The Dieudonné-Schwartz Theorem for p-adic inductive limits

N. De Grande-De Kimpe C. Perez-Garcia*

Abstract

In this paper we investigate the p-adic validity of the classical Dieudonné-Schwartz Theorem proved in [5] and of its main improvements given in the archimedean literature (see e.g. [12], [13], [14], [15] and [19]). We show that the situation differs substantially from the real or complex one.

Introduction

The classical Dieudonné-Schwartz Theorem reads:

"Every strict LF-space is regular"

([5], Proposition 4). This result is still valid in the p-adic context (Corollary 3.3).

Since 1949 lots of extensions and improvements of the above theorem, with or without the condition of strictness, have been obtained (see e.g. [12], Theorem 2.12.2, [13], [14], [15] and [19]).

In this paper we study these results in the p-adic case, where the situation turns out to be quite different. We prove that, under some assumptions, their p-adic versions remain true. We also provide examples showing, on the one hand that if those assumptions are removed then the results fail, and on other hand that the converses of the implications appearing in the positive results are not true, even in very particular cases.

The classes of inductive sequences constructed in the last section are the source of most of our counterexamples, which either do not have a classical counterpart or if they have one it has a typically archimedean character.

*Research partially supported by Ministerio de Ciencia y Tecnologia (MTM2004-02348)
Received by the editors November 2005.
Communicated by F. Bastin.
2000 *Mathematics Subject Classification* : 46S10.
Key words and phrases : Dieudonné-Schwartz Theorem, p-adic inductive limits.

1 Preliminaries

Throughout this paper $K = (K, |\,.\,|)$ is a non-archimedean non-trivially valued field that is complete with respect to the metric induced by the valuation $|\,.\,|$.

Unless explicitly stated otherwise, all vector spaces and locally convex spaces we will consider in this paper are over K.

For fundamentals on normed and locally convex spaces we refer to [21] and [22] respectively.

Let E be a vector space. For a subset A of E we denote by $[A]$ its linear hull. A is **absolutely convex** if $0 \in A$ and $x, y \in A$, $\lambda, \mu \in K$, $\max(|\lambda|, |\mu|) \leq 1$ implies $\lambda\, x + \mu\, y \in A$. For an absolutely convex set $A \subset E$ we define $A^e := A$ if the valuation of K is discrete, $A^e := \bigcap\{\lambda\, A : \lambda \in K, |\lambda| > 1\}$ otherwise. A is called **edged** if $A = A^e$.

Let $E = (E, \tau)$ be a locally convex space. By $\mathcal{U}_E$ we mean the family of all (clopen) absolutely convex zero neighbourhoods in E. For a subset A of E we denote by $\overline{A}^E$ the closure of A in E. A is called **compactoid** if for every $U \in \mathcal{U}_E$ there is a finite set B in E such that $A \subset U + \operatorname{aco} B$, where $\operatorname{aco} B$ is the absolutely convex hull of B, that is, the smallest absolutely convex set containing B. If F is another locally convex space, a linear map $T : E \to F$ is called **compact** if there is a $V \in \mathcal{U}_E$ for which $T(V)$ is a compactoid in F. Also, E is called **nuclear** if for every continuous seminorm p on E there is a continuous seminorm q on E, $q \geq p$, such that the natural map $E_q \to E_p$ is compact, where E_p and E_q are the canonical normed spaces associated to p and q respectively.

By E' we denote the **dual** of E i.e. the vector space of all continuous linear functionals $E \to K$. The **weak topology** $\sigma(E, E')$ is the locally convex topology on E generated by the family of seminorms $\{|f| : f \in E'\}$. Let Z be a subspace of E, endow Z with the induced topology $\tau|Z$. We say that Z has the **Hahn-Banach Extension Property (HBEP)** in E if every $f \in Z'$ has a continuous linear extension to the whole space.

A continuous seminorm p on E is called **polar** if $p = \sup\{|f| : f \in E', |f| \leq p\}$. E is called **polar** if its topology is generated by a family of polar seminorms; **strongly polar** if every continuous seminorm p on E with $p(E) \subset \overline{|K|}$ is polar (where $\overline{|K|}$ is the closure in $\mathbb{R}$ of $\{|\lambda| : \lambda \in K\}$). E is called **of countable type** if for every continuous seminorm p on E the associated normed space E_p is of countable type (recall that a normed space is said to be of countable type if it is the closed linear hull of a countable set). If K is spherically complete, every locally convex space is strongly polar. For any K, strongly polar spaces are polar and spaces of countable type are strongly polar. Also, every nuclear space, in particular $(E, \sigma(E, E'))$, is of countable type. One verifies that each subspace of a strongly polar space (resp. each finite dimensional subspace of a polar space) has the HBEP. But for non-spherically complete fields the HBEP may fails, even there exist normed spaces with trivial dual. For details we refer to [21] and [22].

A very interesting class of locally convex spaces, to which is devoted the present paper, is formed by the locally convex inductive limits. We point out the central role that they play in the definition of a p-adic Laplace and Fourier Transform given in

[9] and [10] respectively and in the index theory of p-adic differential equations (see e.g. [1], [2], [3], [4] and [20]). The last of these references shows also the influence of inductive limits in the study of the p-adic Monsky-Washnitzer cohomology.

An **inductive sequence** is an increasing sequence $E_1 \subset E_2 \subset \ldots$ of subspaces of a vector space E such that $E = \bigcup_n E_n$ and where, for each n, E_n is provided with a locally convex topology τ_n in such a way that each inclusion $E_n \rightarrow E_{n+1}$ is continuous. The **inductive limit** of the sequence $(E_n)_n$ is the space E endowed with the strongest locally convex topology τ_{ind} for which all the inclusions $E_n \rightarrow E$ are continuous.

If the steps E_n are normed (resp. Banach, metrizable, Fréchet) spaces then $(E_n)_n$ is called an LN (resp. LB, LM, LF)-space. As usual, a **Fréchet** space is a metrizable complete locally convex space.

From now on in this paper $(E_n)_n$ is an inductive sequence of locally convex spaces (E_n, τ_n) with inductive limit (E, τ_{ind}).

For each n, let σ_n be the weak topology on E_n. Following [8],

Definition 1.1 We say that $(E_n)_n$ is

(i) **strict** if $\tau_{n+1}|E_n = \tau_n$ for each n,

(ii) **weakly strict** if $\sigma_{n+1}|E_n = \sigma_n$ for each n, or equivalently (apply [8], Lemma 1.4.5,(i)), if:

$$\text{for all } n, \text{ every } f_n \in E_n' \text{ has an extension } \overline{f_n} \in E_{n+1}'. \tag{1}$$

In some special cases there are relations between strictness and weak strictness.

Proposition 1.2 ([8], Theorem 1.4.7)

(i) *If each E_n has the HBEP in E_{n+1} (e.g. when all the E_n are strongly polar) then strictness implies weak strictness.*

(ii) *If all the E_n are metrizable and polar then weak strictness implies strictness.*

But in general strictness and weak strictness are independent properties (see [8], Examples 1.4.10 and 1.4.12).

Now, following [11],

Definition 1.3 We say that $(E_n)_n$ satisfies

(C1) if E_n is closed in E_{n+1} for each n,

(C2) if E_n is closed in E for each n,

(C3) if for each n there exists a $k(n) \in \mathbb{N}$ such that $\overline{E_n}^E \subset E_{n+k(n)}$,

(C4) if, for all n, every closed absolutely convex subset of E_n is closed in E_{n+1} (or equivalently if, for all n, every closed absolutely convex subset of E_n is closed in E).

(C5) if, for all n, every closed absolutely convex and edged subset of E_n is closed in E_{n+1} (or equivalently if, for all n, every closed absolutely convex and edged subset of E_n is closed in E).

Proposition 1.4 ([11], Propositions 3.2, 3.9) (C4) $\Longrightarrow$ (C5) $\Longrightarrow$ (C2) $\Longrightarrow$ (C1),(C3).

Also, the converses of the above implications are not true, and (C1) and (C3) are independent properties, see [11].

Remark 1.5 Looking at 1.1 and 1.3 the reader could ask what happens if in the definitions of the strictness and of the closedness properties (C1), (C4) and (C5) we replace the index $n+1$ by $n+k(n)$ for some $k(n) \in \mathbb{N}$ (as we did for (C3)). But one can easily see that this does not make any difference.

On the other hand, the replacement in the definition of (C3) of the index $n+k(n)$ by $n+1$ leads to a stronger (and different see [11], Remark 3.15) closedness property. We do not treat this extra closedness property here because is does not have any interesting contribution to the purpose of the paper.

The following regularity properties will be crucial in the sequel.

Definition 1.6 We say that $(E_n)_n$ is

(i) **regular** if for every bounded subset D of E there is an n such that $D \subset E_n$ and is bounded in E_n,

(ii) **α-regular** if for every bounded subset D of E there is an n such that $D \subset E_n$.

Remark 1.7 Clearly, regular $\Longrightarrow$ α-regular. But the converse is not true, see Example 2.3 and Remark 2.6.

We finish these Preliminaries with examples of sequence spaces, which will be used through the paper.

Let $B = (b_k^n)_{k,n}$ be an infinite matrix consisting of strictly positive real numbers such that $b_k^n \leq b_k^{n+1}$ for all k, n. For each $n \in \mathbb{N}$,

$$c_0(\mathbb{N}, 1/b^n) := \{(\lambda_k)_k \in K^{\mathbb{N}} : \lim_k |\lambda_k| / b_k^n = 0\},$$

is a Banach space of countable type under the norm $(\lambda_k)_k \mapsto \sup_k |\lambda_k| / b_k^n$. The monotonicity condition we imposed on the matrix B implies that $(c_0(\mathbb{N}, 1/b^n))_n$ is an inductive sequence. Its inductive limit, the so called **Köthe dual space**, is usually denoted by $\Lambda_0(B)$.

Further, for each j,

$$c_0(\mathbb{N}, b^j) := \{(\lambda_k)_k \in K^{\mathbb{N}} : \lim_k |\lambda_k|\ b_k^j = 0\},$$

is a Banach space of countable type under the norm p_j defined by

$$p_j((\lambda_k)_k) := \sup_k |\lambda_k|\ b_k^j, \qquad (\lambda_k)_k \in c_0(\mathbb{N}, b^j). \tag{2}$$

We consider on the so-called **Köthe space** $\Lambda^0(B) := \bigcap_j c_0(\mathbb{N}, b^j)$ the **normal topology**, $n_{0,\infty}$, which is the one defined by the family of norms $\{p_j : j \in \mathbb{N}\}$. Then $(\Lambda^0(B), n_{0,\infty})$ is a Fréchet space of countable type.

When $b_k^n = n^k$, $\Lambda_0(B)$ is the space of germs of analytic functions at zero, and $\Lambda^0(B)$ is the space of entire functions on K. For more details on $\Lambda_0(B)$ and $\Lambda^0(B)$, see Section 3.2 of [8].

2 The p-adic Dieudonné-Schwartz Theorem without strictness

In Proposition 3.4 we will prove that for an LF-space any of the closedness properties (C1)-(C5) implies regularity. In general we can say the following.

Proposition 2.1
(i) (C3) $\Longrightarrow$ α-*regular.*
(ii) *If all the* E_n *are polar,* (C5) $\Longrightarrow$ *regular.*

Proof. The proof of (i) is a simple adaptation of the classical one given in [13], Theorem 1.

(ii) Suppose all the E_n are polar and $(E_n)_n$ satisfies (C5), so also (C3) by Proposition 1.4. Now let $D \subset E$ be a bounded subset of E. Since $(E_n)_n$ is α-regular by (i), there exists an n such that $D \subset E_n$, and clearly D is $\tau_{ind}|E_n$-bounded. Also, (C5) implies that (E_n, τ_n) and $(E_n, \tau_{ind}|E_n)$ have the same closed hyperplanes. So $(E_n, \tau_n)' = (E_n, \tau_{ind}|E_n)'$ ([11], Proposition 1.1). Hence D is weakly bounded in E_n and by polarity and Theorem 7.5 of [22] we finally obtain that D is τ_n-bounded. Thus, $(E_n)_n$ is regular.

Remark 2.2 In Examples 3.8 we will see that the converses of Proposition 2.1 are not true.

Next we show that if in Proposition 2.1 we consider a weaker closedness property or a stronger regularity one then there exist LM-spaces with steps of countable type for which the implications fail, even when they satisfy an extra regularity like property. At the same time, in Example 2.3 and Remark 2.6 we construct α-regular inductive sequences that are not regular, as we have already promised in Remark 1.7.

Example 2.3 *There exist non-regular inductive sequences* $(E_n)_n$ *of metrizable spaces of countable type that satisfy* (C2) (*hence are* α*-regular, Proposition 2.1,*(i)), *and also satisfy the following property:*

(R) *For all* n *and every bounded subset* D *of* E_n *there exists an* $m \geq n$ *such that* $\overline{D}^E \subset E_m$ *and is bounded in* E_m.

Proof. Let $(E_n)_n$ be the inductive sequence of 4.1.1. Suppose the matrix B satisfies

$$\text{for every } j \text{ there exists } j' > j \text{ such that } \lim_k b_k^j / b_k^{j'} = 0. \tag{3}$$

Let Z be an infinite dimensional p_{j_0}-closed subspace of $\Lambda^0(B)$. By τ-closedness of Z and Theorem 4.1,(v) we obtain that $(E_n)_n$ satisfies (C2).

That $(E_n)_n$ is not regular follows from Corollary 4.4. In fact, we have $\tau|Z \neq \tau'|Z$ because, by Proposition 3.5 of [7], $(Z, \tau'|Z)$ is nuclear and, however, $(Z, \tau|Z)$ is an infinite dimensional normed space, which cannot be nuclear.

Finally, in order to prove property (R) for this $(E_n)_n$ we first see that

$$\text{for all } C \subset \Lambda^0(B) \text{ that is } \tau'\text{-bounded, } \overline{C}^X \text{ is } \tau'\text{-bounded.} \tag{4}$$

To this end, let $C \subset \Lambda^0(B)$ be τ'-bounded. Then C is τ'-compactoid, because $(\Lambda^0(B), \tau')$ is nuclear. Hence there exists a $(\mu_k)_k \in \Lambda^0(B)$ such that

$$C \subset C' := \{(\lambda_k)_k \in \Lambda^0(B) : |\lambda_k| \leq |\mu_k| \text{ for all } k\}$$

([16], Proposition 2.1). Since all the coordinate maps on $\Lambda^0(B)$ are p_{j_0}-continuous (=τ-continuous), the set C' is τ-closed, so it contains $\overline{C}^X$. Also, C' is τ'-bounded, hence so is $\overline{C}^X$.

Apart from (4), it is clear that by τ-closedness of Z,

$$\text{for all } C \subset Z \text{ that is } \tau'\text{-bounded, } \overline{C}^X \subset Z. \tag{5}$$

Next we use (4) and (5) to get property (R). Let $n \in \mathbb{N}$ and let $D \subset E_n$ be τ_n-bounded. We prove that $\overline{D}^E$ is contained and bounded in E_n (so property (R) holds for $m = n$). In fact, continuity of the projections $\pi_i : E \rightarrow X$ implies

$$\pi_i(\overline{D}^E) \subset \overline{\pi_i(D)}^X \quad \text{for all } i. \tag{6}$$

Therefore, since each $\pi_i(D)$ (and hence its τ-closure) is τ-bounded, we have that $\pi_i(\overline{D}^E)$ is also τ-bounded.

Now let $i > n$. Then $\pi_i(D)$ is τ'-bounded (and contained in Z when $i > n+1$). So, by (4)$-$(6), $\pi_i(\overline{D}^E)$ is τ'-bounded (and contained in Z when $i > n+1$).

Thus, as

$$\overline{D}^E \subset \pi_1(\overline{D}^E) \times \ldots \times \pi_{n+1}(\overline{D}^E) \times \prod_{i>n+1} \pi_i(\overline{D}^E),$$

and as the set after the inclusion is contained and bounded in E_n, we have the same for $\overline{D}^E$.

Remark 2.4 Clearly regularity implies property (R) considered in Example 2.3. For LN-spaces spaces, and with the same proof as in [15], Theorem 2, one has that regularity and (R) are equivalent properties. But this is not the case for LM-spaces, as we have just seen. There exist even LM-spaces having property (R) and that are not α-regular, as we prove in the following example, which also shows that (i) of Proposition 2.1 is not true for (C1).

Example 2.5 *There exist inductive sequences of metrizable spaces of countable type that satisfy* (C1) *and property* (R)*, but are not α-regular.*

Proof. Let $(E_n)_n$ be the inductive sequence of 4.1.1. Suppose the matrix B satisfies (3) i.e. $(\Lambda^0(B), n_{0,\infty})$ is nuclear ([7], Proposition 3.5). Let Z be a subspace of $\Lambda^0(B)$ that is $n_{0,\infty}$-closed but not p_{j_0}-closed (Such Z exists, otherwise by Proposition 1.1 of [11], $(\Lambda^0(B), n_{0,\infty})' = (\Lambda^0(B), p_{j_0})'$, which would imply that p_{j_0} and $n_{0,\infty}$ have the same bounded sets, [22], Theorem 7.5. Hence $n_{0,\infty}$ would be defined by the norm p_{j_0} ([17], Lemma 4.2): a contradiction because $(\Lambda^0(B), n_{0,\infty})$ is nuclear and infinite dimensional, so it cannot be normable).

Since Z is τ'-closed but not τ-closed it follows respectively from Theorem 4.1,(iv) and Proposition 4.3 that $(E_n)_n$ satisfies (C1) but it is not α-regular.

Finally, looking at the proof of the last part of Example 2.3 we see that property (R) follows as soon as we prove that (5) holds for this $(E_n)_n$.

To this end, let $C \subset Z$ be τ'-bounded ($= n_{0,\infty}$-bounded), we may assume that C is absolutely convex. Since $Y := (\Lambda^0(B), n_{0,\infty})$ is a nuclear Fréchet space we have that $A := \overline{C}^Y$ is absolutely convex, metrizable, compactoid and complete in Y. By [25], Proposition 9.1 we obtain that $\tau|A = \tau'|A$. Hence A is τ-complete, so τ-closed. Also, $C \subset A \subset Z$ (the last inclusion because Z is τ'-closed). Therefore, by τ-closedness of A, $\overline{C}^X \subset Z$.

Remark 2.6 α-regularity and (R) are independent properties. Indeed, this is a consequence of Example 2.5 and the fact that *there exist inductive sequences of metrizable spaces of countable type that satisfy* (C2) (*hence are α-regular, Proposition 2.1,*(i)), *but do dot satisfy* (R).

To find such an inductive sequence, let $(E_n)_n$ be as in 4.1.2, where Z is a $\|\,.\,\|$-closed subspace of F for which the restriction of f to Z, $f|Z$, is not $\|\,.\,\|$-continuous (hence by [11], Proposition 1.1, $\operatorname{Ker} f|Z := \{x \in Z : f(x) = 0\}$ is $\|\,.\,\|$-dense in Z).

From τ-closedness of Z and Theorem 4.1,(v) we have that $(E_n)_n$ satisfies (C2).

Now suppose that (R) holds for $(E_n)_n$; we derive a contradiction. Let $B_Z := \{x \in Z : \|x\| \leq 1\}$. The set $C := B_Z \cap \operatorname{Ker} f|Z$ is τ'-bounded in Z. Hence $C^{\mathbb{N}}$ is bounded in E_1 and, by property (R), there is an m such that $\overline{C^{\mathbb{N}}}^E \subset E_m$ and is bounded in E_m. Applying (8) of the proof of Theorem 4.1 we obtain that $A^{\mathbb{N}} \cap E$ is a bounded subset of E_m (where $A := \overline{C}^X$). If π_{m+1} is the $m+1$-th projection on E_m, then $\pi_{m+1}(A^{\mathbb{N}} \cap E) = A$ is bounded in Y i.e. $\overline{C}^X$ is τ'-bounded. But this is an impossibility. In fact, $\|\,.\,\|$-density of $\operatorname{Ker} f|Z$ leads to $B_Z \subset \overline{C}^X$, so τ'-boundedness of $\overline{C}^X$ would imply that B_Z is τ'-bounded i.e. that $\tau|Z$ and $\tau'|Z$ have the same bounded sets. So these normed topologies on Z coincide, a contradiction because $f|Z$ is τ'-continuous but not τ-continuous.

Remark 2.7 Compare the examples of this section with the classical ones given in [14], in which the steps are not even metrizable.

3 The $\mathfrak{p}$-adic Dieudonné-Schwartz Theorem with strictness

Theorem 3.1 (p-adic Dieudonné-Schwartz Theorem for strict inductive limits)
(i) *Strict* + (C3) $\Longrightarrow$ *regular.*
(ii) *Strict* + (C1) $\Longrightarrow$ (C4) + *regular.*

Proof. (i) By Proposition 2.1,(i) we only have to show that "strict + α-regular $\Longrightarrow$ regular". This follows directly from [8], Theorem 1.4.7,(i).

(ii) Let $(E_n)_n$ be strict and suppose it satisfies (C1). For $n \in \mathbb{N}$, let $A \subset E_n$ be absolutely convex and closed in E_n. Strictness implies that A is $\tau_{n+1}|E_n$-closed, and from (C1) we derive that A is closed in E_{n+1}. Hence (C4) holds and so also (C3) (Proposition 1.4). Now regularity follows from (i).

When we replace "strict" by "weakly strict" the Dieudonné-Schwartz Theorem can be partially saved.

Theorem 3.2 (p-adic Dieudonné-Schwartz Theorem for weakly strict inductive limits)
(i) *If all the E_n are polar, then*

$$\textit{Weakly strict} + \text{(C3)} \Longrightarrow \textit{regular.}$$

(ii) *If all the E_n are strongly polar, then*

$$\textit{Weakly strict} + \text{(C1)} \Longrightarrow \text{(C5)} + \textit{regular.}$$

(iii) *If either*
(iii.a) *K is spherically complete,*
or
(iii.b) *all the E_n are metrizable and polar,*
then

$$\textit{Weakly strict} + \text{(C1)} \Longrightarrow \text{(C4)} + \textit{regular.}$$

Proof. (i) This goes as in Theorem 3.1,(i) making use of the facts that by polarity, weakly bounded sets in E_n are bounded ([22], Theorem 7.5) and that by weak strictness, $\sigma|E_n = \sigma_n$ ([8], Theorem 1.4.(7),(ii)) for all n, where σ is the weak topology on E.

Before continuing the proof note that in (ii) (resp. (iii)) it suffices to see that $(E_n)_n$ satisfies (C5) (resp (C4)). In fact, once we have (C5) or (C4) then regularity follows from (i) and Proposition 1.4. So, let $(E_n)_n$ be weakly strict and satisfy (C1).

(ii) Suppose all the E_n are strongly polar. For $n \in \mathbb{N}$, let $A \subset E_n$ be absolutely convex edged and closed in E_n. Since E_n is strongly polar we have that A is σ_n-closed in E_n ([22], Theorem 4.7). By weak strictness, A is $\sigma_{n+1}|E_n$-closed, hence $\tau_{n+1}|E_n$-closed. From (C1) we obtain that A is closed in E_{n+1}. Thus, $(E_n)_n$ satisfies (C5).

(iii.a) Let K be spherically complete. Then in this case the absolutely convex sets that are closed coincide with the weakly closed ones ([26], Theorem 2). So the same reasoning as in (ii) leads now to (C4).

(iii.b) Suppose all the E_n are metrizable and polar. By Proposition 1.2,(ii) we have that $(E_n)_n$ is strict. Then (C4) follows from Theorem 3.1,(ii).

Corollary 3.3

(i) *Every strict LF-space is regular.*

(ii) *Every weakly strict LF-space with polar steps is regular.*

Proof. Every strict LF-space satisfies (C1), so (i) is a direct consequence of Theorem 3.1,(ii). Also, (ii) follows from Theorem 3.2,(iii.b).

For LF-spaces we also have the following (compare Proposition 2.1).

Proposition 3.4 *Let* $(E_n)_n$ *be an LF-space. Then, any of the closedness properties* (C1)$-$(C5) *implies regularity.*

Proof. By Proposition 1.4 it suffices to prove the result for (C1) and (C3).

First assume that $(E_n)_n$ satisfies (C1). Applying the Open Mapping Theorem ([18], Corollary 2.74) to each inclusion $(E_n, \tau_n) \rightarrow (E_n, \tau_{n+1}|E_n)$ we obtain that $(E_n)_n$ is strict. By Corollary 3.3,(i) $(E_n)_n$ is regular.

Now suppose that $(E_n)_n$ satisfies (C3). There is a sequence $m_1 < m_2 < \ldots$ in $\mathbb{N}$ such that $F_n := \overline{E_n}^E \subset E_{m_n}$ for all n. Then $(F_n, \tau_{m_n}|F_n)_n$ is an LF-space that satisfies (C2) and has the same inductive limit as the original one $(E_n)_n$. By the above $(F_n)_n$ is regular, and then so is $(E_n)_n$.

Remark 3.5 In (i) of Theorems 3.1 and 3.2 we cannot include other closedness properties (apart from the stated one (C3)). In fact, *there exist inductive sequences* $(E_n)_n$ *of normed spaces of countable type that are strict and weakly strict and satisfy* (C3) *but do not satisfy any of the other closedness properties*. An example of such an inductive sequence is given in Remark 4.2.2 of [11]. Note that the steps of this $(E_n)_n$ cannot be Banach, because for LB-spaces each of the properties (C1), (C2), (C4) and (C5) characterizes its strictness ([11], Proposition 4.8).

The example presented here is the p-adic substitute of Example 1 of [13].

In the classical case the implication "Weakly strict + (C1) $\Longrightarrow$ (C4) + regular" of Theorem 3.2 is always true (see [13], where the authors also proved that weak strictness, more concretely its characterization of (1), is equivalent to their property (H8)). In contrast to that, the next examples show that, for p-adic inductive limits with polar steps, the conclusions of Theorem 3.2,(ii),(iii) may fail when K is not spherically complete.

Examples 3.6 *Suppose* K *is not spherically complete.*

(i) *There exist inductive sequences of polar spaces that are weakly strict and satisfy* (C1), *but do not satisfy any of the other closedness properties, and are not* α*-regular.*

(ii) *There exist inductive sequences of polar spaces that are weakly strict and satisfy* (C2) (*hence are regular, Theorem 3.2,*(i)), *but do not satisfy* (C5).

(iii) *There exist inductive sequences of spaces of countable type that are weakly strict and satisfy* (C5) (*hence are regular, Theorem 3.2,*(i)), *but do not satisfy* (C4).

Proof. (i) Let $F := \ell^\infty$. There is a closed subspace Z of F that has the HBEP in F, contains c_0, and is not weakly closed in F ([23], Remark after Proposition 1.5).

Let $(E_n)_n$ be the inductive sequence of 4.1.4 with F and Z as above. By Theorem 4.1 we have that $(E_n)_n$ satisfies all the requirements related to weak strictness and closedness.

Now we see that $(E_n)_n$ is not α-regular. By Theorem 4.1,(viii) it suffices to find a bounded set $C \subset Z$ for which $\overline{C}^X \not\subseteq Z$. For this, take $C := \{x \in Z : \|x\| \leq\}$ and note that, as $(F/c_0)' = \{0\}$ ([21], Corollary 4.3), we have that c_0 (hence Z) is weakly dense in F.

(ii) Let $(E_n)_n$ be the inductive sequence of 4.1.4 with F being not strongly polar and Z a weakly closed subspace of F with the HBEP in F.

By (iii),(v) of Theorem 4.1, $(E_n)_n$ is weakly strict and satisfies (C2). Also, since F is not strongly polar, there exists an absolutely convex and edged set A in F that is closed but not weakly closed ([22], Theorem 4.7) i.e. A is τ'-closed but not τ-closed in F. Applying (vii) of Theorem 4.1 we deduce that (C5) fails for this $(E_n)_n$.

(iii) Let $(E_n)_n$ be the inductive sequence of 4.1.4 with F of countable type and Z a closed subspace of F. It follows from (iii),(iv) of Theorem 4.1 that $(E_n)_n$ is weakly strict and satisfies (C1), hence satisfies (C5) by Theorem 3.2,(ii).

However, as F is infinite dimensional its weak topology is not the norm topology (i.e. $\tau \neq \tau'$). By non-spherical completeness of K there is a τ'-closed absolutely convex subset of F that is not τ-closed ([24], Corollary 1.2). It follows from (vi) of Theorem 4.1 that (C4) does not hold for $(E_n)_n$.

Remarks 3.7

1. For real or complex inductive limits we also have that "(C4) $\Longrightarrow$ Weakly strict + (C1)" (see [13]). The same happens in the non-archimedean case when either K is spherically complete or all the E_n are metrizable and strongly polar ([11], Theorem 4.3). However, this is not the case in general (see [11], Counterexample 4.4.4, where the steps are polar Banach spaces).

This also proves that the converses of (ii) and (iii) of Theorem 3.2 may fail (note that by Proposition 2.1,(ii), for polar steps we always have (C4) $\Longrightarrow$ regular).

2. The converse of Theorem 3.1,(ii) holds when all the E_n are metrizable and strongly polar ([11], Theorem 4.3). But it fails even when K is spherically complete (see [11], Counterexample 4.4.1, where the steps are spaces of countable type).

The failure of the converses in Proposition 2.1 (already announced in Remark 2.2) and in (i) of Theorems 3.1 and 3.2 is shown by the next two examples.

Examples 3.8

(i) *There exist inductive sequences of Banach spaces of countable type that are regular and do no satisfy any of the strictness and any of the closedness properties.*

(ii) *There exist inductive sequences of spaces of countable type that are strict, weakly strict and regular, but do not satisfy any of the closedness properties.*

Proof. (i) Let B be a matrix satisfying (3) and for each n, let $E_n := c_0(\mathbb{N}, 1/b^n)$. Then $(E_n)_n$ is an inductive sequence of Banach spaces of countable type which is compact, that is, for each n there exists $m \geq n$ such that the inclusion $E_n \rightarrow E_m$ is compact ([8], Theorem 3.2.18). Hence $(E_n)_n$ is regular ([8], Theorem 3.1.7,(v)). But it is not strict ([8], Remark 3.1.3.III), so neither weakly strict (Proposition 1.2).

It remains to prove that (C1) and (C3) fail for $(E_n)_n$. Suppose this inductive sequence has any of these two closedness properties; we derive a contradiction. There exist $m_1 < m_2 < \dots$ in $\mathbb{N}$ such that, for each n, $m(n) \geq n$ and $\overline{E_n}^{E_{m_n+1}} \subset E_{m_n}$ (with $m_n = n$ in case (C1) holds). But $\overline{E_n}^{E_{m_n+1}} = E_{m_n+1}$, so $E_{m_n+1} = E_{m_n}$ for all n, and by the Open Mapping Theorem ([18], Corollary 2.74) this equality is topological. In particular, $(E_{m_n})_n$ is strict, a contradiction because it is compact.

(ii) Let $I_1, I_2, \dots$ be infinite countable sets such that $I_n \subset I_{n+1}$ and $I_{n+1} \setminus I_n$ is infinite for all n. Then $F_n := c_0(I_n)$ is a Banach space of countable type which is naturally included in $c_0(I_{n+1})$; we will identify F_n with its image under the inclusion. We have that $(F_n)_n$ is a strict inductive sequence of Banach spaces, hence regular by Corollary 3.3,(i). Let (F, τ_{ind}) be its inductive limit. Let $I := \bigcup_n I_n$, e_i $(i \in I)$ the canonical unitary vectors of $c_0(I)$, and $A := \{e_i : i \in I\}$. For each n, set $E_n := [A \cup F_n]$ and endow E_n with the restricted topology $\tau_{ind}|E_n$. It is easily seen that (F, τ_{ind}) is also the inductive limit of $(E_n)_n$ (observe that by strictness of $(F_n)_n$, the topology on each F_n is $\tau_{ind}|F_n$, [8], Theorem 1.4.7,(i)). Also, since each F_n is of countable type then F is of countable type ([8], Proposition 1.1.10,(i)) and hence so are the E_n ([22], Proposition 4.12,(i)). Further, it is clear by construction that $(E_n)_n$ is strict and, by Proposition 1.2,(i), it is weakly strict.

Next we show regularity of $(E_n)_n$. Let $D \subset F$ be τ_{ind}-bounded. As $(F_n)_n$ is regular there is an n such that $D \subset F_n$ and D is bounded in F_n. Now by continuity of the inclusion $F_n \to E_n$ we deduce that D is contained and bounded in E_n.

Finally we prove that $(E_n)_n$ does not satisfy any of the closedness properties. By Proposition 1.4 and the Dieudonné-Schwartz Theorem 3.1,(ii) it suffices to check that $(E_n)_n$ does not satisfy (C3). For that we first see that, for each n, $\overline{E_n}^F = F$. Indeed, if $x \in F$ then there is an m such that $x \in F_m = \overline{[e_i : i \in I_m]}^{F_m} \subset \overline{[e_i : i \in I_m]}^F \subset \overline{[A]}^F \subset \overline{E_n}^F$. Hence $x \in \overline{E_n}^F$. Therefore, if $(E_n)_n$ would satisfy (C3) there would exist an r such that $E_n = E_{n+1}$ for each $n \geq r$, which is not possible because $E_n \neq E_{n+1}$ for all n.

Remarks 3.9

1. Examples 3.8 are the p-adic substitutes of the classical ones given in Examples 2 and 4 of [13], Example of [15] and Counterexample of [19], all of them with a typically archimedean character.

2. The steps of Example 3.8,(ii) cannot be metrizable. Indeed, for a strict LM-space we have (C3) $\iff$ α-regular ([19], Theorem 2, which also works in the p-adic case).

The results and examples of Section 2 show that the conclusions of the p-adic Dieudonné-Schwartz Theorems 3.1 and 3.2 are not true when the (weak) strictness condition is dropped. The same occurs when the closedness condition is the dropped one. In fact,

Examples 3.10

(i) *There exist inductive sequences of normed spaces of countable type that are strict and weakly strict, but are not α-regular and do not satisfy any of the closedness properties.*

(ii) *There exist inductive sequences of spaces of countable type that are weakly strict, but are neither strict nor α-regular and do not satisfy any of the closedness properties.*

(iiii) *If K is not spherically complete, there exist inductive sequences of polar metrizable spaces that are strict, but are neither weakly strict nor α-regular and do not satisfy any of the closedness properties.*

Proof. Let c_{00} be the linear hull of the canonical unitary vectors of c_0. Let $B_{c_{00}}$ and B_{c_0} be the closed unit balls in c_{00} and c_0 respectively.

(i) The normed space c_0/c_{00} is infinite dimensional (otherwise, c_0 would have countable dimension, which is not the case because of the Baire Category Theorem, see e.g. [6], 3.9.3). Hence there is a sequence $(y_n)_n$ in c_0 such that $y_1 \notin c_{00}$ and $y_{n+1} \notin c_{00} + [y_1, \ldots, y_n]$ for all n. Now put $E_n := c_{00} + [y_1, \ldots, y_n]$ equipped with the norm induced by c_0. Then $(E_n)_n$ is an inductive sequence of normed spaces of countable type which by construction is strict (equivalently weakly strict, by Proposition 1.2). To finish the proof of (i) it suffices to see that $(E_n)_n$ is not α-regular. In fact, by Proposition 1.4 and Theorem 3.1, we then conclude that all the closedness properties fail for $(E_n)_n$.

So let us see that $(E_n)_n$ is not α-regular. Let E be its inductive limit, let $D := B_{c_0} \cap E$. As $D = \overline{B_{c_{00}}}^E$ ([11], Example 4.1.1), D is bounded in E. But if there would be an n for which $D \subset E_n$ then $E = E_m$ for all $m \geq n$, a contradiction. So D is not contained in any step, and we are done.

(ii) Let $(E_n)_n$ be the inductive sequence of 4.1.4 with $F := c_0$, $Z := c_{00}$. By Theorem 4.1 we have that $(E_n)_n$ is weakly strict but not strict and does not satisfy any of the closedness properties. Now let us prove that $(E_n)_n$ is not α-regular. For that, $C := B_{c_{00}} \subset Z$ is a τ-bounded set for which $\overline{C}^X = B_{c_0}$ (note that B_{c_0} is weakly closed in c_0, [22], Theorem 4.7), so $\overline{C}^X \not\subseteq Z$ (otherwise, $c_0 = Z$, a contradiction). From Theorem 4.1,(viii) we deduce that α-regularity fails for $(E_n)_n$.

(iii) Suppose K is not spherically complete. Let $(E_n)_n$ be the inductive sequence of 4.1.3 with $F := \ell^\infty$, $Z := c_{00}$. As Z is not τ'-closed (hence not τ-closed) and does not have the HBEP in F ([21], Theorem 4.15), it follows from Theorem 4.1 and Proposition 4.3 that $(E_n)_n$ meets the requirements.

Remark 3.11 By Corollary 3.3 the steps of (i) cannot be Banach. The example of (i) is the p-adic substitute of the classical one given in [13], Example 3.

Also, applying Proposition 1.2 we obtain that the steps of (ii) cannot be metrizable and that the steps of (iii) cannot be strongly polar. In particular, it is not possible to give an example satisfying the conditions of (iii) when K is spherically complete.

We finish this section by giving an example that on the one hand shows the existence of inductive sequences that do not satisfy any of the strictness, closedness and regularity properties considered in this paper (Example 3.12), and on the other hand reveals a new contrast with the classical case (Remark 3.13).

Example 3.12 *There exist inductive sequences of Banach spaces of countable type that are not α-regular and do not satisfy any of the strictness and closedness properties.*

Proof. Let $(c_0(\mathbb{N}, \frac{1}{b^n}))_n$ be the inductive sequence of Example 3.2.14 of [8]. In the proof of this example there was constructed a bounded sequence in the inductive limit which is not localized in any step. Hence this inductive sequence is not α-regular.

By Corollary 3.3,(i) $(c_0(\mathbb{N}, \frac{1}{b^n}))_n$ is not strict (equivalently, it is not weakly strict, Proposition 1.2). Finally, by Proposition 3.4 we have that this inductive sequence does not satisfy any of the closedness properties.

Remark 3.13 Observe that when K is not spherically complete the spaces $c_0(\mathbb{N}, \frac{1}{b^n})$ of Example 3.12 are reflexive ([21], Corollary 4.18). This fact is in sharp contrast with the classical case ([15], Theorem 4), where it was proved that any real or complex LB-space with reflexive steps is regular.

4 Very useful examples of inductive sequences

Inspired by [11], Theorem 5.1, we construct certain classes of inductive sequences which provide most of the examples needed along the paper.

Theorem 4.1 *Let τ, τ' be Hausdorff locally convex topologies on a vector space F, $\tau \leq \tau'$, let $X := (F, \tau)$, $Y := (F, \tau')$. Let Z be a subspace of F which we equip with the topology $\tau'|Z$. For each $n \in \mathbb{N}$, set*

$$E_n := X^n \times Y \times \prod_{i>n+1} Z,$$

where all the product spaces appearing in the definition of E_n are endowed with the corresponding product topologies. Then we have the following.

(i) *$(E_n)_n$ is an inductive sequence of Hausdorff locally convex spaces. If (E, τ_{ind}) is its inductive limit then $E \subset X^{\mathbb{N}}$ and $\tau_\pi|E \leq \tau_{ind}$, where τ_π is the product topology on $X^{\mathbb{N}}$. In particular, (E, τ_{ind}) is Hausdorff.*

(ii) *$(E_n)_n$ is strict $\Longleftrightarrow \tau = \tau'$.*

(iii) *$(E_n)_n$ is weakly strict $\Longleftrightarrow X' = Y'$ and Z has the HBEP in Y.*

(iv) *$(E_n)_n$ satisfies* (C1) *$\Longleftrightarrow Z$ is τ'-closed.*

(v) *$(E_n)_n$ satisfies* (C2) *$\Longleftrightarrow (E_n)_n$ satisfies* (C3) *$\Longleftrightarrow Z$ is τ-closed.*

(vi) *$(E_n)_n$ satisfies* (C4) *$\Longrightarrow Z$ is τ'-closed and every τ'-closed absolutely convex subset of F is τ-closed.*

(vii) *$(E_n)_n$ satisfies* (C5) *$\Longrightarrow Z$ is τ'-closed and every τ'-closed absolutely convex and edged subset of F is τ-closed.*

(viii) *Z is τ-closed $\Longrightarrow (E_n)_n$ is α-regular $\Longrightarrow$ for all $C \subset Z$ that is τ-bounded, $\overline{C}^X \subset Z$.*

(ix) *$(E_n)_n$ is regular $\Longleftrightarrow (E_n)_n$ is α-regular and every τ-bounded subset of Z is τ'-bounded.*

Proof. Properties (i)$-$(iv) and (vi)$-$(vii) can be proved as their counterparts of Theorem 5.1 of [11]. Also, Proposition 1.4 takes care of (C2) $\Longrightarrow$ (C3) of (v).

Before continuing we prove (a) and (b) below, which will be used in (v) and (viii).

(a) *For every zero neighbourhood W in E there exists an r such that for each $n \geq r$ there are $U_1, \ldots, U_n \in \mathcal{U}_X$ with*

$$U_1 \times \ldots \times U_n \times \prod_{i>n} Z \subset W. \tag{7}$$

Indeed, we may assume that W is absolutely convex. Since $W \cap E_1$ is a zero neighbourhood in E_1, there exists an r such that

$$W \supset W \cap E_1 \supset V_1 \times V_2 \times V_3 \times \ldots \times V_r \times \prod_{i>r} Z,$$

where $V_1 \in \mathcal{U}_X$, $V_2 \in \mathcal{U}_Y$, $V_3, \ldots, V_r \in \mathcal{U}_Z$. Also, for each $n \geq r$, $W \cap E_n$ is a zero neighbourhood in E_n. Hence there exists an s such that

$$W \supset W \cap E_n \supset U_1 \times \ldots \times U_n \times U_{n+1} \times U_{n+2} \times \ldots \times U_{n+s} \times \prod_{i>n+s} Z,$$

where $U_1, \ldots, U_n \in \mathcal{U}_X$, $U_{n+1} \in \mathcal{U}_Y$, $U_{n+2}, \ldots, U_{n+s} \in \mathcal{U}_Z$. This leads to

$$U_1 \times \ldots \times U_n \times \prod_{i>n} Z \subset (U_1 \times \ldots \times U_n \times \{0\}^{\mathbb{N}}) + (\{0\}^n \times \prod_{i>n} Z) \subset W + W \subset W,$$

and we arrive at (7).

(b) *Let $m \in \mathbb{N}$ and let $A_1, A_2, \ldots$ be non-empty subsets of F with $A_i \subset Z$ for $i > m+1$. Then*

$$\overline{\prod_i A_i}^E = (\prod_i \overline{A_i}^X) \bigcap E. \tag{8}$$

In order to see this, note that the inclusion $\overline{\prod_i A_i}^E \subset (\prod_i \overline{A_i}^X) \bigcap E$ follows because the set $(\prod_i \overline{A_i}^X) \bigcap E$ is $\tau_\pi|E$-closed (hence τ_{ind}-closed, see (i)) and contains $\prod_i A_i$. For the opposite inclusion, let $x = (x_1, x_2, \ldots) \in (\prod_i \overline{A_i}^X) \bigcap E$, let W be a zero neighbourhood in E. Choose r satisfying (7) and let n be such that $x \in E_n$ (so, $x_i \in Z$ for $i > n+1$). We may assume $n \geq r, m$. By (7) there are $U_1, \ldots, U_{n+1} \in \mathcal{U}_X$ such that $U_1 \times \ldots \times U_{n+1} \times \prod_{i>n+1} Z \subset W$. For each $i \in \{1, \ldots, n+1\}$ there is an $y_i \in A_i$ such that $x_i - y_i \in U_i$; for $i > n+1$ take $y_i \in A_i \subset Z$. Then $y := (y_1, \ldots, y_{n+1}, y_{n+2}, \ldots) \in \prod_i A_i$ and $x - y \in U_1 \times \ldots \times U_{n+1} \times \prod_{i>n+1} Z \subset W$. Thus, $x \in \overline{\prod_i A_i}^E$, and the proof of (b) is finished.

Now let us prove the rest of (v).

First suppose that $(E_n)_n$ satisfies (C3). Then $\overline{E_1}^E \subset E_{n_1}$ for some n_1. So, by (8), $(F^2 \times \prod_{i>2} \overline{Z}^X) \cap E \subset E_{n_1}$, from which we obtain $\overline{Z}^X \subset Z$ i.e. Z is τ-closed.

Next let Z be τ-closed. Then, again by (8), for each n we have $\overline{E_n}^E = (F^{n+1} \times \prod_{i>n+1} \overline{Z}^X) \cap E = E_n$, that is, E_n is closed in E. Hence $(E_n)_n$ satisfies (C2).

Finally we prove (viii) and (ix).

(viii) To get the first implication, let Z be τ-closed. By (v) we have that $(E_n)_n$ satisfies (C3) and applying Proposition 2.1,(i) we arrive at α-regularity.

For the proof of the second implication, suppose $(E_n)_n$ is α-regular. Let $C \subset Z$ be τ-bounded. First we see that $C^{\mathbb{N}}$ is bounded in E. For that, let W be a zero neighbourhood in E. Let r and $U_1, \ldots, U_r \in \mathcal{U}_X$ be satisfying (7) for $n = r$, that is,

$$U_1 \times \ldots \times U_r \times \prod_{i>r} Z \subset W. \tag{9}$$

Then, from τ-boundedness of C and (9) it follows easily that $C^{\mathbb{N}} \subset \lambda W$ for some $\lambda \in K$. Thus, $C^{\mathbb{N}}$ is τ_{ind}-bounded. Then so is $\overline{(C^{\mathbb{N}})}^E$, and by α-regularity there is an n such that $\overline{(C^{\mathbb{N}})}^E \subset E_n$. From (7) we derive that $(\overline{C}^X)^{\mathbb{N}} \cap E_{n+1} \subset E_n$. Hence $\overline{C}^X \subset Z$ and we finally obtain the desired second implication of (viii).

(ix) First assume that $(E_n)_n$ is regular. Clearly it is α-regular. Let now $C \subset Z$ be τ-bounded. As in (viii) we obtain that $C^{\mathbb{N}}$ is bounded in E. By regularity there is an n such that $C^{\mathbb{N}}$ is bounded in E_n. Then $\pi_{n+1}(C^{\mathbb{N}}) = C$ is bounded in Y i.e. C is τ'-bounded (for each i, π_i is the i-th projection).

Conversely, assume that $(E_n)_n$ is α-regular and every τ-bounded subset of Z is τ'-bounded. Let $D \subset E$ be τ_{ind}-bounded. Since by (i) $\tau_\pi|E \leq \tau_{ind}$, we have that D is $\tau_\pi|E$-bounded. So $\pi_i(D)$ is τ-bounded for all i. Further, by α-regularity $D \subset E_n$ for some n, so $\pi_i(D) \subset Z$ for $i > n+1$. It follows from the assumption that these last sets $\pi_i(D)$ are τ'-bounded. Thus, $D \subset \prod_i \pi_i(D)$ is contained and bounded in E_{n+1}, and $(E_n)_n$ is regular.

Particular cases of Theorem 4.1 The following choices for F, τ and τ' are frequently used through the paper.

4.1.1 F:= the Köthe space $\Lambda^0(B)$ associated to an infinite matrix B, τ=: the topology on F defined by one fixed norm p_{j_0}, as defined in (2) for $j = j_0$, τ':= the normal topology $n_{0,\infty}$.

4.1.2 F:= an infinite dimensional normed space of countable type with norm $\|\,.\,\|$, τ:= the topology on F defined by $\|\,.\,\|$, τ':= the topology on F defined by the norm $\|\,.\,\|_f : x \mapsto \max(\|x\|, |f(x)|)$, where f is a linear functional $F \to K$ that is not $\|\,.\,\|$-continuous.

4.1.3 F:= an infinite dimensional polar normed space, $\tau = \tau'$:= the norm topology on F.

4.1.4 F:= an infinite dimensional polar normed space, τ:= the weak topology $\sigma(F, F')$ on F, τ':= the norm topology on F.

For any of these choices we usually change the subspace Z in the examples given along the paper, according to the purpose of each of these examples.

Clearly the steps of 4.1.1, 4.1.2 and 4.1.3 are metrizable spaces. Using the hereditary properties of spaces of countable type and polar spaces ([22], Propositions 4.12 and 5.3 respectively) we obtain that, for any Z, the steps of 4.1.1 and 4.1.2 are always of countable type, and that the steps of 4.1.3 and 4.1.4 are always polar, being of countable type if and only if F is of countable type.

Remark 4.2 We can give some partial affirmative answers to the validity of the converses of (vi)$-$(viii) of Theorem 4.1. With the same proof as in Propositions 5.3 and 5.4 of [11], we have that the converses of (vi) and (vii) are true when either X and Y are metrizable and polar or K is spherically complete (additionally when Y is of countable type, in case of (vii)).

Also, with respect to (viii) we can say:

Proposition 4.3 *Let X, Y, Z and $(E_n)_n$ be as in Theorem 4.1. Suppose X is metrizable. Then the following are equivalent.*

(α) *Z is τ-closed.*

(β) *$(E_n)_n$ is α-regular.*

(γ) *For all $C \subset Z$ that is τ-bounded, $\overline{C}^X \subset Z$.*

Proof. By Theorem 4.1,(viii) we only have to prove $(\gamma) \Longrightarrow (\alpha)$.

Assume (γ) holds. Let $x \in \overline{Z}^X$. By τ-metrizability there exists a sequence $x_1, x_2, \ldots$ in Z such that $x = \lim_n x_n$ in X. Then $C := \{x_1, x_2, \ldots\}$ is a τ-bounded subset of Z such that $x \in \overline{C}^X$. Hence by (γ) we have $x \in Z$ i.e. Z is τ-closed.

Corollary 4.4 *Let X, Y, Z and $(E_n)_n$ be as in Theorem 4.1. Suppose X and Y are metrizable. Then*

$$(E_n)_n \text{ is regular} \Longleftrightarrow Z \text{ is } \tau\text{-closed and } \tau|Z = \tau'|Z.$$

Proof. By Theorem 4.1,(ix) and Proposition 4.3 we have to see that $\tau|Z = \tau'|Z$ if and only if every τ-bounded subset of Z is τ'-bounded. The "only if" is clear. To prove the "if", suppose that $\tau|Z$ and $\tau'|Z$ have the same bounded sets. Then by Lemma 4.2 of [17] these metrizable topologies coincide, and we are done.

Remark 4.5 Let $m \in \mathbb{N}$. In Remark 3.15 and Section 5 of [11] we proved that *if τ, τ' and X, Y, Z are as in Theorem 4.1 and for each n we set*

$$E_n := X^n \times Y \times Z^m \times \{0\}^{\mathbb{N}} \tag{10}$$

(*as above all the product spaces are endowed with the corresponding product topologies*), *then properties* (i)−(iv) *and* (vi)−(vii) *of Theorem 4.1 are true, whereas we have:*

(v)′ *$(E_n)_n$ always satisfies* (C3). *$(E_n)_n$ satisfies* (C2) $\Longleftrightarrow$ *Z is τ-closed.*

With respect to regularity we now prove that *if $(E_n)_n$ is as in* (10) *then it is regular* (compare Theorem 4.1,(viii),(ix)). In order to see this, let E be the inductive limit of $(E_n)_n$ and let $D \subset E$ be bounded in E. Since E is contained in the locally convex direct sum $\bigoplus_n X$ and the inclusion $E \rightarrow \bigoplus_n X$ is continuous, we have that D is bounded in $\bigoplus_n X$. So there is an n such that $\pi_i(D)$ is bounded in X for $i \leq n$ and $\pi_i(D) = 0$ for $i > n$ (where π_i is the i-th projection). Therefore, $D \subset E_n$ and it is bounded in E_n. Thus, $(E_n)_n$ is regular.

References

[1] G. Christol, Z. Mebkhout, *Sur le théorème de l'indice des équations différentielles p-adiques* I, Ann. Inst. Fourier **43** (1993), 1545-1574.

[2] G. Christol, Z. Mebkhout, *Sur le théorème de l'indice des équations différentielles p-adiques* II, Ann. of Math. **146** (1997), 345-410.

[3] G. Christol, Z. Mebkhout, *Sur le théorème de l'indice des équations différentielles p-adiques* III, Ann. of Math. **151** (2000), 385-457.

[4] G. Christol, Z. Mebkhout, *Sur le théorème de l'indice des équations différentielles p-adiques* IV, Invent. Math. **143** (2001), 629-672.

[5] J. Dieudonné, L. Schwartz, *la dualité dans les espaces* $(\mathcal{F})$ *and* $(\mathcal{LF})$, Ann. Inst. Fourier Grenoble **1** (1949), 61-101.

[6] R. Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

[7] N. De Grande-De Kimpe, *Non-archimedean Fréchet spaces generalizing spaces of analytic functions*, Indag. Math. **44** (1982), 423-439.

[8] N. De Grande-De Kimpe, J. Kąkol, C. Perez-Garcia, W.H. Schikhof, *p-adic locally convex inductive limits*, Lecture Notes in Pure and Appl. Math., **192**, Marcel Dekker, New York, 1997, 159-222.

[9] N. De Grande-De Kimpe, A. Khrennikov, *The non-archimedean Laplace Transform*, Bull. Belg. Math. Soc. Simon Stevin **3** (1996), 225-237.

[10] N. De Grande-De Kimpe, A. Khrennikov, L. van Hamme, *The Fourier Transform for p-adic tempered distributions*, Lecture Notes in Pure and Appl. Math., **207**, Marcel Dekker, New York, 1999, 97-112.

[11] N. De Grande-De Kimpe, C. Perez-Garcia, *Strictness and closedness in p-adic inductive limits*, Contemporary Mathematics, **384**, American Mathematical Society, Providence, 2005, 79-100.

[12] J. Horvath, *Topological vector spaces and distributions*, Vol I, Addison Wesley, 1966.

[13] J. Kucera, C. Bosch, *Dieudonné-Schwartz Theorem on bounded sets in inductive limits* II, Proc. Amer. Math. Soc. **86** (1982), 392-394.

[14] J. Kucera, K. McKennon, *Bounded sets in inductive limits*, Proc. Amer. Math. Soc. **69** (1978), 62-64.

[15] J. Kucera, K. McKennon, *Dieudonné-Schwartz Theorem on bounded sets in inductive limits*, Proc. Amer. Math. Soc. **78** (1980), 366-368.

[16] C. Perez-Garcia, *On compactoidity in non-Archimedean locally convex spaces with a Schauder basis*, Indag. Math. **50** (1988), 85-88.

[17] C. Perez-Garcia, W.H. Schikhof, *Compact operators and the Orlicz-Pettis property in p-adic Analysis*, Report 9101, Department of Mathematics, University of Nijmegen, The Netherlands, 1991, 1-27.

[18] J.B. Prolla, *Topics in Functional Analysis over Valued Division Rings*, North-Holland, Amsterdam, 1982.

[19] Qui Jing-Hui, *Dieudonné-Schwartz Theorem in inductive limits of metrizable spaces*, Proc. Amer. Math. Soc. **92** (1984), 255-257.

[20] P. Robba, G. Christol, *Équations différentielles p-adiques*, Hermann, Paris, 1994.

[21] A.C.M. van Rooij, *Non-Archimedean Functional Analysis*, Marcel Dekker, New York, 1978.

[22] W.H. Schikhof, *Locally convex spaces over nonspherically complete valued fields* I-II, Bull. Soc. Math. Belg. Sér. B **38** (1986), 187-224.

[23] W.H. Schikhof, *The complementation property of ℓ^∞ in p-adic Banach spaces*, Lecture Notes in Math., **1454**, Springer Verlag, Berlin, 1990, 342-350.

[24] W.H. Schikhof, *The p-adic Krein-Šmulian Theorem*, Lecture Notes in Pure and Appl. Math., **137**, Marcel Dekker, New York, 1992, 177-189.

[25] W.H. Schikhof, *A perfect duality between p-adic Banach spaces and compactoids*, Indag. Math. N.S. **6** (1995), 325-339.

[26] J. van Tiel, *Ensembles pseudo-polaires dans les espaces localement K-convexes*, Indag. Math. **28** (1966), 369-373.

N. De Grande-De Kimpe
Groene Laan 36 (302)
B 2830 Willebroek
Belgium

C. Perez-Garcia
Department of Mathematics
Facultad de Ciencias
Universidad de Cantabria
Avda. de los Castros s/n
39005 Santander
Spain
E-mail address: perezmc@unican.es