Ω-Algebras over Henselian Discrete Valued Fields with Real Closed Residue Field

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Let $\mathbb{R}((t))$ be the fraction field of the complete discrete valuation ring $\mathbb{R}[[t]]$, of formal power series over $\mathbb{R}$, and let $F = \mathbb{R}((t))(x)$ be the field of rational functions in one variable over $\mathbb{R}((t))$. Let A be a central simple algebra over F of exponent 2. The quadratic extension $\mathbb{C}((t))(x)$ of F is a C_2-field (cf. [Ser, Chap. II, section 3.3]) and therefore $A \otimes_F \mathbb{C}((t))(x)$ is an algebra of index ≤ 2, cf. [Art, theorem 6.2]. It follows that the index of A over F is less than or equal to 4. A well known theorem of Albert implies that A is Brauer equivalent to a biquaternion algebra, i.e., a tensor product of two quaternion algebras. *Can one describe the algebras of exponent 2 which are exactly of index 2, i.e. which are Brauer equivalent to quaternion division algebras?* To make the question more precise we recall that the Brauer group of a rational function field $K(x)$ over any field K, which we may assume to be of characteristic not equal to 2, is described to some extent by its ramification data. We recall what is meant by this.

One interprets $K(x)$ as the function field of the projective line $\mathbb{P}^1_K$. The closed points y of $\mathbb{P}^1_K$ correspond to the K-discrete valuations of $K(x)$, either y is the point at infinity of $\mathbb{P}^1_K$ or y corresponds to a monic irreducible polynomial in $K[x]$. The Brauer group of $K(x)$ is described by an exact sequence of cohomology groups, due to Fadeev, cf. [Fad, theorem 15.2, theorem 15.3],[Ser, chap. II, App. sec. 5]. We are only interested in algebras of exponent 2 so we only consider the sequence restricted to the 2-components of the different groups;

$$0 \to {}_2\mathrm{Br}(K) \to {}_2\mathrm{Br}(K(x)) \overset{\oplus\partial_y}{\to} \bigoplus_{y \in \mathbb{P}^1_K} H^1(K(y), \mathbb{Z}/2\mathbb{Z}) \overset{\sum \mathrm{cor}}{\to} H^1(K, \mathbb{Z}/2\mathbb{Z}) \to 0. \qquad (FES)$$

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Here $K(y)$ is the residue field of the discrete valuation corresponding to the closed point y and ∂_y is the associated ramification map. The map $\sum \mathrm{cor}$ is the sum of the values of the corestriction maps $H^1(K(y), \mathbb{Z}/2\mathbb{Z}) \to H^1(K, \mathbb{Z}/2\mathbb{Z})$ induced by the inclusion of the absolute Galois groups $\mathrm{Gal}(\overline{K}/K(y)) \subset \mathrm{Gal}(\overline{K}/K)$, (cf. [Ser, Chap. II, App. sec. 3]). Note that $H^1(K(y), \mathbb{Z}/2\mathbb{Z}) \cong K(y)^*/K(y)^{*2}$ and $H^1(K, \mathbb{Z}/2\mathbb{Z}) \cong K^*/K^{*2}$, moreover these isomorphisms are canonical since -1 is the only primitive 2th-root of unity. The corestriction map corresponds to the norm map:

$$N_{K(y)/K} : K(y)^*/K(y)^{*2} \to K^*/K^{*2},$$

if the cohomology groups and the groups of square classes are identified. There is an explicit formula for the value of the ramification map. Let v be the valuation corresponding to y and let $(f, g)_{K(x)}$ be a quaternion algebra over $K(x)$, then

$$\partial_y((f, g)_{K(x)}) = (-1)^{v(f)v(g)} \overline{\left(\frac{f^{v(g)}}{g^{v(f)}}\right)} \in K(y)^*/K(y)^{*2}. \tag{RAM}$$

Since any $A \in {}_2\mathrm{Br}(K(x))$ is Brauer equivalent to a tensor product of quaternions algebras (for general K this is Merkurjev's theorem, for $K = \mathbb{R}((t))$ we saw already that the algebras of exponent 2 are Brauer equivalent to a tensor product of 2 quaternion algebras.), the formula enables to calculate the ramification by linearity. The ramification map ∂_y factors through ${}_2\mathrm{Br}(K(x)_y)$, where $K(x)_y$ is the completion of $K(x)$ with respect to the valuation v corresponding to y. This directly implies that a central simple algebra A over $K(x)$ which is trivial over $K(x)_y$ is unramified, i.e., $\partial_y(A) = 1 \bmod K(y)^{*2}$.

The fact that $\mathrm{im}(\oplus\partial_y) \subset \ker(\sum \mathrm{cor})$ is called Faddeev's reciprocity law. The exact sequence (FES) says that the Brauer class of an algebra A of exponent 2 over a rational function field $K(x)$ is "almost" given by a finite set of local data, namely its non-trivial ramification. The ramification data of A consists of a finite set of closed points $\mathrm{Ram}(A) := \{y \in \mathbb{P}^1_K | \partial_y(A) \neq 0 \in H^1(K(y), \mathbb{Z}/2\mathbb{Z})\}$, called the *ramification locus of* A, and the set $\{\partial_y(A) | y \in \mathrm{Ram}(A)\}$. Faddeev's reciprocity law implies that $\sum_{y \in \mathrm{Ram}(A)} \mathrm{cor}(\partial_y(A)) = 0$ in $H^1(K, \mathbb{Z}/2\mathbb{Z})$. The exactness of the sequence (FES) in ${}_2\mathrm{Br}(K(x))$ and in $\bigoplus_{y \in \mathbb{P}^1_K} H^1(K(y), \mathbb{Z}/2\mathbb{Z})$ implies that data consisting of a finite set of points $S = \{y \in \mathbb{P}^1_K\}$ and a set of non-trivial elements $\delta_y \in H^1(K(y), \mathbb{Z}/2\mathbb{Z}), y \in S$ satisfying Faddeev's reciprocity law are exactly the ramification data of some algebra A of exponent 2 over $K(x)$, i.e., $S = \mathrm{Ram}(A)$ and $\delta_y = \partial_y(A)$ for all $y \in S$. The Brauer class of this algebra A is defined up to a factor in the Brauer group of K. More precisely two algebras A and A' have the same ramification data if and only if $A \sim A' \otimes_{K(x)} B$ where $B \cong b \otimes_K K(x)$ with b a constant algebra (by this we mean an algebra defined over K). We can now rephrase the above question in terms of ramification data:

> *Which ramification data (of algebras of exponent 2 over $\mathbb{R}((t))(x)$) correspond to the ramification data of a quaternion algebra?*
>
> *Can one describe a quaternion algebra A over $\mathbb{R}((t))(x)$ explicitly in terms of its ramification data, i.e., can one construct explicitly a quadratic splitting field for A in terms of the ramification data of A?*

Our main results (cf. section 1) provide some partial answers to these questions. (We also note that using the correspondence between quaternion division algebras over a rational function field $K(x)$ and conic bundle surfaces over K, see [CS, section 2], [Isk1, section 3], [Isk2, lemma 6, corollary], our results provide information on conic bundle surfaces over $\mathbb{R}((t))$.)

Survey of the main results

Let Ω be the set of all orderings in a real field E. Consider the homomorphism

$$\psi : {}_2\mathrm{Br}(E) \to \prod_{\omega\in\Omega} {}_2\mathrm{Br}(E_\omega)$$

where E_ω is the real closure of E for the ordering ω.

Definition 0.1. Let E be any real field. We denote the kernel of the map

$$\psi : \mathrm{Br}(E) \to \prod_{\omega\in\Omega} \mathrm{Br}(E_\omega)$$

by $\Omega\,\mathrm{Br}(E)$ and the 2-component of this groups by ${}_2\Omega\,\mathrm{Br}(E)$.
We call central simple algebras over E whose Brauer equivalence class is in $\Omega\,\mathrm{Br}(E)$, Ω-algebras.
In case E is non-real the kernel is by definition all of $\mathrm{Br}(E)$

In [Pfis2] Pfister studied the maximal dimension of anisotropic torsion quadratic forms over fields of transcendence degree 2 over a real closed field. It is conjectured that the u-invariant (this is the maximal dimension of anisotropic torsion quadratic forms over such a field, cf., [Pfis3]) is ≤ 4. In [Pfis2] Pfister shows that this conjecture is equivalent with the following,

Conjecture (Pfister's conjecture) *Let F be a field of transcendence degree 2 over a real closed field R. If F is real then every Ω-division algebra of exponent 2 over F is a quaternion algebra. If F is non-real then every central division algebra of exponent 2 is a quaternion algebra.*

Now consider the case where F is a purely transcendental extension of degree one over the function field of a (smooth projective) curve over the real numbers $\mathbb{R}$, so $F = \mathbb{R}(C)(x)$. Let v be the discrete valuation corresponding to a closed point $y \in C$ and let $\mathbb{R}(C)_v$ be the completion of $\mathbb{R}(C)$ with respect to v. It is well known that $\mathbb{R}(C)_v$ is of the form $\mathbb{C}((t))$ or of the form $\mathbb{R}((t))$. Since $\mathbb{C}((t))(x)$ is a C_2-field the algebras of exponent 2 over $\mathbb{C}((t))(x)$ are also of index 2 (cf. [Art, theorem 6.2]). So in view of the above conjecture it is natural to ask, *is the index of Ω-algebras of exponent 2 over $\mathbb{R}((t))(x)$ equal to 2*?
After discussions we had with Karim Becher on the results of a preliminary version of this paper, he was able to prove, using quadratic form theoretic arguments, that $u(\mathbb{R}((t))(x) = 4$ and therefore also that the answer to this question is positive, (cf., [Bech, corollary 3.3, examples 3.4 (2)]). So the Ω-algebras of exponent 2 over

$\mathbb{R}((t))(x)$ are quaternion algebras. In this paper we give a description of the ramification data of these Ω-algebras. This leads to a characterization of such algebras by local data (cf., proposition 1.8) and to the isomorphism (cf., theorem 1.11)

$$_2\Omega\,\mathrm{Br}(\mathbb{R}((t))(x)) \cong \oplus_{\substack{y\in\mathbb{P}^1_{\mathbb{R}((t))}\\ y \text{ non-real}}} \mathbb{R}((t))(y)^*/\mathbb{R}((t))(y)^{*2}.$$

We further consider four different types of monic irreducible polynomials in $\mathbb{R}((t))[x]$ (i.e., finite points on $\mathbb{P}^1_{\mathbb{R}((t))}$). For Ω-algebras of exponent 2 whose ramification laws consists only of points of the first three types we determine explicitly a quadratic splitting field in the terms of the ramification data (cf., section 1, theorem 1.8). (Although our theorems can be obtained without using Becher's results we use the fact Ω-algebras of exponent 2 over $\mathbb{R}((t))(x)$ are quaternion algebras since it does simplify certain parts. (We indicate how to avoid Becher's result, cf. remark 1.9.

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Notation and terminology.

Throughout the paper we will use the following terminology. We call two central simple algebras A and B over a field K equivalent if they are Brauer equivalent over K, i.e., if they define the same element in $\mathrm{Br}(K)$, we use the notation $A \sim B$. We call a central simple algebra over K trivial if its class in $\mathrm{Br}(K)$ is trivial .

Quaternion algebras over a field K with a K-basis of the form $1, i, j, k$ satisfying $i^2 = a$, $j^2 = b$ and $ij = -ji = k$ with $a, b \in K$, will be denoted by the symbol $(a, b)_K$. If there is no confusion possible we will omit in proofs and in calculations the field in the index of this symbol.

Since we are only interested in algebras of exponent 2 we will use in the rest of the paper the term Ω-algebra for Ω-algebras of exponent ≤ 2.

1 Ω-algebras over rational function fields over Henselian discrete valued fields with real closed residue field

Our results concerning Ω-algebras over $\mathbb{R}((t))(x)$ only use the fact that $\mathbb{R}((t))$ is a Henselian discrete valued field with real closed residue field. So we will formulate and prove the results in this generality.

1.1 Preliminary facts and results

In the sequel of the paper K will be a Henselian discrete valued field with real closed residue field denote by k. We may assume that k is a subfield of K. The field K is the fraction field of a Henselian discrete valuation ring $\mathcal{O}_K$ and we fix a uniformizing element π in $\mathcal{O}_K$. The algebraic closure of k is $k(i)$, i being the square root of -1. If E is any field extension of k, we write $E(i)$ for $E \otimes_k k(i)$.

A finite extension L of K is itself a Henselian discrete valued field; we denote its valuation ring by $\mathcal{O}_L$. The residue field of L is either the real closed field k or its algebraic closure $k(i)$. Any finite extension L/K can be split in a tower $K \subset N \subset L$

where N/K is an unramified extension and L/N is a totally ramified extension of K. The extension N is either equal to K (in this case L/K is totally ramified) or $N = K(i)$. This follows from the fact that the unramified algebraic extensions of a Henselian discrete valued field are unique "lifts" of the residue field extensions of the real closed field k, of course the latter only has two extensions k and $k(i)$. The totally ramified part L/N has the form $L = N(\sqrt[n]{\pi'})$ with $\pi' = u\pi$ and u a unit in $\mathcal{O}_K$. Since k is real closed it follows that the units in $\mathcal{O}_K$ are all $\pm n$-th powers in K and this for all $n \in \mathbb{N}$, $n \geq 1$. So the totally ramified extensions of N are all of the form $N(\sqrt[n]{\pi})$ or $N(\sqrt[n]{-\pi})$. For instance the only quadratic extensions of K are $K(i), K(\sqrt{\pi}), K(\sqrt{-\pi})$, this implies that $(-1,-1)_K, (-1,\pi)_K, (-1,-\pi)_K$ represent the only non-trivial elements in ${}_2\mathrm{Br}(K)$. It is also important to note that K is a hereditarily pythagorean field, i.e., all finite real extension of K are pythagorean (i.e., a real field in which every sum of squares is a square) or equivalently all non-real field extensions of K contain $K(i)$ as a subfield (cf. [Beck, theorem III.1.1]). Hereditarily pythagorean fields are also be characterized by

Lemma 1.1. *([Beck, theorem III.1.4]) E is a hereditarily pythagorean field if and only if the rational function field in one variable over E, $E(x)$, has Pythagoras number 2.*

Corollary 1.2. *Let E be a hereditarily pythagorean field. Any polynomial $f \in E[x]$ which is a sum of squares in $E(x)$ is a sum of two squares in $E[x]$.*

Proof: This follows from lemma 1.1 together with a well known result of Cassels saying that a polynomial in one variable over a field represented by a quadratic form over the rational function field is also represented by that quadratic form over the polynomial ring, cf. [Pfis3, Chap. 1, theorem 2.2]. ■

The fact that the Pythagoras number of $K(x)$ is 2, plays a basic role in the study of division algebras over $\mathbb{R}((t))$ of exponent 2. It is also essential in Becehr's proof of theorem 1.6), (see remark 1.7). The polynomials in $K(x)$ that are equal to a sum of two squares have a nice characterization.

Lemma 1.3. (a) *Let $f \in K[x]$ be a square free polynomial which is a sum of two squares in $K(x)$. Then the monic irreducible factors of f are sums of two squares in $K(x)$.*
(b) *A monic irreducible polynomial $p \in K[x]$ is a sum of two squares in $K[x]$ if and only if $K[x]/(p)$ is a non-real field.*

Proof: (a) Let $f \in K[x]$ be a sum of squares in $K(x)$, by the above corollary 1.2 we know that f is a sum of two squares in $K[x]$. Therefore the leading coefficient of f is a sum of squares in K, this implies, since K is pythagorean, that the leading coefficient is a square in K. So we may assume without loss of generality that f is a monic square free polynomial which is a sum of two squares in $K[x]$. But then f is a norm of $K(i)[x]/K[x]$ and the multiplicativity of the norm implies that all the monic irreducible factors of f are equal to a sum of two squares in $K[x]$.
(b) Let $p \in K[x]$ be a monic irreducible polynomial in $K[x]$ which is a sum of two squares in $K[x]$, say $p = g_1^2 + g_2^2$. Then $-1 \equiv \frac{g_1^2}{g_2^2} \bmod (p)$, so -1 is a square in $K[x]/(p)$ implying that $K[x]/(p)$ is non-real.

Conversely let p is a monic irreducible polynomial in $K[x]$ such that $K[x]/(p)$ is non-real. Then since K is hereditarily pythagorean we have $i \in K[x]/(p)$. It follows that the factorisation of p over the quadratic extension $K(i)$ is of the form $p = q\overline{q}$, with q a monic irreducible polynomial in $K(i)[x]$ and $\overline{q}$, the image of q under the non-trivial automorphism of $K(i)(x)/K(x)$. So p is a norm from $K(i)[x]/K[x]$ and therefore a sum of two squares. ■

The rational function field $K(x)$ is the function field of the projective line $\mathbb{P}^1_K$ over K. The valuation defined by the degree map on $K(x)$ corresponds to a closed point of $\mathbb{P}^1_K$ which we call the point at infinity and which we denote by ∞. The "finite" closed points of $\mathbb{P}^1_K$ are parameterized by monic irreducible polynomials of $K[x]$. The order functions corresponding to these polynomials define K-discrete valuations on $K(x)$. Throughout the rest of the paper we will identify the closed points of $\mathbb{P}^1_K$, the corresponding discrete valuations and (for finite points) the monic irreducible polynomials in $K[x]$. We distinguish real and non-real closed points in $\mathbb{P}^1_K$. The real points are the closed points $y \in \mathbb{P}^1_K$ with real residue field $K(y)$. We denoted the set of real points by $\mathbb{P}^1_{K,r}$ and the non-real points by $\mathbb{P}^1_{K,nr}$. So the residue fields of the non-real points $z \in \mathbb{P}^1_{K,nr}$ contain the square root of -1, i.e., $i \in K(z)$. Lemma 1.3 tells us that the points in $\mathbb{P}^1_{K,nr}$ are exactly the points that correspond to monic irreducible polynomials that are equal to a sum of two squares in $K[x]$.
We start with some facts on central simple algebras of exponent 2 over $K(x)$ and on their ramification. In the introduction we remarked that all central simple algebras of exponent 2 over $\mathbb{R}((t))(x)$ are of index less than or equal to 4 since $\mathbb{C}((t))(x)$ is a C_2-field. The same is true for central simple algebras of exponent 2 over $K(x)$, since $K(i)(x)$ is also a C_2-field, [Ser, chap. II, section 3.3]. So we have

Lemma 1.4. *Let A be a central simple algebra of exponent 2 over $K(x)$. Then A is equivalent to a biquaternion algebra.*

In general it is not so that all central simple algebras of exponent 2 over $K(x)$ are of index 2. We refer to the examples in [KRTY, lemma 3.10], there it are examples over $F(x)$ with F a $\wp$-adic field, but it easy to see how to obtain analogue examples over $\mathbb{R}((t))(x)$.
The following lemma will be helpful to calculate the ramification of certain elements in ${}_2\mathrm{Br}(K(x))$.

Lemma 1.5. (a) *Let f be a sum of squares in $K(x)$. For all the real points $y \in \mathbb{P}^1_{K,r}$, every quaternion algebra of the form $(f, g)_{K(x)}$ is trivial over the completion $K(x)_y$. In particular this holds for the point at infinity of $\mathbb{P}^1_K$.*
(b) *Consider a quaternion algebra $(g, x)_{K(x)}$, with g a square free polynomial over K not divisible by x. Let p be a monic irreducible factor of g. Then $(g, x)_{K(x)}$ is ramified in the point $y \in \mathbb{P}^1_K$ corresponding to p if and only if p has a root θ which is not a square in $K(y)(\cong K(\theta))$.*
(c) *Consider a quaternion algebra $(g, \pi)_{K(x)}$, with g a square free polynomial over K. Let q be a monic irreducible factor of g. Then $(g, \pi)_{K(x)}$ is ramified in the point $y \in \mathbb{P}^1_K$ corresponding to q if and only if π is not a square in $K(y)$.*

Proof: (a) Let $y \in \mathbb{P}^1_{K,r}$. So the completion $K(x)_y$ is a real field. The residue field of this completion has Pythagoras number one by Hensel's lemma the same holds for $K(x)_y$. This implies that f is a square in $K(x)_y$, so $(f, g)_{K(x)_y}$ is trivial.

(b) Since $v(x) = 0$ and $v(g) = 1$ (where v is the valuation corresponding to p), we calculate the ramification using the ramification formula (RAM)

$$\partial_y((g,x)_{K(x)}) = (-1)^{v(g)v(x)}\overline{\left(\frac{g^{v(x)}}{x^{v(g)}}\right)} \equiv \overline{x} \equiv \theta \bmod K(y)^{*2}.$$

So the ramification in y is non-trivial if and only if θ is not a square in $K(y)$.
(c) Since $v(\pi) = 0$ and $v(g) = 1$ (where v is the valuation corresponding to q), the ramification formula (RAM) yields

$$\partial_y((g,\pi)_{K(x)}) = (-1)^{v(g)v(\pi)}\overline{\left(\frac{g^{v(\pi)}}{\pi^{v(g)}}\right)} \equiv \overline{\pi} \bmod K(y)^{*2}.$$

So the ramification in y is non-trivial if and only if π is not a square in $K(y)$. ∎

The algebras in point (a) of the lemma are Ω-algebras, it is namely clear that any quaternion algebra of the form $(f,g)_{K(x)}$ with f a sum of squares in $K(x)$ is trivial over all real closures $K(x)_\omega$, $\omega \in \Omega$ since f is a square in $K(x)_\omega$. Becher's result gives us the converse of this fact.

Theorem 1.6 (K. Becher)**.** *Every Ω-algebra A over $K(x)$ of exponent 2 is Brauer equivalent to a quaternion division algebra of the form (e^2+g^2, h) with $e, g, h \in K[x]$.*

Remark 1.7. We note that in [Bech] the following more general result is shown. Let K be a field with Pythagoras number ≤ 2 such that $u(K(\sqrt{-1})) = 4$ then $u(K) \leq 4$. This result implies that if $K = \mathbb{R}((t))(C)$ is the function field of a curve over $\mathbb{R}((t))$ such that $p(K) = 2$ then the u-invariant of K is 4. As we noted before this yields that Ω-algebras over K of exponent 2 are Brauer equivalent to quaternion algebras. In [TVGY] the Pythagoras number of function fields of hyperelliptic curves over $\mathbb{R}((t))$ is studied. It is shown there that if C is a curve with good reduction and if $\mathbb{R}((t))(C)$ is a real field then $p(\mathbb{R}((t))(C)) = 2$. Becher's result then implies that the u-invariant of such fields is 4.

1.2 Ω-algebras over $K(x)$

We characterize in this subsection the Ω-algebras over $K(x)$ by their local data. Theorem 1.6 together with lemma 1.5 tells us that a Ω-algebra can only be ramified in non-real points $y \in \mathbb{P}^1_{K,nr}$ or equivalently in points $y \in \mathbb{P}^1_K$ corresponding to irreducible polynomials which are equal to a sum of two squares. This leads to a full characterization of Ω-algebras over $K(x)$ in terms of local data.

Proposition 1.8. *(a) Let B be a central simple algebra over $K(x)$ which is only ramified in non-real points. Then B is Brauer equivalent to $C \otimes A$ with C a constant algebra (i.e., C is a central simple algebra over K) and A an Ω-algebra over $K(x)$.*
(b) Let A be a central simple algebra of exponent 2 over $K(x)$. Then A is an Ω-algebra if and only if the following two properties hold
(i) $A_\infty = A \otimes K(x)_\infty$ *is trivial, so A is unramified in infinity.*

(ii) *The ramification locus of A only consists of non-real points $y \in \mathbb{P}^1_{K,nr}$. (Equivalently the monic irreducible polynomials at which A is ramified are equal to a sum of two squares).*

Proof: (a) Let B be a central simple $K(x)$-algebra which is only ramified in non-real points. Write B, up to Brauer equivalence, as a tensor product of quaternion algebras $\prod_i (a_i f_i, b_i g_i)_{K(x)}$ with f_i, g_i monic polynomials over K and $a_i, b_i \in K^*$. (Lemma 1.4 implies that it is a product of two such factors we do not need this for the argument.) Since B is by assumption unramified in infinity the points in the ramification locus of B correspond to irreducible factors of f_i and g_i. We can expand the product $\prod_i (a_i f_i, b_i g_i)_{K(x)}$ to a product $\prod_i (a_i, b_i)_{K(x)} \otimes \prod_i (a_i, g_i)_{K(x)} \otimes \prod_i (b_i, f_i)_{K(x)} \otimes \prod_j (p_j, q_j)_{K(x)}$ with p_j and q_j monic irreducible factors of the f_i and g_i. We collect the first three parts of this expansion together with all the factors $(p_r, q_r)_{K(x)}$ with p_r and q_r not equal to a sum of squares and call this product C. The remaining factors are clearly all Ω-algebras since either p_i or q_j is a sum of squares. Therefore there product, say A, is also an Ω-algebra. As mentioned before it follows from theorem 1.6 and lemma 1.5 that A is only ramified in non-real points. Since $C \sim B \otimes A$ it follows that the ramification locus of C is a subset of the union of the ramification locus of A and that of B. This implies that also C is only ramified in non-real points. In particular we have that C is unramified in infinity. Now all the factors $(p_i, q_j)_{K(x)}$ in the product defining C can only ramify in finite points corresponding to monic irreducible polynomials which are not equal to a sum of two squares, i.e., in finite real points, so the same holds for C. It follows from these observations that C is unramified in all points of $\mathbb{P}^1_K$. Faddeev's exact sequence (cf. (FES)) then implies that C is a constant algebra. We obtain that $B \sim C \otimes_{K(x)} A$ with C a constant algebra and A an Ω-algebra, what we needed tot prove.
(b) The "only if" part follows from theorem 1.6 and lemma 1.5 as we noted before. The "if" part of the statement follows from point (a) and the fact that Ω-algebras are trivial over $K(x)_\infty$ (cf. lemma 1.5(a)). Namely let B be an algebra satisfying conditions (i) and (ii). Applying point (a), (condition (ii) tells us that the hypotheses is satisfied), we obtain $B \sim C \otimes_{K(x)} A$ with C a constant algebra and A an Ω-algebra. So $1 \sim B \otimes_{K(x)} K(x)_\infty \sim C \otimes_{K(x)} A \otimes K(x)_\infty \sim C \otimes_{K(x)} K(x)_\infty$. The latter is only possible if C is a trivial algebra. So $B \sim A$ is a Ω-algebra. ∎

Remark 1.9. It is possible to prove the above proposition without using Becher's theorem. To do this one needs another argument to show that Ω-algebras are split over the completions of $K(x)$ at real points.
It is known that A is equivalent to a tensor product of quaternion algebras $\prod_i (e_i, f_i)_{K(x)}$ where the e_i are monic polynomials over K which are equal to a sum of squares. (For fields F such that $I^3(F(\sqrt{-1})) = 0$ an argument can be found in [BP], where the statement follows from the end of the proof of proposition 2.9.). Lemma 1.5 (a) tells us that all the factors $(e_i, f_i)_{K(x)}$ of A are split over the completions of $K(x)$ at real points.

The referee of a previous version of this paper suggested the following argument.
Let A be a Ω-algebra and $y \in \mathbb{P}^1_{K,r}$ a real point (possibly the point at infinity). The completion $K(x)_y$ of $K(x)$ in y has a Henselian valuation with value group $\mathbb{Z} \times \mathbb{Z}$ and with a real residue field. Using the fact that the number of square classes in

$K(x)_y$ is 8 one can check that every nonzero element in ${}_2\operatorname{Br}(K(x)_y)$ is represented by a quaternion algebra which is nonsplit at a real closure of $K(x)_y$ with respect to at least one of its four orderings. Therefore ${}_2\Omega\operatorname{Br}(K(x)_y) = 0$. This implies that $A \otimes_{K(x)} K(x)_y \sim 1$.

Corollary 1.10. *An Ω-algebra A over $K(x)$ with empty ramification locus is trivial. Two Ω-algebras A and A' over $K(x)$ with the same ramification locus are equivalent.*

Proof: It follows from proposition 1.8 (b) that the ramification locus of an Ω-algebra only consists of points y with non-real residue fields. The square class group $K(y)^*/K(y)^{*2}$ of such points is a group of order two. So the ramification locus determines the ramification completely. This implies that the second statement follows immediately from applying the first one to $A \otimes_{K(x)} A'$.
To prove the first statement assume A is an Ω-algebras with empty ramification locus, so A is everywhere unramified and the exact sequences (FES) then yields that A is a constant algebra. Proposition 1.8(b) implies that an Ω-algebra cannot be Brauer equivalent to a non-trivial constant algebra. So A must be the trivial algebra. ∎

From the above observation we obtain the following theorem.

Theorem 1.11.

$$ {}_2\Omega\operatorname{Br}(K(x)) \cong \bigoplus_{y\in\mathbb{P}^1_{K,nr}} K(y)^*/K(y)^{*2}. $$

Proof: Let ${}_2\operatorname{Br}(K(x)) \xrightarrow{\Delta} \bigoplus_{y\in\mathbb{P}^1_K} K(y)^*/K(y)^{*2}$ be the sum of the ramification maps as given in the exact sequence (FES). We first show that the image of ${}_2\Omega\operatorname{Br}(K(x))$ under this map is the subgroup $\bigoplus_{y\in\mathbb{P}^1_{K,nr}} K(y)^*/K(y)^{*2}$.
Proposition 1.8 (b) says that $\Delta({}_2\Omega\operatorname{Br}(K(x))) \subset \bigoplus_{y\in\mathbb{P}^1_{K,nr}} K(y)^*/K(y)^{*2}$.
Let $(\alpha_y)_{y\in\mathbb{P}^1_{K,nr}} \in \oplus_{y\in\mathbb{P}^1_{K,nr}} K(y)^*/K(y)^{*2}$, put $\alpha = (\alpha_z)_{z\in\mathbb{P}^1_K}$ with $\alpha_z = 1$ if $z \in \mathbb{P}_{K,r}$. Note that we have $i \in K(y)$, for $y \in \mathbb{P}^1_{K,nr}$, so by the transitivity of the norm

$$ \begin{aligned} \operatorname{cor}_{K(y)/K}(K(y)^*/K(y)^{*2}) &= N_{K(y)/K}(K(y)^*/K(y)^{*2}) \\ &= N_{K(i)/K}(N_{K(y)/K(i)}(K(y)^*/K(y)^{*2})) \\ &\subset K^{*2} + K^{*2} = K^{*2}. \end{aligned} $$

It follows that $\operatorname{cor}(\alpha_y)$ is trivial in K^*/K^{*2} for all $y \in \mathbb{P}^1_{K,nr}$. So $\sum \operatorname{cor}_{z\in\mathbb{P}^1_K}(\alpha_z)$ is also trivial. Faddeev's exact sequence implies the existence of an algebra $B \in {}_2\operatorname{Br}(K(x))$ which ramifies exactly in the points y with $\alpha_y \notin K(y)^{*2}$, i.e., such that $\Delta(B) = (\alpha_z)_z$. Proposition 1.8(a) says that $B \sim C \otimes_{K(x)} A$ with C a constant algebra and A an Ω-algebra. But then $\Delta(B) = \Delta(A) = (\alpha_y)_y$. We proved that $\Delta({}_2\Omega\operatorname{Br}(K(x))) = \oplus_{\substack{y\in\mathbb{P}^1_K \\ y \text{ non-real}}} K(y)^*/K(y)^{*2}$.
The injectivity of the restriction of Δ to ${}_2\Omega\operatorname{Br}(K(x))$ follows directly from corollary 1.10. ∎

We saw that any Ω-algebra is a quaternion algebra of the form (e^2+g^2, h) with $e, g, h \in K[x]$ so it is an algebra with the totally real extension $K(x)(\sqrt{e^2+g^2})/K(x)$

as a splitting field. Is it possible to construct such a splitting in explicit way in terms of the ramification data of the algebra? We first define 4 types of monic polynomials over K. We will give a positive answer to this question in case the ramification locus of the Ω-algebra does not contain points corresponding to monic irreducible polynomials of type (4).

Definition 1.12. Type (1) The monic polynomials $Q \in K[x]$ which are sums of 2 squares and whose monic irreducible factors Q_i over K are of degree $2q_i$ with q_i an odd number. If x_i is a root of Q_i (in some algebraic closure of K) then $x_i \in K(x_i)^{*2}$ (i.e., x_i is a square in its root field.)

Type (2) The monic polynomials $R \in K[x]$ which are sums of 2 squares and whose monic irreducible factors R_j over K are of degree $2r_j$ with r_j an odd number. If y_j is a root of R_j (in some algebraic closure of K) then $y_j \notin K(y_j)^{*2}$ (i.e., y_j is not a square in its root field.)

Type (3) The monic polynomials $P \in K[x]$ which are sums of 2 squares and whose monic irreducible factors P_k over K are of degree $2^{u_k}p_k$ with p_k an odd number, $u_k \in \mathbb{N}$ and $u_k > 1$. If z_k is a root of P_k (in some algebraic closure of K) then $z_k \notin K(z_k)^{*2}$ (i.e., z_k is not a square in its root field.)

Type (4) The monic polynomials $S \in K[x]$ which are sums of 2 squares and whose monic irreducible factors S_l over K are of degree $2^{v_l}s_l$ with s_l an odd number, $v_l \in \mathbb{N}$ and $v_l > 1$. If z_l is a root of S_l (in some algebraic closure of K) then $z_l \in K(z_l)^{*2}$ (i.e., z_l is a square in its root field.)

Note that the monic irreducible factors of polynomials of type (1), (2), (3) or (4) are also equal to a sum of two squares in $K[x]$.

Theorem 1.13. *Let A be an Ω-algebra over $K(x)$ of exponent 2. Let A be ramified exactly in the points corresponding to monic irreducible polynomials $Q_i, i = 1, \ldots, a$, of type (1), $R_j, j = 1, \ldots, e$, of type (2) and $P_k, k = 1, \ldots, l$, of type (3) (where any of the three sets of polynomials may be empty). Then there is a polynomial $h \in K[x]$ such that A is equivalent to a quaternion algebra of the form: $(\pi h, PQR)$, with $P = \prod_k P_k$, $Q = \prod_i Q_i$ and $R = \prod_j R_j$ and with π a uniformizing element for the discrete valuation on K.*

The polynomial h occurring here will be constructed explicitly in terms of the ramification data of A.

Remark 1.14. (a) The only Ω-algebras for which theorem 1.13 does not give an explicit description are those which are ramified in some monic irreducible polynomial S of degree $2^t r$, $t > 1$, r an odd number, which is a sum of two squares and with a root which is a square in the root field of S.
(b) In previous notes the authors obtained some special cases of theorem 1.13. In [BY] the cases $\mathrm{Ram}(A) = \{Q_i\}_i$, $\mathrm{Ram}(A) = \{R_j\}_j$, $\mathrm{Ram}(A) = \{P_k\}_k$ or $\mathrm{Ram}(A) = \{R_j, P_k\}_{j,k}$ where treated. In [BVY] the case $\mathrm{Ram}(A) = \{Q_i, R_j\}_{i,j}$ and in [Baz] the case $\{Q_i, P_k\}_{i,k}$ was proved. The latter, i.e., an Ω-algebras A with ramification of type (1) and (3) say in points $\{Q_i, P_k\}_{i,k}$, follows from the fact that the quaternion algebra $(QP, \pi x)$, with $Q = \prod_i Q_i$ and $P = \prod_k P_k$ has the required ramification type. This can be seen by expanding $(QP, \pi x)$ into

$$(Q, \pi) \otimes (Q, x) \otimes (P, \pi) \otimes (P, x).$$

Lemma 1.5 implies that (Q, x) and (P, π) are trivial algebras. The same lemma yields that the ramification locus of (Q, π) consists exactly of the points Q_i and that the ramification locus of (P, x) consists exactly of the points P_k. Since $(QP, \pi x)$ is an Ω-algebra with exactly the same ramification as A, corollary 1.10 implies that $A \sim (QP, \pi x)$.
We mention further that in [BTY] a result complementary to the theorem stated above is given. There an explicit description of all Ω-algebras ramified in at most two points of $\mathbb{P}^1_K$ is obtained (cf., [BTY, theorem 3]).

2 Proof of theorem 1.13

The proof of theorem 1.13 will be given in the next subsections. It is organized as follows. We start with a subsection containing some technical lemmas (cf. subsection 2.1) In the following subsection the polynomial $h \in K[x]$ occurring in the statement of the theorem is constructed. The last subsection contains a final lemma from which the theorem follows. Throughout the notation given in theorem 1.13 remains fixed.

2.1 Some lemmas

The following lemma allows us to reduce certain arguments to the case where the polynomials R, P, Q are of degree a power of 2.

Lemma 2.1. *Let L be an odd degree extension of K and let F be a finite non-real extension of K. then $[LF : F]$ is odd.*

Proof: Let $[L : K] = d$, d odd. Note that $L = K(\sqrt[d]{\pi})$ so a non-real extension F of K is of the form $F = K(i)(\sqrt[e]{\pi})$. It follows that $LF = K(i)(\sqrt[l]{\pi})$ with l the least common multiple of d and e. So $[LF : F] = \frac{l}{e}$, which is an odd number since it divides d. ∎

Lemma 2.2. (1) *Let $g \in K[x]$ be a monic irreducible polynomial of non-zero degree divisible by 4 and such that g is a sum of squares in $K(x)$. Then the quaternion algebra $(g, \pi)_{K(x)}$ is trivial.*
(2) *Let $f, g \in K[x]$ be monic irreducible polynomials, both sums of squares in $K(x)$. Let $\deg f = 2$ and $4 | \deg g$. Let y_0 be a root of g and assume that $y_0 \notin K(y_0)^{*2}$. Then the quaternion algebra $(f, g)_{K(x)}$ is trivial.*

Proof: (1) Note that the algebra (g, π) is an Ω-algebra, g being a sum of squares. It follows that the ramification can only occur at g. Let θ be a root of g in some algebraic closure of K. Since $\deg g = 2^s m$ with $s > 1$, m an odd number, and since g is a sum of squares in $K(x)$ it follows that $K(\theta) = K(i)(\sqrt[l]{\pi})$, with $l = 2^{s-1}m$. So π is a square in $K(\theta)$. Consequently, by lemma 1.5 (b), (g, π) is unramified everywhere, hence trivial (cf. corollary 1.10).
(2) Since f is a monic quadratic polynomial, lemma 1.5 (a) implies that $(f, g) \otimes K(x)_\infty$ is trivial. Let x_0 be a root of f. Put $\deg g = 4m$, $m \in \mathbb{N} \setminus \{0\}$. Since f and g are both a sum of squares in $K(x)$ their root fields contain $k(i)$. It follows

that $K(x_0) = K(i)$ and that $K(y_0)$ is a totally ramified extension of $K(x_0)$ of degree $2m$. Hence x_0 is a square in $K(y_0)$ because the latter contains the unique quadratic extension of $K(x_0)$. Since y_0 is not a square in $K(y_0)$ and all units are squares in $K(y_0)$, the values $v(x_0)$ and $v(y_0)$ (with v the valuation on $K(y_0)$) are distinct.
First assume that $v(x_0) > v(y_0)$. We have $f(y_0) = (y_0 - x_0)(y_0 - x_0^\tau)$, with τ the automorphism induced by sending i to $-i$. Since $v(x_0^\tau) = v(x_0) > v(y_0)$ it follows that $f(y_0) \equiv y_0^2 \equiv 1 \bmod K(y_0)^{*2}$.
Now let $v(x_0) < v(y_0)$ then $f(y_0) = (y_0 - x_0)(y_0 - x_0^\tau) \equiv (-x_0)(-x_0^\tau) \equiv f(0) \equiv 1 \bmod K(y_0)^{*2}$ (f is a sum of squares in $K[x]$, so its constant term is a square). It follows from the ramification formula that (f, g) is unramified in g.
Since $K(y_0)$ is the unique totally ramified extension of $K(i)$, $K(y_0)/K$ is a Galois extension. Let $G = \mathrm{Gal}(K(y_0)/K) = \{\sigma_1, \ldots, \sigma_{4m}\}$. Then $g(x_0) = \prod_{i=1}^{4m}(x_0 - y_0^{\sigma_i})$, where for all $i = 1, \ldots, 4m$ the elements $y_0^{\sigma_i}$ have equal values with respect to the valuation v of $K(y_0)$.
If $v(x_0) < v(y_0)$ then $v(g(x_0)) = v(\prod_{i=1}^{4m}(x_0 - y_0^{\sigma_i})) = v(x_0^{4m}) = 4mv(x_0)$. Hence $g(x_0) \equiv 1 \bmod K(x_0)^{*2}$.
If $v(x_0) > v(y_0)$ then $v(g(x_0)) = v(\prod_{i=1}^{4m}(x_0 - y_0^{\sigma_i})) = v(\prod_{i=1}^{4m}(-y_0^{\sigma_i})) = v(g(0))$ so $g(x_0) \equiv g(0) \equiv 1 \bmod K(x_0)^{*2}$, because g is a sum of squares in $K[x]$ and so $g(0)$ is a square in K. Hence the formula for the ramification also yields that (f, g) does not ramify at f.
It follows that the Ω-algebra (f, g) is unramified everywhere and corollary 1.10 implies that (f, g) is trivial. ∎

Remark 2.3. Let E be the splitting field of the polynomials Q, R and P, it is a Galois extension of K. Let H be the 2-Sylow subgroup of $\mathrm{Gal}(E/K)$. The fixed field $L = E^H$ of H is an odd degree extension of K. Since E contains all the roots of the polynomials Q_i, R_j and P_k and since $[E : L] = 2^m$ for some $m \geq 1$, it follows that all the irreducible factors over L of the polynomials Q_i, R_j and P_k have degree a power of 2 (they cannot be of degree one since the degrees of the polynomials Q_i, R_j and P_k are all even and so they cannot have a root in an odd degree extension). Moreover we have,

Corollary 2.4. *Let L be as above. The irreducible factors over L of Q_i, R_j and P_k, have degrees $2, 2$ and 2_k^s, $s_k > 1$ respectively. They are monic irreducible polynomials over L of type (1), (2) and (3) respectively.*

Proof: The only thing we still need to show is that the polynomials are of the given type. Let $Q_{i,L}$ be an irreducible factor of Q_i over L and let x_i be a root of $Q_{i,L}$, it is also a root of Q_i so it is a square in the root field $K(x_i)$. It follows that x_i is also a square in the larger field $L(x_i)$. So $Q_{i,L}$ is of type (1).
Let $R_{j,L}$ be an irreducible factor of R_j over L and let y_j be a root of $R_{j,L}$, it is also a root of R_j so it is not a square in the root field $K(y_j)$. Lemma 2.1 implies that the degree $[L(y_j) : K(y_j)]$ is odd. It follows that y_j is not a square in $L(y_j)$. So $R_{j,L}$ is of type (2). In the same way it follows that the irreducible factors of P_k over L are of type (3). ∎

These observations will allows us to reduce some arguments to the case where the polynomials Q_i, R_j, P_k are of degree a power of 2.

Lemma 2.5. *Let A be the Ω-algebra over $K(x)$ as in theorem 1.13. Then:*

(1) $A \sim (QR, \pi)_{K(x)} \otimes_{K(x)} (P, x)_{K(x)}$

(2) $A \sim (xQR, \pi P)_{K(x)} \otimes_{K(x)} (\pi, x)_{K(x)}$

Proof: We first prove the lemma in the special case that the polynomials Q_i and R_j are of degree 2 and that the polynomials P_k are of degree 2^{s_k}.
(1) Consider the algebra $(QR, \pi) \otimes_{K(x)} (P, x)$. We calculate its ramification. Since QR and P are polynomials which are equal to a sum of squares in $K(x)$ lemma 1.5 (a) implies that $((QR, \pi) \otimes_{K(x)} (P, x))_{K(x)_\infty}$ is trivial.
The algebra $(QR, \pi) \otimes_{K(x)} (P, x)$ can only ramify in finite points corresponding to irreducible factors of Q, R, P and x.
Since $\deg Q_i = \deg R_j = 2$ and since Q_i and R_j are both sums of squares we have $K(x_i) = K(y_j) = K(i)$. So π is not a square in $K(x_i) = K(y_j)$. Lemma 1.5 (c) implies that (QR, π) is ramified in the points corresponding to the polynomials Q_i and R_j for all i and all j. The same is true for $(QR, \pi) \otimes_{K(x)} (P, x)$ since (P, x) is unramified in these points.
Since z_k is not a square in $K(x)$ we conclude (by lemma 1.5 (b) that (P, x) is ramified in the irreducible polynomials P_k. The same is true for $(QR, \pi) \otimes_{K(x)} (P, x)$ since (QR, π) is unramified in these points.
The polynomial P is a sum of squares in $K[x]$ so $P(0) \equiv 1 \bmod K^{*2}$. Hence, by the ramification formula, (P, x) and therefore also $(QR, \pi) \otimes_{K(x)} (P, x)$ is not ramified in x.
It follows that A and $(QR, \pi) \otimes_{K(x)} (P, x)$ have the same ramification so corollary 1.10 yields that they are equivalent. This finishes the proof of (1).
(2) Lemma 2.2 implies that the quaternion algebras (Q_i, P_k) and (R_j, P_k) are trivial. Hence $(Q, P) \sim \prod_{i,k}(Q_i, P_k)$ and $(R, P) \sim \prod_{j,k}(R_j, P_k)$ are trivial. This and part (1) of the proof implies

$$\begin{aligned} A &\sim (QR, \pi) \otimes (P, x) \\ &\sim (QR, \pi) \otimes (x, \pi) \otimes (P, x) \otimes (P, Q) \otimes (P, R) \otimes (x, \pi) \\ &\sim (xQR, \pi) \otimes (xQR, P) \otimes (\pi, x) \\ &\sim (xQR, \pi P) \otimes (\pi, x). \end{aligned}$$

We now show that the general case can be reduced to the special case above. Let L/K be the odd degree extension described in remark 2.3, say with $[L : K] = d$. Choose a uniformizing element π_L in L such that $\pi = (\pi_L)^d$ (this is possible since $L = K(\sqrt[d]{\pi})$).
Since Q, R and P are also over L of type (1), (2) and (3) respectively. And since the degree of the irreducible factors of Q and R over L is 2 and since the degree of the irreducible factors of P is a power of 2. It follows from the above that $A \otimes L \sim (QR, \pi_L)_{L(x)} \otimes_{L(x)} (P, x)_{L(x)} = ((QR, \pi) \otimes (P, x)) \otimes L(x)$. But $L(x)/K(x)$ being of odd degree implies that the natural map ${}_2\mathrm{Br}(K(x)) \to {}_2\mathrm{Br}(L(x))$ is injective. So the equivalence $A \sim (QR, \pi) \otimes_{K(x)} (P, x))$ follows. The second equivalence follows in the same way. ∎

Lemma 2.6. *Let f, g be monic irreducible polynomials in $K[x]$ such that f is a sum of squares in $K(x)$. Let x_0 be a root of f and y_0 a root of g such that $K(y_0)$ is a non-real field.*
If $f(y_0) \not\equiv 1 \bmod K(y_0)^{*2}$ then $v(x_0) = v(y_0)$ with v the valuation of the field $K(x_0, y_0)$.
If in addition f is a quadratic polynomial then $x_0 = \pi^a u_0$ with $a \in \mathbb{Z}$ and u_0 a unit in $\mathcal{O}_{K(x_0)}$. Then $w_0 = y_0\pi^{-a}$ is a unit in $\mathcal{O}_{K(x_0,y_0)}$. Let $\overline{u}_0$ and $\overline{w}_0$ be the residues in $k(i)$ of respectively u_0 and w_0, then $\overline{u}_0$ and $\overline{w}_0$ are equal or conjugated under the automorphism τ defined by $\tau(i) = -i$.

Proof: By assumption f is of even degree, say $\deg f = 2m$. Let x_0 be a root of f and let $\sigma_1, \dots \sigma_{2m}$ be the automorphisms of the splitting field L of f. So the elements $x_0^{\sigma_1}, \dots, x_0^{\sigma_{2m}}$ are exactly the $2m$ different roots of f, and $f(x) = \prod_{i=1}^{2m}(x - x_0^{\sigma_i})$. The values of the roots $x_0^{\sigma_1}, \dots, x_0^{\sigma_{2m}}$ with respect to the valuation on L are all equal.
Suppose for the sake of contradiction that $v(x_0) \neq v(y_0)$, where v is the valuation on $K(x_0, y_0)$. If $v(x_0) > v(y_0)$ then $v(f(y_0)) = v(\prod_{i=1}^{2m}(y_0 - x_0^{\sigma_i})) = v(y_0^{2m})$. Hence $f(y_0) \equiv 1 \bmod K(y_0)^{*2}$, contradicting the hypotheses.
If $v(x_0) < v(y_0)$ then $v(f(y_0)) = v(\prod_{i=1}^{2m}(y_0 - x_0^{\sigma_i})) = v(\prod_{i=1}^{2m}(-x_0^{\sigma_i})) = v(f(0))$, hence $f(y_0) \equiv f(0) \equiv 1 \bmod K^{*2}$ (since f is a sum of squares), again contradicting the hypotheses.
So $v(x_0) = v(y_0)$. In case $m = 1$, i.e., f is a quadratic polynomial, $f(x) = (x - x_0)(x - x_0^\tau)$, with τ inducing the non-trivial automorphism on $L = K(x_0) = K(i)$, so that $\tau(i) = -i$.
We can put $x_0 = \pi^a u_0$ with u_0 a unit in $\mathcal{O}_{K(x_0)}$ and it follows that $w_0 = y_0\pi^{-a}$ is a unit in $\mathcal{O}_{K(y_0)}$ (note that $K(x_0) = K(y_0)$ in this case). We have $f(y_0) = (\pi^a w_0 - \pi^a u_0)(\pi^a w_0 - \pi^a u_0^\tau) \equiv (w_0 - u_0)(w_0 - u_0^\tau) \bmod K(y_0)^{*2}$. By assumption $K(y_0)$ contains $k(i)$ and therefore the units in $\mathcal{O}_{K(y_0)}$ are all squares. For the sake of contradiction assume that $\overline{w_0} \neq \overline{u_0}$ and $\overline{w_0} \neq \overline{u_0}^\tau$. Then $w_0 - u_0$ and $w_0 - u_0^\tau$ are units in $\mathcal{O}_{K(y_0)}$ so it are squares in $K(y_0)$. The above calculation implies that $f(y_0) \equiv 1 \bmod K(y_0)^{*2}$ contradicting the hypotheses. This proves the lemma. ∎

Lemma 2.7. *Let δ be a root of the polynomial $x^n - a^2$, where $n \equiv 1 \bmod 2$ and $a \in K^*$. If $K(\delta)$ is non-real then $R(\delta) \equiv 1 \bmod K(\delta)^{*2}$ and $P(\delta) \equiv 1 \bmod K(\delta)^{*2}$ (where R and P are products of monic irreducible polynomials of type (2) and (3), respectively).*

Proof: Since $K(\delta)$ is a non-real extension over K, it contains $K(i)$,hence $K(\delta) = K(i)(\sqrt[m]{\pi})$ for some $m \in \mathbb{N}$. So $\delta = (\sqrt[m]{\pi})^p u$, with $p \in \mathbb{Z}$ and u a unit in $\mathcal{O}_{K(i)(\sqrt[m]{\pi})}$. From $\delta^n = a^2$, it follows that $\pi^{\frac{pn}{m}} u^n = a^2$. This implies $\frac{pn}{2m} \in \mathbb{Z}$ since $a \in K^*$. Let $d = \gcd(m, n)$, put $m = dm_1, n = dn_1$, with $\gcd(m_1, n_1) = 1$, note that d is odd since n is odd. Since $\frac{pn_1}{2m_1} = \frac{pn}{2m} \in \mathbb{Z}$, $n \equiv 1 \bmod 2$ and $\gcd(m_1, n_1) = 1$, it follows that $2m_1$ is a divisor of p, i.e. $p = 2m_1p_1$, with $p_1 \in \mathbb{Z}$. Since u is a unit in $\mathcal{O}_{K(i)(\sqrt[m]{\pi})}$ it is a square in $K(i)(\sqrt[m]{\pi}) = K(\delta)$. Hence, $\delta = (\sqrt[m]{\pi})^p u = (\sqrt[m]{\pi})^{2m_1p_1} u \equiv 1 \bmod K(\delta)^{*2}$.
Let w be a root of an irreducible factor R_j of R or of an irreducible factor P_k of P. Then $i \in K(w)$ so $K(w) = K(i)(\sqrt[g]{\pi})$, with $g = 2^{s-1}r, s \in \mathbb{N}, s > 1$, and r an odd number.
We first show that the values of the elements δ, w with respect to the valuation of the field $K(\delta, w)$ are distinct. To do this assume $v(\delta) = v(w)$. Now $K(\delta) = K(i)(\sqrt[m]{\pi})$

and $K(w) = K(i)(\sqrt[g]{\pi})$ implies that $K(w,\delta) = K(i)(\sqrt[h]{\pi})$), with $h = \mathrm{lcm}(g,m)$. Since $w \not\equiv 1 \bmod K(w)^{*2}$, we have $w = \pi^{(2q+1)/g}\varepsilon$, where $q \in \mathbb{Z}$ and ε is a unit in $\mathcal{O}_{K(w)}$. Hence, from $w = (\sqrt[h]{\pi})^{(2q+1)h/g}\varepsilon$ and $\delta = (\sqrt[h]{\pi})^{ph/m}u = (\sqrt[h]{\pi})^{2p_1m_1h/m}u$ it follows that $\frac{(2q+1)h}{g} = \frac{2p_1m_1h}{m}$, since we assumed that $v(w) = v(\delta)$. So we have $\frac{2q+1}{g} = \frac{2p_1}{d}$, implying that $(2q+1)d = 2p_1g$. This is impossible, since $(2q+1)d$ is odd and $2p_1g$ is even.
Since the values of the elements δ and w are distinct, lemma 2.6 implies that $S(\delta) \equiv 1 \bmod K(\delta)^{*2}$ for all irreducible factors $S(x)$ of $R(x)$ or of $P(x)$. It follows that $R(\delta) = \prod_{j=1}^{l} R_j(\delta) \equiv 1 \bmod K(\delta)^{*2}$ and that $P(\delta) = \prod_{k=1}^{l} P_k(\delta) \equiv 1 \bmod K(\delta)^{*2}$. ■

2.2 The construction of the polynomial h

Let A be the Ω-algebra over $K(x)$ as given in theorem 1.13. We now construct the polynomial $h(x) \in K[x]$ in terms of the ramification of A given by the monic irreducible polynomials $Q_i, i = 1,\dots a$, $R_j, j = 1,\dots,e$ and $P_k, k = 1,\dots,l$ of type (1), (2) and (3) respectively.
Consider the field extension $M = K(y_1,\dots,y_e,z_1,\dots,z_l)$, with for $j = 1,\dots,e$, y_j a root of the polynomial Q_j, and for $k = 1,\dots,l$, z_k a root of the polynomials P_k. Then $M = K(i)(\sqrt[d]{\pi})$ for some $d \in \mathbb{N}$, define $n := 4d+1$. We denote $Y = \{y_1,\dots,y_e\}$ and $Z = \{z_1,\dots,z_l\}$.
Using that y_j is a square in $K(y_j)$ and z_k is not a square in $K(z_k)$. And using that $K(y_j,z_k) = K(i)(\sqrt[m]{\pi})$, with $m = 2^{s_k-1}r$ where r is odd, and s_k is such that $2^{s_k}p_k = [K(z_k):K]$, with p_k odd. A similar argument as the one used in the proof of lemma 2.7 yields that for the valuation on M all the elements of Y have a value different to the values of all the elements in Z.

In order to define the polynomial h we have to consider four different cases. In what follows we use the following notation. For any finite set W of elements in L, let $m(W)$ denote the element of W with the smallest value and let $M(W)$ denote the element of W with the largest value.

(i) *The element of $Y \cup Z$ with the smallest valuation is an element of Y and the element of $Y \cup Z$ with the largest value is an element of Z.*
In this case we partition Y and Z respectively in subsets $Y_r \subset Y$, $r = 1,\dots,b$ and $Z_t \subset Z$, $t = 1,\dots,b$ such that for all elements $\tilde{y}_r \in Y_r$ and all elements $\tilde{z}_t \in Z_t$ the following holds:
$$v(\tilde{y}_1) < v(\tilde{z}_1) < \ldots < v(\tilde{y}_b) < v(\tilde{z}_b).$$

Put (where square brackets indicate taking integer parts)

$$\begin{array}{lllll} c_1 & = & [\frac{4d+1}{2d}v(M(Y_1))] + 1, & \dots & c_{2b-1} = [\frac{4d+1}{2d}v(M(Y_b))] + 1 \\ c_2 & = & [\frac{4d+1}{2d}v(M(Z_1))] + 1 & \dots & c_{2b} = [\frac{4d+1}{2d}v(M(Z_b))] + 1 \end{array}$$

Then for $i = 1,\dots,b$ we have $v(M(Y_i)) < v(m(Z_i))$ and the following inequalities hold:
$$\frac{4d+1}{2d}(v(m(Z_i)) - v(M(Y_i))) \geq \frac{4d+1}{2d} > 2.$$

This implies

$$\frac{4d+1}{2d}v(M(Y_i)) < c_{2(i-1)+1} < \frac{4d+1}{2d}v(M(Y_i)) + 2 < \frac{4d+1}{2d}v(m(Z_i)),$$

and therefore

$$(4d+1)v(M(Y_i)) < 2dc_{2(i-1)+1} < (4d+1)v(m(Z_i)).$$

We obtain

$$v(M(Y_i)^n) < v(\pi^{2c_{2(i-1)+1}}) < v(m(Z_i)^n).$$

By definition we have for $i = 1, \dots, b-1$, (since the values of the elements in Y are different form the values of the elements in Z), that $v(M(Z_i)) < v(m(Y_{i+1}))$. We get

$$\frac{4d+1}{2d}(v(m(Y_{i+1}) - v(M(Z_i))) \geq \frac{4d+1}{2d} > 2,$$

so

$$(4d+1)v(M(Z_i)) < 2dc_{2i} < (4d+1)v(m(Y_{i+1})),$$

yielding

$$v(M(Z_i)^n) < v(\pi^{2c_i}) < v(m(Y_{i+1})^n).$$

Finally for $j = b$ we get directly from the definition that

$$v(M(Z_b)^n) < v(\pi^{2c_{2b}}).$$

So we verified that

$$\begin{aligned} v((M(Y_1))^n) \quad < \quad & v(\pi^{2c_1}) < v((m(Z_1))^n) \leq v((M(Z_1))^n) < v(\pi^{2c_2}) < \\ & v((m(Y_2))^n) \leq v(M(Y_2))^n) < \ldots \leq v((M(Z_b))^n) < v(\pi^{2c_{2b}}). \end{aligned}$$

(ii) *The element of $Y \cup Z$ with the smallest valuation is an element of Y and the element of $Y \cup Z$ with the largest value is an element of Y.*
In this case we can partition Y and Z respectively in subsets $Y_r \subset Y$, $r = 1, \dots, b+1$ and $Z_t \subset Z$, $t = 1, \dots, b$ such that for all elements $\widetilde{y}_r \in Y_r$ and all elements $\widetilde{z}_t \in Z_t$ the following holds:

$$v(\widetilde{y}_1) < v(\widetilde{z}_1) < \ldots < v(\widetilde{y}_b) < v(\widetilde{z}_b) < v(\widetilde{y}_{b+1}).$$

As above we define c_j $(j = 1 \dots, 2b)$,

$$\begin{aligned} c_{2(i-1)+1} \quad &= \quad [\tfrac{4d+1}{2d}M(Y_i)] + 1 \\ c_{2i} \quad &= \quad [\tfrac{4d+1}{2d}M(Z_i)] + 1, \end{aligned}$$

and verify in a similar way that

$$\begin{aligned} v((M(Y_1))^n) \quad < \quad & v(\pi^{2c_1}) < v((m(Z_1))^n) \leq v((M(Z_1))^n) < v(\pi^{2c_2}) < v((m(Y_2))^n) \leq \\ & \ldots \leq v((M(Z_b))^n) < v(\pi^{2c_{2b}}) < v((m(Y_{b+1}))^n). \end{aligned}$$

(iii) *The element of $Y \cup Z$ with smallest value is an element of Z and the element of $Y \cup Z$ with largest value is an element of Y.* We partition $Y \cup Z$ in $2b$ sets $Y_1, \ldots, Y_b$ and $Z_1, \ldots, Z_b$, such that

$$v(\tilde{z}_1) < v(\tilde{y}_1) < \ldots < v(\tilde{z}_b) < v(\tilde{y}_b).$$

And we define c_j, $(j = 1, \ldots, 2b-1)$ as follows:

$$\begin{aligned} c_{2(i-1)+1} &= [\tfrac{4d+1}{2d} M(Z_i)] + 1 \\ c_{2i} &= [\tfrac{4d+1}{2d} M(Y_i)] + 1, \end{aligned}$$

here $i =, 1 \ldots b-1$. One can verify the following inequalities,

$$\begin{aligned} v((M(Z_1))^n) &< v(\pi^{2c_1}) < v((m(Y_1))^n) \leq v((M(Y_1))^n) < v(\pi^{2c_2}) < v((m(Z_2))^n) \leq \\ &\ldots < v(\pi^{2c_{2b-1}}) < v((m(Y_b))^n). \end{aligned}$$

(iv) *The element of $Y \cup Z$ with smallest value is an element of Z and the element of $Y \cup Z$ with largest value is also an element of Z.*
We partition $Y \cup Z$ in $2b+1$ sets $Y_1, \ldots, Y_b$ and $Z_1, \ldots, Z_{b+1}$. We define the elements c_i $(i = 1, \ldots, 2b+1)$:

$$\begin{aligned} c_{2(i-1)+1} &= [\tfrac{4d+1}{2d} M(Z_i)] + 1 \\ c_{2i} &= [\tfrac{4d+1}{2d} M(Y_i)] + 1, \end{aligned}$$

with $i = 1, \ldots, b$. And one verifies the inequalities,

$$\begin{aligned} &v((M(Z_1))^n) < v(\pi^{2c_1}) < v((m(Y_1))^n) \leq v((M(Y_1))^n) < v(\pi^{2c_2}) < v((m(Z_2))^n) \leq \\ &\qquad \ldots < v(\pi^{2c_{2b}}) < v((m(Z_{b+1}))^n) \leq v((M(Z_{b+1}))^n) < v(\pi^{2c_{2b+1}}). \end{aligned}$$

Definition of h.

Define $m := 2b$ if (i) or (ii) holds, $m := 2b-1$ if (iii) holds and $m := 2b+1$ if (iv) holds.
Let L be the odd degree extension of K defined in remark 2.3. Every polynomial Q_i, $i = 1, \ldots, a$ splits over L in q_i irreducible factors of degree 2, say $Q_{i,t}$, $i = 1, \ldots a$ and $t = 1, \ldots q_i$. Let $x_{i,t}$ be a root of $Q_{i,t}$. Then $x_{i,t} = \pi^{\frac{v(x_{i,t})}{q_i}} w_{i,t}$ where $v(x_{i,t})$ is the value of $x_{i,t}$ in the field $L(x_{i,t})$, $w_{i,t}$ is a unit in $\mathcal{O}_{L(x_{i,t})}$, and q_i is as defined in definition 1.12. Since the residue field of K is infinite we can choose units u_s in K in such a way that $(\overline{u_s})^2 \neq (\overline{w_{i,t}})^n$, $(\overline{u_s})^2 \neq (\overline{w_{i,t}^{\tau}})^n$ for all $s = 1, \ldots m$, and $j = 1, \ldots, a$, where $\overline{u_s}$ be the residue of u_s in k, $\overline{w_{i,t}}$ be the residue of $w_{i,t}$ in $k(i)$, and $\overline{w_{i,t}^{\tau}}$ is an element conjugated to $\overline{w_{i,t}}$ under τ (the automorphism defined by $i \mapsto -i$).

Definition 2.8. Let m, c_s and u_s for $s = 1, \ldots, m$ be as defined above. Define $a_s = \pi^{c_s} u_s$, $s = 1, \ldots, m$, and define

$$h(x) := \prod_{s=1}^{m} (a_s^2 - x^n).$$

2.3 Proof of theorem 1.13

With h as in definition 2.8 the following holds:

Lemma 2.9. *The quaternion algebras* $(xQR, \pi P)_{K(x)}$ *and* $(\pi h(x), xPQR)_{K(x)}$ *are isomorphic.*

Proof: First note that since $n-1$ is even, $a_s^2 - x^n = a_s^2 - (x^{\frac{n-1}{2}})^2 x$ is a norm of the quadratic extension $K(x)(\sqrt{x})$. It follows that for each s the quaternion algebra $(a_s^2 - x^n, x)$ is trivial. Hence also (h, x) is trivial. Lemma 1.5 (1) implies that the algebra $(P, \pi) = \otimes_{k=1}^{l}(P_k, \pi)$ is trivial. Expanding $(\pi h, xPQR)$ then yields

$$\begin{aligned}(\pi h, xPQR) &\sim (\pi, xPQR) \otimes (h, xPQR)\\ &\sim (\pi, x) \otimes (QR, \pi) \otimes (h, PQR).\end{aligned}$$

Lemma 1.5 (b) says that (Q, P) and (R, P) are trivial so the expansion of $(xQR, \pi P)$ gives

$$\begin{aligned}(xQR, \pi P) &\sim (x, \pi) \otimes (x, P) \otimes (QR, \pi) \otimes (Q, P) \otimes (R, P)\\ &\sim (x, \pi) \otimes (P, x) \otimes (QR, \pi).\end{aligned}$$

It follows that the isomorphism, $(xQR, \pi P) \cong (\pi h, xPQR)$, which we have to prove, is established if we show that

$$(x, \pi) \otimes (P, x) \otimes (QR, \pi) \cong (\pi, x) \otimes (QR, \pi) \otimes (h, PQR),$$

or equivalently that

$$(P, x) \cong (h, PQR). \tag{1}$$

Note that both are Ω-algebras. The ramification locus of (P, x) consists exactly of the points corresponding to $P_1, \ldots, P_l$. So the isomorphism (1) holds if the ramification locus of the right hand side is also equal to $P_1, \ldots, P_l$ (cf. corollary 1.10). This holds true if

$$\begin{cases} h(x_i) \equiv 1 \bmod K(x_i)^{*2} & \text{for all } i = 1, \ldots, a\\ h(y_j) \equiv 1 \bmod K(y_j)^{*2}, & \text{for all } j = 1, \ldots, e\\ h(z_k) \not\equiv 1 \bmod K(z_k)^{*2}, & \text{for all } k = 1, \ldots, l\\ P(\delta)Q(\delta)R(\delta) \sim 1 \text{ in } K(\delta) & \text{for all roots } \delta \text{ of } h. \end{cases} \tag{2}$$

We first verify the last condition in (2). Let δ be a root of h. Note that if $K(\delta)$ is a real field, then we have that $P(\delta)Q(\delta)R(\delta) \equiv 1 \bmod K(\delta)^{*2}$ (since the polynomials P, Q, R are sums of squares in $K[x]$). So we may assume that $K(\delta)$ is a non-real field. According to lemma 2.7 we have $R(\delta) \equiv 1 \bmod K(\delta)^{*2}$, $P(\delta) \equiv 1 \bmod K(\delta)^{*2}$ in $K(\delta)$.

Assume that $Q(\delta) \not\equiv 1 \bmod K(\delta)^{*2}$. Then it follows from lemma 2.1 that $Q(\delta) \not\equiv 1 \bmod L(\delta)^{*2}$ where L is the odd degree extension of K defined in remark 2.3. Then $Q_{i,t}(\delta) \not\equiv 1 \bmod L(\delta)^{*2}$ for some (i, t), where $Q_{i,t}$ are the monic irreducible factors of Q over L (as defined in subsection 2.2). We also fixed a root $x_{i,t}$ of $Q_{i,t}$. According to lemma 2.6, $v(\delta) = v(x_{i,t})$ in $L(x_{i,t}, \delta)$. Let $\delta = \pi^{v(\delta)}\varepsilon$, with $h_1 \in \mathbb{Z}$ and ε a unit in $\mathcal{O}_{L(x_{i,t},\delta)}$, since $\delta^n = a_s^2$, for some $s \in \{1, \ldots, m\}$, we have $\varepsilon = \sqrt[n]{u_s^2}$. And by the choice of the units u_s, $s = 1, \ldots, m$ we see that the residues of ε^n and $w_{i,t}^n$ are not

equal and not conjugated under the automorphism τ, therefore the same holds for the residues of ε and $w_{i,t}$. Lemma 2.6 then implies $Q_i(\delta) \equiv 1 \bmod L(\delta)^{*2}$, and we have a contradiction. Hence $Q(\delta) \equiv 1 \bmod K(\delta)^{*2}$.
So $P(\delta)Q(\delta)R(\delta) \equiv 1 \bmod K(\delta)^{*2}$, and we proved that the last condition of (2) is satisfied.
We now verify the first condition of (2). To do this we verify for all $s = 1, \ldots, m$ and for all indices i that $a_s^2 - x_i^n \equiv 1 \bmod K(x_i)^{*2}$ in the three possible cases $v(a_s^2) < v(x_i^n)$, $v(a_s^2) > v(x_i^n)$ and $v(a_s^2) = v(x_i^n)$, where v is the valuation of $K(x_i)$.
If $v(a_s^2) < v(x_i^n)$ then $a_s^2 - x_i^n \equiv a_s^2 \equiv 1 \bmod K(x_i)^{*2}$. If $v(a_s^2) > v(x_i^n)$ then $a_s^2 - x_i^n \equiv -x_i^n \equiv 1 \bmod K(x_i)^{*2}$ since $x_i \equiv 1 \bmod K(x_i)^{*2}$ and $K(i) \subset K(x_i)^2$. Finally if $v(a_s^2) = v(x_i^n)$ then $a_s^2 - x_i^n = \pi^{2v(a_s)}(u_s^2 - w_i^n) \equiv 1 \bmod K(x_i)^{*2}$ because $u_s^2 - w_i^n$ is a unit in $\mathcal{O}_{K(x_i)}$ (by construction of the elements u_s).

Finally we verify the second and the third condition of (2), i.e., $h(y) \equiv 1 \bmod K(y)^{*2}$ for all $y \in Y$ and $h(z) \not\equiv 1 \bmod K(z)^{*2}$ for all $z \in Z$. To do this we have to consider cases (i) - (iv), on which the definition of h depend, separately.

We consider case (i). Let $y \in Y_1$ then $h(y) = \prod_{s=1}^{2b}(a_s^2 - y^n) \equiv \prod_{s=1}^{2b}(-y^n) \equiv y^{2bn} \equiv 1 \bmod K(y)^{*2}$. Let $y \in Y_j$ with $j > 1$, then $h(y) = \prod_{s=1}^{2b}(a_s^2 - y^n) \equiv \prod_{s=1}^{2j-2} a_s^2 \prod_{s=2j-1}^{2b}(-y^n) \equiv a_s^{4(j-1)} y^{2(b-j+1)n} \equiv 1 \bmod K(y)^{*2}$. This settles the second condition of (2).
Let $z \in Z_k$ then $h(z) = \prod_{s=1}^{2b}(a_s^2 - z^n) \equiv \prod_{s=1}^{2k-1} a_s^2 \prod_{s=2k}^{2b}(-z^n) \equiv -a_s^{2(2k-1)} z^{(2b-2k+1)n} \equiv z \not\equiv 1 \bmod K(z)^{*2}$, since $(2b - 2k + 1)n \equiv 1 \bmod 2$ and $z \not\equiv 1 \bmod K(z)^{*2}$. Which proves that the third condition of (2) holds for h.
In a similar way on can verify in each of the three other cases ((ii), (iii) and (iv)) that the polynomial h satisfies the second and the third condition of (2). ∎

We can now prove our main result.

Proof of theorem 1.13: Let A be an Ω-algebra over $K(x)$ of exponent 2 and with ramification locus as in theorem 1.13. Lemma 2.5 implies that $A \sim (xQR, \pi P) \otimes (\pi, x)$.
According to Lemma 2.9 there exist elements $a_1, \ldots, a_m \in K^*$ and an odd number n, such that $(xQR, \pi P) \sim (\pi \prod_{s=1}^{m}(a_s^2 - x^n), xPQR)$. Since $(a_s^2 - x^n, x) \sim (a_s^2 - (x^{\frac{n-1}{2}})^2 x, x) \sim 1$, we have $(h, x) \sim (\prod_{s=1}^{m}(a_s^2 - x^n), x) \sim 1$. This implies $(\pi, x) \sim (\pi h, x)$. Therefore

$$A \sim (xQR, \pi P) \otimes (\pi, x) \sim (\pi h, xPQR) \otimes (\pi h, x) \sim (\pi h, PQR),$$

as stated in the theorem. ∎

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