On an Elliptic Equation Involving a Kirchhoff Term and a Singular Perturbation

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Abstract

In this paper we consider the existence of positive solutions for the following class of singular elliptic nonlocal problems of Kirchhoff-type

$$\begin{cases} -M(\|u\|^2)\Delta u & = & \frac{h(x)}{u^\gamma} + k(x)u^\alpha & \text{in} & \Omega, \\ u & > & 0 & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \partial\Omega, \end{cases}$$

where $\Omega \subset I\!R^N, N \geq 2$, is a bounded smooth domain, $M : \mathbb{R} \to \mathbb{R}$ is a continuous function and $\|u\|^2 = \int_\Omega |\nabla u|^2$ is the usual norm in $H_0^1(\Omega)$. The main tools used are the Galerkin method and a Hardy-Sobolev inequality.

1 Introduction.

In recent years much attention has been devoted to nonlocal problems due two basic aspects of mathematical research:

(i) Such problems arise in significant physical situations as, for example, nonlinear elasticity theory, Biology, heat transfer, among others. In particular, in Biology, such kind of problems appears mainly in phenomena in which there is migration represented by a term which is nonlocal.

(ii) The presence of a nonlocal term poses some interesting and nontrivial difficulties.

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The interested reader may consult Chipot[2], Chipot-Lovat[3], Corrêa[4], Alves-Corrêa-Ma[1], Ma[10] and the references therein, where there is some detailed information on nonlocal problems and their applications.

In particular, in this paper, we are interested in the following elliptic problem

$$\begin{cases} -M(\|u\|^2)\Delta u &= \frac{h(x)}{u^\gamma} + k(x)u^\alpha & \text{in} \quad \Omega, \\ u &> 0 & \text{in} \quad \Omega, \\ u &= 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $h, k \in C(\overline{\Omega}), h, k \geq 0$ in Ω, $h, k \not\equiv 0$, $\alpha, \gamma \in (0,1), M : \mathbb{R} \to \mathbb{R}$ is a given function, whose properties will be introduced later, $\|u\|^2 = \int_\Omega |\nabla u|^2$ is the usual norm in $H_0^1(\Omega)$ and $M(\|u\|^2)\Delta u$ is the Kirchhoff operator. This problem is the stationary counterpart of the Kirchhoff hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - M\left(\int_\Omega |\nabla u(x,t)|^2 dx\right)\Delta u = f(x,u), \tag{1.2}$$

which is motivated in the mathematical description of vibrations of an elastic stretched string. For more information the reader may consult Chipot[2] and Lions[8], [9].

With respect to the problem

$$\begin{cases} -M(\|u\|^2)\Delta u &= f(u) & \text{in} \quad \Omega, \\ u &= 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

we have to mention that this kind of singular equation has not yet been considered.

Since we allow the function M to attain negative values, the best way to treat this problem is to use the Galerkin Method, like it was done [5]. This application of the Galerkin Method relies on a variant of the Brouwer Fixed Point Theorem which is established below. The proof can be found in Lions[8], p. 53.

Proposition 1.1. *Suppose that $F : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function such that $< F(\xi), \xi > \geq 0$ on $|\xi| = r$, where $< \cdot, \cdot >$ is the usual inner product in $\mathbb{R}^m$ and $|\cdot|$ its corresponding norm. Then there exists $\xi_0 \in \overline{B}_r(0)$ such that $F(\xi_0) = 0$.*

Recall that a solution of (1.1) means a weak solution, that is, a function $u \in H_0^1(\Omega)$ such that

$$M(\|u\|^2)\int_\Omega \nabla u \cdot \nabla\varphi = \int_\Omega \left(\frac{h(x)}{u^\gamma} + k(x)u^\alpha\right)\varphi, \text{for all } \varphi \in H_0^1(\Omega).$$

Another result which will play a fundamental role in our approach is a Hardy-Sobolev-type inequality . Let us denote by φ_1 a positive eigenfunction of $(-\Delta, H_0^1(\Omega))$ corresponding to the first eigenvalue λ_1.

Proposition 1.2. (Hardy-Sobolev Inequality) *If $u \in H_0^1(\Omega)$, then $\frac{u}{\varphi_1^\gamma} \in L^q(\Omega)$, where $\frac{1}{q} = \frac{1}{2} - \frac{(1-\gamma)}{N}, 1 \leq \gamma \leq 1$, and there is a constant $C > 0$ such that*

$$\|\frac{u}{\varphi_1^\gamma}\|_{L^q} \leq C\|\nabla\|_{L^2}, \textit{ for all } u \in H_0^1(\Omega). \tag{1.4}$$

In this inequality the extreme case $\gamma = 0$ is the Sobolev imbedding theorem $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, where $2^* = \frac{2N}{N-2}$. The other extreme case $\gamma = 1$ is a fact already observed in Hardy-Littlewood-Polya[7], that the behavior of a function $u \in H_0^1(\Omega)$ near the boundary $\partial\Omega$ is such that $\frac{u}{\varphi_1}$ belongs to $L^2(\Omega)$, see de Figueiredo[6].

Let $M : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

(M_1) There exist $m_0 > 0$ and $\theta_1 > 0$ such that $M(t) \geq m_0$ if $t \geq \theta_1$.

(M_2) $\theta_2 = \sup\{t > 0; M(t) \leq 0\} > 0$.

In view of (M_1) we have that θ_2 is finite. Under these assumptions we state the main result of this paper.

Theorem 1.1. *Let $h, k : \overline{\Omega} \to \mathbb{R}$ be positive and continuous functions, α and γ real numbers belonging to the interval $(0,1)$ and $M : \mathbb{R}^+ \to \mathbb{R}, \mathbb{R}^+ = [0,\infty)$, a continuous function satisfying $(M_1) - (M_2)$. Then problem (1.1) possesses a positive solution.*

2 Proof of Theorem 1.1.

In order to improve the exposition we split the proof of Theorem 1.1 in some lemmas. First, for each fixed number $\epsilon > 0$, let us consider the auxiliary problem

$$\left\{ \begin{array}{rcll} -M(\|u\|^2)\Delta u & = & \frac{h(x)}{(\epsilon+u)^\gamma} + k(x)u^\alpha & \text{in} \quad \Omega \\ u & > & 0 & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega. \end{array} \right. \tag{2.1}$$

Lemma 2.1. *For each fixed $\epsilon > 0$, problem (2.1) possesses a solution u_ϵ.*

Proof. We begin by focusing our attention on the problem

$$\left\{ \begin{array}{rcll} -M^+(\|u\|^2)\Delta u & = & \frac{h(x)}{(\epsilon+|u|)^\gamma} + k(x)|u|^\alpha & \text{in} \quad \Omega \\ u & > & 0 & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{2.2}$$

where $M^+ : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$M^+(t) = \left\{ \begin{array}{ccc} 0 & \text{if} & 0 \leq t \leq \theta_2 \\ M(t) & \text{if} & t > \theta_2. \end{array} \right.$$

Since we are going to use the Galerkin method let us consider $\mathbb{B} = \{\psi_1, \psi_2, \ldots\}$ a Hilbertian basis of $H_0^1(\Omega)$ and for each fixed $m \in \mathbb{N}$, let us denote by $\mathbb{V}_m = span\{\psi_1, \ldots, \psi_m\}$. It is well known that $\mathbb{V}_m$ is isomorphic and isometric to $\mathbb{R}^m$ in the following way: $\mathbb{V}_m \longleftrightarrow \mathbb{R}^m$, $u = \sum_{j=1}^m \xi_j \psi_j \longleftrightarrow \xi = (\xi_1, \ldots, \xi_m)$, $\|u\|^2 = \sum_{j=1}^m \xi_j^2 = |\xi|^2$, where $\|u\|^2 = \int_\Omega |\nabla u|^2$ is the usual norm in $H_0^1(\Omega)$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^m$.

From now on we make the identifications $u \longleftrightarrow \xi$ and $\mathbb{V}_m \longleftrightarrow \mathbb{R}^m$, as above, with no additional comments. In order to use Proposition 1.1 we construct the map $F : \mathbb{R}^m \to \mathbb{R}^m$, by considering the aforementioned identifications, $F(\xi) = (F_1(\xi), \ldots, F_m(\xi))$, as follows:

$$F_i(\xi) = M^+(\|u\|^2)\int_\Omega \nabla u \cdot \nabla\psi_i - \int_\Omega \frac{h(x)\psi_i}{(\epsilon+|u|)^\gamma} - \int_\Omega k(x)|u|^\alpha\psi_i, i = 1, \ldots, m.$$

Thus, denoting by $< \cdot, \cdot >$ the usual inner product in $H_0^1(\Omega)$, one has

$$< F(\xi), \xi >= M^+(\|u\|^2)\|u\|^2 - \int_\Omega \frac{h(x)u}{(\epsilon+|u|)^\gamma} - \int_\Omega k(x)|u|^\alpha u.$$

We note that:

$$\int_\Omega \frac{h(x)u}{(\epsilon+|u|)^\gamma} \le \|h\|_\infty \int_\Omega \frac{|u|}{\epsilon^\gamma} \le C_\epsilon\|u\|$$

and

$$\int_\Omega k(x)|u|^\alpha u \le \|k\|_\infty \int_\Omega |u|^{\alpha+1} \le C\|u\|^{\alpha+1},$$

where C and C_ϵ are constants which do not depend on u and m. This implies

$$< F(\xi), \xi >\ge M^+(\|u\|^2)\|u\|^2 - C_\epsilon\|u\| - C\|u\|^{\alpha+1}.$$

We now take $\|u\|^2 \ge \theta_1$ so that $M^+(\|u\|^2) = M(\|u\|^2) \ge m_0$ and therefore

$$< F(\xi), \xi >\ge m_0\|u\|^2 - C_\epsilon\|u\| - C\|u\|^{\alpha+1},$$

$\|u\|^2 \ge \theta_1$. Thus, if $\|u\| = |\xi| = r$, with r large enough, we have

$$< F(\xi), \xi >> 0,$$

where r does not depend on m. From Proposition 1.1 we find $u_m \in \mathbb{V}, \xi^{(m)} \leftrightarrow u_m, \xi^{(m)} \in \mathbb{R}^m, |\xi^{(m)}| = \|u_m\| \le r$, satisfying $F(u_m) = 0$. Hence

$$M^+(\|u_m\|^2)\int_\Omega \nabla u_m \cdot \nabla\psi_i = \int_\Omega \frac{h(x)\psi_i}{(\epsilon+|u_m|)^\gamma} + \int_\Omega k(x)|u_m|^\alpha\psi_i, i = 1, \ldots, m$$

which yields

$$M^+(\|u_m\|^2)\int_\Omega \nabla u_m \cdot \nabla\psi = \int_\Omega \frac{h(x)\psi}{(\epsilon+|u_m|)^\gamma} + \int_\Omega k(x)|u_m|^\alpha\psi, \forall\psi \in \mathbb{V}_m. \quad (2.3)$$

We now fix $l \le m, \mathbb{V}_l \subset \mathbb{V}_m$, and $\psi \in \mathbb{V}_l$. In view of boundedness of $(\|u_m\|)$, one has $\|u_m\|^2 \to t_0,\ u_m \rightharpoonup u,$ in $H_0^1(\Omega),\ u_m \to u,$ in $L^2(\Omega),\ u_m(x) \to u(x)$ a.e. in Ω, perhaps for subsequences. Thus

$$M^+(\|u_m\|^2) \to M^+(t_0),$$

because M^+ is continuous, and

$$\int_\Omega \nabla u_m \cdot \nabla\psi \to \int_\Omega \nabla u \cdot \nabla\psi \text{ as } m \to \infty,$$

for all $\psi \in \mathbb{V}_l$.

Furthermore

$$\left|\frac{h(x)\psi}{(\epsilon+|u_m|)^\gamma}\right| \le \frac{C}{\epsilon^\gamma}|\psi| \in L^1(\Omega), \text{ for all } m \in \mathbb{N},$$

$$\frac{h(x)\psi}{(\epsilon+|u_m|)^\gamma} \to \frac{h(x)\psi}{(\epsilon+|u|)^\gamma} \text{ a.e. in } \Omega$$

and by invoking the Lebesgue Dominated Convergence Theorem, we get

$$\int_\Omega \frac{h(x)\psi}{(\epsilon+|u_m|)^\gamma} \to \int_\Omega \frac{h(x)\psi}{(\epsilon+|u|)^\gamma} \text{ for all } \psi \in \mathbb{V}_l.$$

To prove that

$$\int_\Omega k(x)|u_m|^\alpha\psi \to \int_\Omega k(x)|u|^\alpha\psi \text{ for all } \psi \in \mathbb{V}_l,$$

we proceed in the following way: $u_m \rightharpoonup u$ in $H_0^1(\Omega)$, which implies $u_m \to u$ in $L^1(\Omega)$, $|u_m| \to |u|$ in $L^1(\Omega), |u_m|^\alpha \to |u|^\alpha$ in $L^{\frac{1}{\alpha}}(\Omega)$, because the mapping $L^1(\Omega) \to L^{\frac{1}{\alpha}}(\Omega), |u| \mapsto |u|^\alpha$, is well defined, hence continuous. Also, because $\frac{1}{\alpha} > 1$ one has $L^{1/\alpha}(\Omega) \hookrightarrow L^1(\Omega)$ and such a immersion is continuous. Consequently, $|u_m|^\alpha \to |u|^\alpha$ in $L^1(\Omega)$.

Hence

$$\left|\int_\Omega k(x)|u_m|^\alpha\psi - \int_\Omega k(x)|u|^\alpha\psi\right| = \left|\int_\Omega k(x)[|u_m|^\alpha - |u|^\alpha]\right| \le$$

$$\int_\Omega |k(x)|||u_m|^\alpha - |u|^\alpha||\psi|.$$

We recall that ψ is a linear combination of $\psi_1, \ldots, \psi_m$ and each $\psi_i, i = 1, \ldots, m$. Therefore ψ is continuous because we may take ψ_i as eigenfunctions of $(-\Delta, H_0^1(\Omega))$. Thus

$$\left|\int_\Omega k(x)|u_m|^\alpha\psi - \int_\Omega k(x)|u|^\alpha\psi\right| \le C\int_\Omega ||u_m|^\alpha - |u|^\alpha| \to 0,$$

where C is a positive constant. Taking limits on both sides of the equality (2.3) one gets

$$M^+(t_0)\int_\Omega \nabla u \cdot \nabla\psi = \int_\Omega \frac{h(x)\psi}{(\epsilon+|u|)^\gamma} + \int_\Omega k(x)|u|^\alpha\psi, \tag{2.4}$$

for all $\psi \in \mathbb{V}_l$. Since $l \in \mathbb{N}$ is arbitrary, equality (2.4) remains valid for all $\psi \in H_0^1(\Omega)$ and so

$$M^+(t_0)\int_\Omega \nabla u \cdot \nabla\psi = \int_\Omega \frac{h(x)\psi}{(\epsilon+|u|)^\gamma} + \int_\Omega k(x)|u|^\alpha\psi, \tag{2.5}$$

for all $\psi \in H_0^1(\Omega)$. In view of this one has that $M^+(t_0) > 0$ and so $M^+(t_0) = M(t_0)$ which implies

$$M(t_0)\|u\|^2 = \int_\Omega \frac{h(x)u}{(\epsilon+|u|)^\gamma} + \int_\Omega k(x)|u|^\alpha u, \tag{2.6}$$

We now take $u_m = \psi$ in equation (2.3) to obtain

$$M^+(\|u_m\|^2)\|u_m\|^2 = \int_\Omega \frac{h(x)u_m}{(\epsilon+|u|)^\gamma} + \int_\Omega k(x)|u_m|^\alpha u_m, \tag{2.7}$$

and taking limits on both sides of this last equation, one gets

$$M(t_0)t_0 = \int_\Omega \frac{h(x)u}{(\epsilon + |u|)^\gamma} + \int_\Omega k(x)|u|^\alpha u. \tag{2.8}$$

By comparing equations in (2.6) and (2.8) we conclude that

$$M(t_0)t_0 = M(t_0)\|u\|^2$$

and because $M(t_0) > 0$ we have that $M(t_0) = M(\|u\|^2)$ which implies that u is a solution of the auxiliary problem (2.1). This concludes the proof of Lemma 2.1. ∎

For each $n \in \mathbb{N}$ set $\epsilon = \frac{1}{n}$ and $u_{\frac{1}{n}} = u_n$ where $u_{\frac{1}{n}}$ is obtained in the preceding lemma.

Lemma 2.2. *There is $\delta > 0$ such that $M(\|u_n\|^2) \geq \delta > 0$, for all $n \in \mathbb{N}$.*

Proof. We reason by contradiction. Suppose that $\liminf M(\|u_n\|^2) = 0$. If this is the case we infer that $(\|u_n\|^2)$ is bounded, due to assumption (M_1), and so

$$\|u_n\|^2 \to \theta_0, \; u_n \rightharpoonup u \text{ in } H_0^1(\Omega),$$

perhaps for subsequences. In view of the continuity of M

$$0 = \liminf M(\|u_n\|^2) = \lim M(\|u_n\|^2) = M(\theta_0).$$

We now note that

$$\frac{h(x)}{(1+t)^\gamma} + k(x)t^\alpha \geq C\left[\frac{1}{(1+t)^\gamma} + t^\alpha\right] \geq m_0 > 0,$$

for all $x \in \overline{\Omega}$ and $t \geq 0$, where C is a constant. Since

$$-M(\|u_n\|^2)\Delta u_n \geq m_0 > 0 \text{ in } \Omega,$$

and $M(\|u_n\|^2) > 0$, we may take $\varphi > 0, \varphi \in C_0^1(\overline{\Omega})$, so that

$$M(\|u_n\|^2)\int \nabla u_n \cdot \nabla\varphi \geq m_0 \int_\Omega \varphi > 0$$

which implies $0 \geq m_0 \int_\Omega \varphi > 0$, a contradiction. This completes the proof of Lemma 2.2. ∎

Lemma 2.3. $(\|u_n\|)$ *is bounded.*

Proof. Indeed, we have

$$M(\|u_n\|^2)\|u_n\|^2 = \int_\Omega \frac{h(x)u_n}{\left(\frac{1}{n} + u_n\right)^\gamma} + \int_\Omega k(x)u_n^{\alpha+1}.$$

Also

$$\int_\Omega \frac{h(x)u_n}{\left(\frac{1}{n} + u_n\right)^\gamma} \leq \|h\|_\infty \int_\Omega u_n^{1-\gamma} \leq \|h\|_\infty |\Omega|^\gamma \left(\int_\Omega u_n\right)^{1-\gamma} \leq C_1\|u_n\|^{1-\gamma}$$

and

$$\int_\Omega k(x)u_n^{\alpha+1} \leq \|k\|_\infty \int_\Omega u_n^{\alpha+1} \leq \|u_n\|^{\alpha+1},$$

where C_1 and C_2 are constants do not depend on n. Hence

$$\delta\|u_n\|^2 \leq M(\|u_n\|^2)\|u_n\|^2 \leq C_1\|u_n\|^{1-\gamma} + C_2\|u_n\|^{\alpha+1}$$

and because $1-\gamma < 1$ and $1+\alpha < 2$, one has that $(\|u_n\|^2)$ is bounded. Consequently

$$0 < \delta \leq M(\|u_n\|^2) \leq M_\infty, \text{ for all } n = 1, 2, \ldots$$

and the proof of Lemma 2.3 is over. ∎

Lemma 2.4. *The sequence (u_n), obtained in Lemma 2.1, converges to a solution of problem (1.1).*

Proof. As we have seen, the sequence (u_n) is bounded and so $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, $u_n \to u$ in $L^q(\Omega), 1 \leq q < \frac{2N}{N-2}, N \geq 3$, $u_n(x) \to u(x)$ a.e. in Ω, up to subsequences.

We now take $\psi_1 > 0$ an eigenfunction of $(-\Delta, H_0^1(\Omega))$ associated to the first eigenvalue λ_1, in such a way that

$$m_0 > \lambda_1 M_\infty \psi_1(x), \text{ for all } x \in \overline{\Omega},$$

where m_0 and M_∞ were introduced, respectively, in Lemmas 2.2 and 2.3.

$$\begin{cases} -M(\|u\|^2)\Delta u_n & = & \frac{h(x)}{\left(\frac{1}{n}+u_n\right)^\gamma} & + & k(x)u_n^\alpha & \text{in} & \Omega, \\ & \geq & \frac{h(x)}{(1+u_n)^\gamma} & + & k(x)u_n^\alpha & \text{in} & \Omega, \\ & \geq & C\left[\frac{1}{(1+u_n)^\gamma} + u_n^\alpha\right] & & & \text{in} & \Omega, \\ & \geq & m_0 & > & \lambda_1 M_\infty \psi_1 & \text{in} & \Omega, \\ u_n & = & \psi_1 = 0 & & & \text{on} & \partial\Omega. \end{cases}$$

Thus

$$\begin{cases} -\Delta(M(\|u_n\|^2 u_n)) & > & -\Delta(M_\infty\psi_1) & \text{in} & \Omega, \\ M(\|u_n\|^2)u_n & = & M_\infty\psi_1 = 0 & \text{on} & \partial\Omega, \end{cases}$$

and by invoking the maximum principle, we get

$$M(\|u_n\|^2)u_n > M_\infty\psi_1 \text{ in } \Omega$$

and so

$$u_n(x) > \frac{M_\infty}{M(\|u_n\|^2)}\psi_1(x) \text{ in } \Omega.$$

Let us show that $u_n \to u$ in $H_0^1(\Omega)$. Since

$$-M(\|u_n\|^2)\Delta u_n = \frac{h(x)}{\left(\frac{1}{n}+u_n\right)^\gamma} + k(x)u_n^\gamma \text{ in } \Omega,$$

we take u_n as a test function in order to obtain

$$M(\|u_n\|^2)\int|\nabla u_n|^2 = \int_\Omega \frac{h(x)u_n}{\left(\frac{1}{n}+u_n\right)^\gamma} + \int_\Omega k(x)u_n^{\alpha+1}.$$

Let us estimate the two integrals in the right-hand side of the last equality:

$$\int_\Omega \frac{h(x)u_n}{\left(\frac{1}{n}+u_n\right)^\gamma} \le \|h\|_\infty \int_\Omega \frac{|u_n|}{u_n^\gamma} \le \frac{\|h\|_\infty}{C}\int_\Omega \frac{|u_n|}{\psi_1^\gamma} \le C'\|u_n\|,$$

where in the last expression we used the Hardy-Sobolev inequality, and

$$\int_\Omega k(x)u_n^\alpha u_n \le \|k\|_\infty \int_\Omega u_n^\alpha u_n \le C\|u_n\|^{\alpha+1},$$

which implies

$$\delta\|u_n\|^2 \le C\|u_n\| + C'\|u_n\|^{1+\alpha}$$

and, since $0<\alpha<1$, we conclude that the real sequence $(\|u_n\|)$ is bounded. We obtain the following convergence, perhaps for subsequences,

$$\|u_n\|^2 \to t_0 \Rightarrow M(\|u_n\|^2) \to M(t_0),$$

$u_n \rightharpoonup u$ in $H_0^1(\Omega)$, $\int_\Omega \nabla u_n \nabla\psi \to \int_\Omega \nabla u\nabla\psi$, $\frac{h(x)\psi}{\left(\frac{1}{n}+u_n\right)^\gamma} \to \frac{h(x)\psi}{u^\gamma}$ a.e. in Ω.

Because

$$\left|\frac{h(x)\psi}{\left(\frac{1}{n}+u_n\right)^\gamma}\right| \le h(x)\left|\frac{\psi}{u_n^\gamma}\right| \le \left|\frac{\psi}{\psi_1^\gamma}\right| \in L^1(\Omega),$$

by Lebesgue Dominated Convergence Theorem, one has

$$\int_\Omega \frac{h(x)\psi}{\left(\frac{1}{n}+u_n\right)^\gamma} \to \int_\Omega \frac{h(x)\psi}{u^\gamma}.$$

We also have

$$\int_\Omega k(x)u_n^\alpha\psi \to \int_\Omega k(x)u^\alpha\psi.$$

Consequently,

$$M(t_0)\int_\Omega \nabla u\nabla\psi = \int_\Omega \frac{h(x)\psi}{u^\gamma} + \int_\Omega k(x)u^\alpha\psi, \text{ for all } \psi\in H_0^1(\Omega).$$

We also note that

$$M(\|u_n\|^2)\|u_n\|^2 = \frac{h(x)u_n}{\left(\frac{1}{n}+u_n\right)^\gamma} + \int_\Omega k(x)u_n^{\alpha+1}.$$

As we have done before, we have

$$\frac{h(x)u_n}{\left(\frac{1}{n}+u_n\right)^\gamma} \to h(x)u^{1-\gamma}$$

$$\int_\Omega k(x)u_n^{\alpha+1} \to \int_\Omega k(x)u^{\alpha+1}$$

and by using again the Lebesgue Dominated Convergence Theorem we get

$$M(t_0)t_0 = \int_\Omega h(x)u^{1-\gamma} + \int_\Omega k(x)u^{\alpha+1} \tag{2.9}$$

But,

$$M(t_0)\|u\|^2 = \int_\Omega h(x)u^{1-\gamma} + \int_\Omega k(x)u^{\alpha+1}. \tag{2.10}$$

Comparing equalities (2.9) and (2.10) we obtain

$$M(t_0)t_0 = M(t_0)\|u\|^2 \Rightarrow \|u\|^2 = t_0$$

because, in view of equality (2.9), $M(t_0) \neq 0$. Then

$$M(\|u\|^2)\int_\Omega \nabla u \nabla \psi = \int_\Omega \frac{h(x)}{u^\gamma}\psi + \int_\Omega k(x)u^\alpha u^\alpha \psi,$$

for all $\psi \in H_0^1(\Omega)$ and so u is a weak solution of problem (1.1). ∎

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