation group of Π turns out to be $PGL(3,q)$. In the previous case, is it possible to determine the plane Π?

References

[1] M. Biliotti, E. Francot, Two-transitive orbits in finite projective planes, J. Geom. **82** (2005), 1-24.

[2] M. Biliotti, A. Montinaro, Finite projective planes of order n with a 2-transitive orbit of length $n-3$, Adv. Geom. **6** (2005), 15-37.

[3] D. M. Bloom, The subgroups of $PSL(3,q)$ for odd q, Trans. Amer. Math. Soc. **127** (1967), 150-178.

[4] J. H. Conway, R. T. Curtis, R. A. Parker, R. A. Wilson, Atlas of Finite Groups. Maximal subgroups and ordinary characters for simple groups, Oxford University Press 1985.

[5] U. Dempwolff, $PSL(3,q)$ on projective planes of order q^3, Geom. Dedicata **18** (1985), 101-112.

[6] L. E. Dickson, Linear Groups with an Exposition of the Galois Field Theory, Dover, New York 1958.

[7] R. Figueroa, A Family of not (V,l)-Transitive projective Planes of Order q^3, $q \not\equiv 1 \mod 3$ and $q > 2$, Math. Z. **181** (1982), 471-479.

[8] D. Gorestein, Finite groups, Chelsea Publishing Company, New York 1980.

[9] R. W. Hartley, Determination of the ternary collineation groups whose coefficients lie in $GF(2^n)$, Ann. Math. **27** (1926), 140-158.

[10] C. Hering, On the structure of finite collineation groups of projective planes, Abh. Math. Sem. Univ. Hamburg **49** (1979), 155-182.

[11] C. Hering, H. J. Schaeffer, On the new projective planes of R. Figueroa. In: Combinatorial Theory, Proc. Schloss Rauischholzhausen 1982, D. Jungnickel and K. Wedder (eds), Springer, Berlin, (1982) 187-190.

[12] D. R. Hughes, F. C. Piper, Projective Planes, Springer Verlag, New York - Berlin 1973.

[13] B. Huppert, Endliche Gruppen I, Springer Verlag, New York - Berlin 1967.

[14] Z. Janko, V. Cigić, On planar collineation of order 13 acting on projective plane of order 16, Rad. Mat. **1** (1985), 163-172.

[15] G. Karpilovsky, The Schur Multiplier, Clarendon Press, Oxford 1987.

[16] P. B. Kleidman, M. Liebeck, The subgroup structure of the finite classical groups, Cambridge University Press, Cambridge 1990.

[17] H. Lüneburg, Characterization of the Generalized Hughes Planes, Can. J. Math. **28** (1976), 376-402.

[18] H. H. Mitchell, Determination of ordinary and modular ternary linear groups, Trans. Amer. Math. Soc. ***12*** (1911), 207-242.

[19] G. E. Moorhouse, $PSL(2,q)$ as a collineation group of projective planes of small order, Geom. Dedicata **31** (1989), 63-88.

[20] T. G. Ostrom, A. Wagner, On projective and affine planes with transitive collineation groups, Math. Z. **71** (1959), 186-199.

Dipartimento di Matematica, Università degli Studi di Lecce
Via per Arnesano
73100 Lecce
Italy
alessandro.montinaro@unile.it