Stability for generalized Jensen functional equations and isomorphisms between C^*-algebras

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Abstract

Let $\mathcal{A}$ be a unital C^*-algebra and let M_1 and M_2 be Banach left $\mathcal{A}$-modules. In this paper, we prove the generalized Hyers-Ulam-Rassias stability for a generalized form,

$$g\Big(\sum_{i=1}^{n} r_i x_i\Big) = \sum_{i=1}^{n} s_i g(x_i)$$

of a Cauchy-Jensen functional equation $2g(\frac{x+y}{2}) = g(x) + g(y)$ for a mapping $g : M_1 \to M_2$. As an application, we show that every approximate C^*-algebra isomorphism $h : \mathcal{A} \to \mathcal{B}$ between unital C^*-algebras is a C^*-algebra isomorphism when h satisfies some regular conditions.

1 Introduction

In 1940, S.M. Ulam [16] raised the following problem: Under what conditions does there exist an additive mapping near an approximately additive mapping?

In 1941, D.H. Hyers [4] gave a first affirmative answer to the question of Ulam for Banach spaces: Let E_1 and E_2 be Banach spaces, $\varepsilon \geq 0$ and let $f : E_1 \to E_2$ satisfy

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

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for all $x, y \in E_1$. Then the limit

$$T(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E_1$ and the mapping $T : E_1 \to E_2$ is the unique additive mapping such that

$$\|f(x) - T(x)\| \le \varepsilon$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the mapping T is linear. Th.M. Rassias [13] succeeded in extending the result of Hyers' theorem by weakening the condition for the Cauchy difference controlled by $\|x\|^p + \|y\|^p$, $p \in [0, 1)$ to be unbounded. Thereafter, P. Găvruta [3] generalized the stability result of Th.M. Rassias to the case of the unbounded mapping φ as follows: Let G be an abelian group, E a Banach space and let $\varphi : G^2 \to [0, \infty)$ be a mapping such that

$$\Phi(x, y) := \sum_{n=0}^{\infty} 2^{-(n+1)} \varphi(2^n(x), 2^n(y)) < \infty$$

for all $x, y \in G$. If a mapping $f : G \to E$ satisfies

$$\|f(x + y) - f(x) - f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$, then there exists a unique additive mapping $T : G \to E$ such that

$$\|f(x) - T(x)\| \le \Phi(x, x)$$

for all $x \in G$.

Let X be a Banach space. Let G be an abelian group and E a subset of G such that $nx \in E$ for any integer n and all $x \in E$, and $2x \ne 0$ and $3x \ne 0$ for all $x \in E \setminus \{0\}$. Assume that $f : E \to X$ is a mapping for which there exists a mapping $\varphi : E \setminus \{0\} \times E \setminus \{0\} \to [0, \infty)$ such that

$$\left\| 2f\left(\frac{x + y}{2}\right) - f(x) - f(y) \right\| \le \varphi(x, y)$$

for all $x, y \in E \setminus \{0\}$ with $\frac{x+y}{2} \in E$. Lee and Jun [9] showed that if the series

$$\tilde{\varphi}(x, y) := \sum_{k=0}^{\infty} 3^{-k} \varphi\Big(3^k x, 3^k y\Big) < \infty$$

for all $x, y \in E \setminus \{0\}$, then there exists a unique additive mapping $T : E \to X$ satisfying the inequality

$$\|f(x) - f(0) - T(x)\| \le 3^{-1} \left(\tilde{\varphi}(x, -x) + \tilde{\varphi}(-x, 3x)\right)$$

for all $x \in E \setminus \{0\}$. A large list of references concerning the stability problem of functional equations can be found in [5, 11, 14].

Th.M. Rassias and J. Tabor [15] asked about the stability problem for the following general linear functional equation

$$g(ax + by + c) = Ag(x) + Bg(y) + C \tag{1.1}$$

with $abAB \neq 0$. The equation (1.1) is generalized to the following equation

$$g\left(\sum_{i=1}^{n} r_i x_i + c\right) = \sum_{i=1}^{n} s_i g(x_i) + C, \tag{1.2}$$

where at least two of $\{r_i \in \mathcal{R} : i = 1, \cdots, n\}$ are nonzero, $s_i \in \mathcal{R}$, and $c \in M_1$, $C \in M_2$ are vectors. It was shown in [6] that a mapping $g : M_1 \to M_2$ with $g(0) = 0$ satisfies the functional equation (1.2) if and only if the mapping g is Cauchy additive. In this case we obtain that $g(r_i x) = s_i g(x)$ for each $i = 1, \cdots, n$. Moreover, the authors established the generalized Hyers-Ulam-Rassias stability problem for an approximate mapping $g : M_1 \to M_2$ of the functional equation (1.2) in case that $\sum_{i=1}^{n} r_i \neq 0$ and $\sum_{i=1}^{n} s_i \neq 0$ are not simultaneously equal to 1. They asked about the generalized Hyers-Ulam-Rassias stability problem of (1.2) for the case either $\sum_{i=1}^{n} r_i = 0 = \sum_{i=1}^{n} s_i$ or $\sum_{i=1}^{n} r_i = 1 = \sum_{i=1}^{n} s_i$.

Now, let $x_1, x_2, \cdots, x_n (n \geq 2)$ be distinct vectors in a finite dimensional vector space and let $r_i \in (0, \infty)$ be a weight associated with each x_i. We set $N := \sum_{i=1}^{n} r_i$ for the notational convenience. Then for a mean value $M := \frac{\sum_{i=1}^{n} r_i x_i}{N}$ a mapping $g(x) = x$ satisfies a equation $Ng(M) = \sum_{i=1}^{n} r_i g(x_i)$, which yields the following generalized functional equation

$$Ng\left(\frac{\sum_{i=1}^{n} r_i x_i}{N}\right) = \sum_{i=1}^{n} r_i g(x_i) \tag{1.3}$$

of a Cauchy-Jensen functional equation $2g(\frac{x+y}{2}) = g(x) + g(y)$. For much more general functional equation than (1.3), we are going to investigate an approximate mapping of the following functional equation

$$g\left(\sum_{i=1}^{n} r_i x_i\right) = \sum_{i=1}^{n} s_i g(x_i), \tag{1.4}$$

with $\sum_{i=1}^{n} r_i = 1 = \sum_{i=1}^{n} s_i$, where at least two of $\{r_i \in \mathbb{R} : i = 1, \cdots, n\}$ are nonzero, $s_i \in \mathbb{R}$, $n > 1$.

Throughout this paper, let $\mathcal{A}$ be a unital C^*-algebra with norm $|\cdot|$ and let $U(\mathcal{A})$ the unitary group of $\mathcal{A}$, $\mathcal{A}_{in}$ the set of invertible elements in $\mathcal{A}$, $\mathcal{A}_{sa}$ the set of self-adjoint elements in $\mathcal{A}$, $\mathcal{A}_1 := \{a \in \mathcal{A} \mid |a| = 1\}$, $\mathcal{A}_1^+$ the set of positive elements in $\mathcal{A}_1$. Let M_1 and M_2 be Banach left $\mathcal{A}$-modules unless we give any specific reference. Recently, Park [10, 12] applied the stability results to investigate C^*-algebra isomorphisms between unital C^*-algebras. Now, in the present paper we are going to investigate the generalized Hyers-Ulam-Rassias stability problem for the equation (1.4) in Banach modules over a unital C^*-algebra acting on $U(\mathcal{A})$ or $\mathcal{A}_1^+$. These results are applied to investigate C^*-algebra isomorphisms between unital C^*-algebras. By $\mathbb{R}_+$ and $\mathbb{N}$ we denote the sets of nonnegative real numbers and of positive integers, respectively.

2 Stability of (1.4)

In this section, we are going to prove the generalized Hyers-Ulam-Rassias stability for the equation (1.4) where at least two of $\{r_i \in \mathbb{R} : i = 1, \cdots, n\}$ are nonzero, $s_i \in \mathbb{R}$ for all $i = 1, \cdots, n(n > 1)$ and $\sum_{i=1}^{n} r_i = 1 = \sum_{i=1}^{n} s_i$.

Theorem 2.1. *Let $f : M_1 \to M_2$ be a mapping with $f(0) = 0$ for which there exists a mapping $\phi : {M_1}^n \to \mathbb{R}^+$ and an r_ℓ ($\ell \in \{1, \cdots, n\}$) such that*

$$\left\| D_u f(x_1, \cdots, x_n) := f\Big(\textstyle\sum_{i=1}^n r_i u x_i\Big) - \sum_{i=1}^n s_i u f(x_i) \right\| \le \phi(x_1, \cdots, x_n), \quad (2.1)$$

$$\Phi_\ell(x_1, \cdots, x_n) := \textstyle\sum_{j=0}^\infty \frac{1}{|s_\ell|^{j+1}} \phi\left({r_\ell}^j x_1, \cdots, {r_\ell}^j x_n\right) < \infty \quad (2.2)$$

$$\Big(\Phi_\ell(x_1, \cdots, x_n) := \textstyle\sum_{j=1}^\infty |s_\ell|^{j-1} \phi\left({r_\ell}^{-j} x_1, \cdots, {r_\ell}^{-j} x_n\right) < \infty, \ \textit{respectively,}\Big) \quad (2.3)$$

for all $x_1, \cdots, x_n \in M_1$ and all $u \in U(\mathcal{A})$. Then there exists a unique $\mathcal{A}$-linear mapping $g : M_1 \to M_2$ near f, defined by

$$g(x) = \lim_{m\to\infty} {s_\ell}^{-m} f\left({r_\ell}^m x\right), \quad (2.4)$$

$$\Big(g(x) = \lim_{m\to\infty} {s_\ell}^m f({r_\ell}^{-m} x), \ \textit{respectively,}\Big) \quad (2.5)$$

which satisfies the equation (1.4) and the inequality

$$\|f(x) - g(x)\| \le \Phi_\ell(0, \cdots, 0, \underbrace{x}_{\ell-th}, 0, \cdots, 0) \quad (2.6)$$

for all $x \in M_1$.

Proof. Put $x_\ell := {r_\ell}^j x$ and $x_i := 0$ for all $i \neq \ell$ in (2.1). Then, the inequality (2.1) is rewritten in the form

$$\left\| f\left(r_\ell^{j+1} u x\right) - s_\ell u f({r_\ell}^j x) \right\| \le \phi(0, \cdots, 0, \underbrace{{r_\ell}^j x}_{\ell-th}, 0, \cdots, 0) \quad (2.7)$$

for all $x \in M_1$ and all $u \in U(\mathcal{A})$. Define a sequence $f_m : M_1 \to M_2$ by

$$f_m(x) := s_\ell^{-m} f\left({r_\ell}^m x\right), \quad x \in M_1$$

for all $m \in \mathbb{N}$. Then we figure out by (2.7)

$$\begin{aligned} \|f_{j+1}(ux) - u f_j(x)\| &\le |s_\ell|^{-(j+1)} \left\| f\left(r_\ell^{j+1} u x\right) - s_\ell u f({r_\ell}^j x) \right\| \quad (2.8) \\ &\le |s_\ell|^{-(j+1)} \phi(0, \cdots, 0, \underbrace{{r_\ell}^j x}_{\ell-th}, 0, \cdots, 0) \end{aligned}$$

for all $x \in M_1$ and all $u \in U(\mathcal{A})$. Set $u = 1 \in U(\mathcal{A})$ in (2.1). Then it follows by the convergence of (2.2) that for all nonnegative integers k, m with $m > k \ge 0$,

$$\begin{aligned} \|f_m(x) - f_k(x)\| &\le \sum_{j=k}^{m-1} \|f_{j+1}(x) - f_j(x)\| \quad (2.9) \\ &\le \sum_{j=k}^{m-1} |s_\ell|^{-(j+1)} \phi(0, \cdots, 0, \underbrace{{r_\ell}^j x}_{\ell-th}, 0, \cdots, 0) \\ &\to 0 \quad \text{as} \quad k \to \infty, \end{aligned}$$

which shows that the sequence $\{f_m(x)\}_{m\in\mathbb{N}}$ is a Cauchy sequence, and thus converges in M_2. Therefore a mapping $g : M_1 \to M_2$ given by

$$g(x) := \lim_{m\to\infty} s_\ell^{-m} f\left(r_\ell{}^m x\right)$$

is well defined. Now, letting $k = 0$ in (2.9) and letting $m \to \infty$, we get the approximation (2.6) for a mapping g near f.

We prove that the mapping g satisfies the equation (1.4). Replacing x_i by $r_\ell{}^m x_i$ for all $i = 1, \cdots, n$ in (2.7), we get

$$\begin{aligned}\left\| f\Big(r_\ell{}^m\Big(\sum_{i=1}^n r_i x_i\Big)\Big) - \sum_{i=1}^n s_i f(r_\ell{}^m x_i)\right\| &= \left\| f\Big(\sum_{i=1}^n r_i(r_\ell{}^m x_i)\Big) - \sum_{i=1}^n s_i f(r_\ell{}^m x_i)\right\| \\ &\le \phi(r_\ell{}^m x_1, \cdots, r_\ell{}^m x_n)\end{aligned}$$

for all $x_1, \cdots, x_n \in M_1$. Dividing the last inequality by $|s_\ell|^m$ and taking the limit as $m \to \infty$, we see that g satisfies the equation (1.4). Hence, the mapping g is additive satisfying the relation $g(r_i x) = s_i g(x)$ for each $i = 1, \cdots, n$ by [6, Lemma 2.1].

Now, we prove the uniqueness of g satisfying the equation (1.4) and the inequality (2.6). Assume that h is an arbitrary solution of (1.4) such that the mapping $x \mapsto \|f(x) - h(x)\|$ is bounded by the inequality (2.6). Then, it follows by induction that

$$s_\ell{}^{-m} g(r_\ell{}^m x) = g(x), \quad s_\ell{}^{-m} h(r_\ell{}^m x) = h(x)$$

for all $x \in M_1$. Thus for every $x \in M_1$ we figure out by (2.6)

$$\begin{aligned}\|h(x) - f_m(x)\| &= \|s_\ell{}^{-m} h(r_\ell{}^m x) - s_\ell{}^{-m} f(r_\ell{}^m x)\| \\ &\le |s_\ell|^{-m} \|h(r_\ell{}^m x) - f(r_\ell{}^m x)\| \\ &\le \sum_{j=0}^{\infty} \frac{1}{|s_\ell|^{m+j+1}} \phi\left(0, \cdots, 0, \underbrace{r_\ell{}^{m+j} x}_{\ell-th}, 0, \cdots, 0\right).\end{aligned}$$

By passing the limit as $m \to \infty$ in the above inequality, we obtain $h(x) = g(x)$ for all $x \in M_1$. This proves the uniqueness of g.

On the other hand, taking the limit as $j \to \infty$ in (2.8), we get

$$g(ux) - ug(x) = 0 \tag{2.10}$$

for all $x \in M_1$ and all $u \in U(\mathcal{A})$. It is clear that $g(0x) = 0 = 0g(x)$ for all $x \in M_1$. Now, let a be a nonzero element in $\mathcal{A}$ and K a positive integer greater than $4|a|$. Then we have $|\frac{a}{K}| < \frac{1}{4} < 1 - \frac{2}{3}$. By [7, Theorem 1], there exist three elements $u_1, u_2, u_3 \in U(\mathcal{A})$ such that $3\frac{a}{K} = u_1 + u_2 + u_3$. Thus we calculate by (2.10)

$$\begin{aligned}g(ax) &= g\left(\frac{K}{3} \cdot 3\frac{a}{K} x\right) = \left(\frac{K}{3}\right) g(u_1 x + u_2 x + u_3 x) \\ &= \left(\frac{K}{3}\right) \Big(g(u_1 x) + g(u_2 x) + g(u_3 x)\Big) \\ &= \left(\frac{K}{3}\right) (u_1 + u_2 + u_3) g(x) = \left(\frac{K}{3}\right) \cdot 3\frac{a}{K} g(x) = ag(x)\end{aligned}$$

for all $a \in \mathcal{A}$ ($a \neq 0$) and all $x \in M_1$. So the unique additive mapping $g : M_1 \to M_2$ is an $\mathcal{A}$-linear mapping, as desired.

The proof of assertion indicated by parentheses is similarly proved by the inequalities due to (2.7)

$$\|f({r_\ell}^{-m}ux) - s_\ell u f({r_\ell}^{-m-1}x)\| \leq \phi(0, \cdots, 0, \underbrace{{r_\ell}^{-m-1}x}_{\ell-th}, 0, \cdots, 0),$$

$$\|{s_\ell}^m f({r_\ell}^{-m}x) - f(x)\| \leq \sum_{j=1}^{m} |s_\ell|^{j-1}\phi(0, \cdots, 0, \underbrace{{r_\ell}^{-j}x}_{\ell-th}, 0, \cdots, 0),$$

for all $x \in M_1$, $m \in \mathbb{N}$ and all $u \in U(\mathcal{A})$. The proof is now complete. ■

The following theorem is an alternative result of Theorem 2.1 depending on the action of u in $D_u f$.

Theorem 2.2. *Let $f : M_1 \to M_2$ be a mapping with $f(0) = 0$ for which there exists a mapping $\phi : {M_1}^n \to \mathbb{R}^+$ and an r_ℓ ($\ell \in \{1, \cdots, n\}$) such that*

$$\|D_u f(x_1, \cdots, x_n)\| \leq \phi(x_1, \cdots, x_n),$$

$$\Phi_\ell(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} \frac{1}{|s_\ell|^{j+1}} \phi\left({r_\ell}^j x_1, \cdots, {r_\ell}^j x_n\right) < \infty$$

$$\left(\Phi_\ell(x_1, \cdots, x_n) := \sum_{j=1}^{\infty} |s_\ell|^{j-1} \phi\left({r_\ell}^{-j} x_1, \cdots, {r_\ell}^{-j} x_n\right) < \infty, \text{ respectively,}\right)$$

for all $x_1, \cdots, x_n \in M_1$ and all $u \in {\mathcal{A}_1}^+ \cup \{i\}$. If f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M_1$, then there exists a unique $\mathcal{A}$-linear mapping $g : M_1 \to M_2$, defined by

$$g(x) = \lim_{m\to\infty} {s_\ell}^{-m} f\left({r_\ell}^m x\right), \quad \left(g(x) = \lim_{m\to\infty} {s_\ell}^{m} f\left({r_\ell}^{-m} x\right), \text{ respectively,}\right)$$

which satisfies the equation (1.4) and the inequality

$$\|f(x) - g(x)\| \leq \Phi_\ell(0, \cdots, 0, \underbrace{x}_{\ell-th}, 0, \cdots, 0)$$

for all $x \in M_1$.

Proof. By the same reasoning as the proof of Theorem 2.1, it follows from $u = 1 \in {\mathcal{A}_1}^+ \cup \{i\}$ in (2.1) that there exists a unique additive mapping $g : M_1 \to M_2$, defined by (2.4), which satisfies the equation (1.4) and the inequality (2.6). Under the assumption that f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M_1$, the additive mapping g satisfies $g(tx) = tg(x)$ for all $t \in \mathbb{R}$ and each fixed $x \in M_1$. That is, g is $\mathbb{R}$-linear [5].

Next, it follows from (2.10) subject to $u \in {\mathcal{A}_1}^+ \cup \{i\}$ that $g(ax) = g\left(|a|\frac{a}{|a|} \cdot x\right) = |a| g\left(\frac{a}{|a|} \cdot x\right) = |a| \left(\frac{a}{|a|}\right) \cdot g(x) = a g(x)$ for all nonzero $a \in \mathcal{A}^+ \cup \{i\}$ and all $x \in M_1$. Now, for any element $a \in A$, $a = a_1 + i a_2$, where $a_1 := \frac{a+a^*}{2} \in \mathcal{A}_{sa}$ and $a_2 := \frac{a-a^*}{2i} \in$

$\mathcal{A}_{sa}$ are self-adjoint elements; furthermore, $a = a_1^+ - a_1^- + ia_2^+ - ia_2^-$, where a_1^+, a_1^-, a_2^+ and a_2^- are all positive elements (see [2, Lemma 38.8]). Thus we obtain

$$\begin{aligned} g(ax) &= g\Big(a_1^+x - a_1^-x + ia_2^+x - ia_2^-x\Big) \\ &= g(a_1^+x) - g(a_1^-x) + ig(a_2^+x) - ig(a_2^-x) \\ &= \Big(a_1^+ - a_1^- + ia_2^+ - ia_2^-\Big)g(x) \\ &= ag(x) \end{aligned}$$

for all $a \in \mathcal{A}$ and all $x \in M_1$. Thus the additive mapping g is $\mathcal{A}$-linear, as desired. The proof of the theorem is complete. ∎

As an application, we obtain the generalized Hyers-Ulam-Rassias stability of the equation (1.3), where $N := \sum_{i=1}^n r_i$.

Corollary 2.3. *Let $f : M_1 \to M_2$ be a mapping with $f(0) = 0$ for which there exists a mapping $\varphi : {M_1}^n \to \mathbb{R}^+$ such that*

$$\left\| Nf\left(\frac{\sum_{i=1}^n r_i u x_i}{N}\right) - \sum_{i=1}^n r_i u f(x_i) \right\| \le \varphi(x_1, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in M_1$ and all $u \in U(\mathcal{A})$. Assume that the following series

$$\Phi(x_1, \cdots, x_n) := \sum_{j=0}^{\infty} \left(\frac{N}{r_\ell}\right)^j \varphi\left(\left(\frac{r_\ell}{N}\right)^j x_1, \cdots, \left(\frac{r_\ell}{N}\right)^j x_n\right)$$

$$\left(\Phi(x_1, \cdots, x_n) := \sum_{j=1}^{\infty} \left(\frac{r_\ell}{N}\right)^j \varphi\left(\left(\frac{N}{r_\ell}\right)^j x_1, \cdots, \left(\frac{N}{r_\ell}\right)^j x_n\right), \text{ respectively,}\right)$$

converges for some $\ell = 1, \cdots, n$ and all $x_1, \cdots, x_n \in M_1$. Then there exists a unique $\mathcal{A}$-linear mapping $g : M_1 \to M_2$, defined by

$$g(x) := \lim_{m\to\infty} \left(\frac{N}{r_\ell}\right)^m f\left(\left(\frac{r_\ell}{N}\right)^m x\right),$$

$$\left(g(x) := \lim_{m\to\infty} \left(\frac{r_\ell}{N}\right)^m f\left(\left(\frac{N}{r_\ell}\right)^m x\right), \text{ respectively,}\right)$$

which satisfies the equation (1.4) and the inequality

$$\|f(x) - g(x)\| \le \frac{1}{r_\ell}\Phi(0, \cdots, 0, \underbrace{x}_{\ell-th}, 0, \cdots, 0)$$

for all $x \in M_1$.

Proof. We observe that

$$\left\| f\left(\sum_{i=1}^n \frac{r_i}{N} u x_i\right) - \sum_{i=1}^n \frac{r_i}{N} u f(x_i) \right\| \le \frac{1}{N}\varphi(x_1, \cdots, x_n)$$

for all $x_1, \cdots, x_n \in M_1$ and all $u \in U(\mathcal{A})$. Define $\phi(x_1, \cdots, x_n) := \frac{1}{N}\varphi(x_1, \cdots, x_n)$, and apply Theorem 2.1 (Theorem 2.2, respectively) to obtain the conclusion. ∎

In the following we consider a mapping H satisfying some specific conditions. In particular, we obtain a special case of it if $\varphi(\lambda) := \lambda^p$ and H is a homogeneous mapping of degree $p > 0$ with $|r_\ell|^p \neq |s_\ell|$.

Corollary 2.4. *Let $f : M_1 \to M_2$ be a mapping with $f(0) = 0$ and a mapping $H : {\mathbb{R}_+}^n \to \mathbb{R}_+$ satisfy*

$$\left\| f\Big(\sum_{i=1}^{n} r_i u x_i\Big) - \sum_{i=1}^{n} s_i u f(x_i) \right\| \leq H(\|x_1\|, \cdots, \|x_n\|)$$

for all $x_1, \cdots, x_n \in M_1$ and all $u \in U(\mathcal{A})$. Assume that there exists a mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

(i) $\varphi(|r_\ell|) \neq |s_\ell|$ *for some* ℓ, *and* $\varphi(\lambda) > 0$ *for all* $\lambda > 0$,

(ii) $\varphi\left(\lambda\Big|r_\ell\Big|\right) = \varphi(\lambda)\varphi\left(\Big|r_\ell\Big|\right)$ *for all* $\lambda > 0$,

(iii) $H(\lambda t_1, \cdots, \lambda t_n) \leq \varphi(\lambda) H(t_1, \cdots, t_n)$ *for all* $t_i \in \mathbb{R}_+$, *and all* $\lambda > 0$.

Then there exists a unique $\mathcal{A}$-linear mapping $g : M_1 \to M_2$ which satisfies the equation (1.4) and the inequality

$$\|f(x) - g(x)\| \leq \frac{H(0, \cdots, 0, \overbrace{\|x\|}^{\ell-th}, 0, \cdots, 0)}{\Big||s_\ell| - \varphi(|r_\ell|)\Big|}$$

for all $x \in M_1$. The mapping g is defined by

$$g(x) := \begin{cases} \lim_{m\to\infty} {s_\ell}^{-m} f\left({r_\ell}^m x\right), & \text{if } |s_\ell| > \varphi(|r_\ell|) \\ \lim_{m\to\infty} {s_\ell}^{m} f\left({r_\ell}^{-m} x\right), & \text{if } |s_\ell| < \varphi(|r_\ell|) \end{cases}$$

for all $x \in M_1$.

It follows by condition (ii) of Corollary 2.4 that $\varphi\left(|r_\ell|^j\right) = \varphi\left(|r_\ell|\right)^j$ for any integer j. We obtain the Hyers-Ulam stability problem for the equation (1.4) as a corollary.

Corollary 2.5. *Assume that there exist constants $\varepsilon \geq 0$ and s_ℓ with $0 < |s_\ell| \neq 1$ for some $\ell = 1, \cdots, n$ for which a mapping $f : M_1 \to M_2$ with $f(0) = 0$ satisfies*

$$\left\| f\left(\sum_{i=1}^{n} r_i u x_i\right) - \sum_{i=1}^{n} s_i u f(x_i) \right\| \leq \varepsilon$$

for all $(x_1, \cdots, x_n) \in {M_1}^n$ and all $u \in U(\mathcal{A})$. Then there exists a unique $\mathcal{A}$-linear mapping $g : M_1 \to M_2$ satisfying the equation (1.4) and the inequality

$$\|g(x) - f(x)\| \leq \frac{\varepsilon}{\Big||s_\ell| - 1\Big|}$$

for all $x \in M_1$. The mapping g is defined by (2.4) if $|s_\ell| > 1$, and by (2.5) if $0 < |s_\ell| < 1$.

3 C^*-algebra isomorphisms between unital C^*-algebras

Throughout this section, assume that $r_i = s_i$ are all rational numbers for all $i = 1, \cdots, n$. Assume that $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras. As an application, we are going to investigate C^*-algebra isomorphisms between unital C^*-algebras. We denote $\mathbb{N}_0$ by the set of nonnegative integers.

Theorem 3.1. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping with $h(0) = 0$ for which there exist mappings $\phi : \mathcal{A}^n \to \mathbb{R}^+$ satisfying (2.2), $\psi_1 : \mathcal{A} \times \mathcal{A} \to \mathbb{R}^+$, and $\psi : \mathcal{A} \to \mathbb{R}^+$ such that*

$$\left\| h\left(\sum_{i=1}^{n} r_i \lambda x_i\right) - \sum_{i=1}^{n} r_i \lambda h(x_i) \right\| \leq \phi(x_1, \cdots, x_n), \tag{3.1}$$

$$\|h({r_\ell}^m ux) - h({r_\ell}^m u)h(x)\| \leq \psi_1({r_\ell}^m u, x), \tag{3.2}$$

$$\|h({r_\ell}^m u^*) - h({r_\ell}^m u)^*\| \leq \psi({r_\ell}^m u) \tag{3.3}$$

for all $\lambda \in S^1 := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$, all $u \in U(\mathcal{A})$, all $x, x_1, \cdots, x_n \in \mathcal{A}$ and all $m \in \mathbb{N}_0$. Assume that

$$\lim_{m\to\infty} {r_\ell}^{-m} \psi_1({r_\ell}^m u, x) = 0, \quad \textit{for all} \quad u \in U(\mathcal{A}), x \in \mathcal{A}, \tag{3.4}$$

$$\lim_{m\to\infty} {r_\ell}^{-m} \psi({r_\ell}^m u) = 0, \quad \textit{for all} \quad u \in U(\mathcal{A}), \tag{3.5}$$

$$\lim_{m\to\infty} {r_\ell}^{-m} h({r_\ell}^m u_0) \in \mathcal{A}_{in}, \quad \textit{for some} \quad u_0 \in \mathcal{A}. \tag{3.6}$$

Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof. Consider the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ as Banach left modules over the unital C^*-algebra $\mathbb{C}$. We note that $S^1 = U(\mathbb{C})$. By Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$, defined by $H(x) := \lim_{m\to\infty} {r_\ell}^{-m} h({r_\ell}^m x)$, satisfying the inequality

$$\|h(x) - H(x)\| \leq \sum_{j=0}^{\infty} \frac{1}{|r_\ell|^{j+1}} \phi(0, \cdots, 0, \underbrace{{r_\ell}^j x}_{\ell-th}, 0, \cdots, 0)$$

for all $x \in \mathcal{A}$.

By (3.3) and (3.5), we have

$$\begin{aligned} H(u^*) &= \lim_{m\to\infty} {r_\ell}^{-m} h({r_\ell}^m u^*) = \lim_{m\to\infty} {r_\ell}^{-m} h({r_\ell}^m u)^* \\ &= \left(\lim_{m\to\infty} {r_\ell}^{-m} h({r_\ell}^m u)\right)^* = H(u)^* \end{aligned} \tag{3.7}$$

for all $u \in U(\mathcal{A})$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements ([8, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^{m} c_j u_j \quad (c_j \in \mathbb{C}, u_j \in U(\mathcal{A}))$, we get by (3.7)

$$\begin{aligned} H(x^*) &= H\left(\sum_{j=1}^{m} \bar{c_j} u_j^*\right) = \sum_{j=1}^{m} \bar{c_j} H(u_j^*) = \sum_{j=1}^{m} \bar{c_j} H(u_j)^* = \left(\sum_{j=1}^{m} c_j H(u_j)\right)^* \\ &= H\left(\sum_{j=1}^{m} c_j u_j\right)^* = H(x)^* \end{aligned}$$

for all $x \in \mathcal{A}$.

Using the relations (3.2) and (3.4), we get

$$\begin{aligned} H(ux) &= \lim_{m\to\infty} {r_\ell}^{-m} h\left({r_\ell}^m ux\right) && (3.8)\\ &= \lim_{m\to\infty} {r_\ell}^{-m} h\left({r_\ell}^m u\right) h(x) = H(u)h(x) \end{aligned}$$

for all $u \in U(\mathcal{A})$ and all $x \in \mathcal{A}$. On the other hand, it follows from (3.8) and the additivity of H that the equation

$$\begin{aligned} H(ux) &= {r_\ell}^{-m} H\left({r_\ell}^m ux\right) = {r_\ell}^{-m} H\left(u{r_\ell}^m x\right) \\ &= {r_\ell}^{-m} H(u) h\left({r_\ell}^m x\right) = H(u){r_\ell}^{-m} h\left({r_\ell}^m x\right) \end{aligned}$$

holds for all $u \in U(\mathcal{A})$ and all $x \in \mathcal{A}$. Taking the limit as $m \to \infty$ in the last equation, we obtain

$$H(ux) = H(u)H(x) \tag{3.9}$$

for all $u \in U(\mathcal{A})$ and all $x \in \mathcal{A}$. Now, let $z \in \mathcal{A}$ be an arbitrary element. Then $z = \sum_{j=1}^m c_j u_j \quad (c_j \in \mathbb{C}, u_j \in U(\mathcal{A}))$, and it follows from (3.8) that

$$\begin{aligned} H(zx) &= H\left(\sum_{j=1}^m c_j u_j x\right) = \sum_{j=1}^m c_j H(u_j x) = \sum_{j=1}^m c_j H(u_j) h(x) && (3.10)\\ &= H\left(\sum_{j=1}^m c_j u_j\right) h(x) = H(z)h(x) \end{aligned}$$

for all $z, x \in \mathcal{A}$. Similarly, we see from (3.9) that

$$H(zx) = H(z)H(x) \tag{3.11}$$

for all $z, x \in \mathcal{A}$. It follows from (3.10) and (3.11) that

$$H(u_0)H(x) = H(u_0 x) = H(u_0)h(x)$$

for all $x \in \mathcal{A}$. Since $H(u_0) = \lim_{m\to\infty} {r_\ell}^{-m} h\left({r_\ell}^m u_0\right)$ is invertible by assumption, we see that $H(x) = h(x)$ for all $x \in \mathcal{A}$. Hence the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism, as desired. ∎

Theorem 3.2. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and (3.6) for which there exist a mapping $\phi : \mathcal{A}^n \to \mathbb{R}^+$ satisfying (2.2), and mappings ψ_1, ψ such that*

$$\begin{aligned} \left\| h\left(\sum_{i=1}^n r_i \lambda x_i\right) - \sum_{i=1}^n r_i \lambda h(x_i) \right\| &\le \phi(x_1, \cdots, x_n), && (3.12)\\ \|h({r_\ell}^m ux) - h({r_\ell}^m u)h(x)\| &\le \psi_1({r_\ell}^m u, x), && (3.13)\\ \left\|h\left({r_\ell}^m u^*\right) - h\left({r_\ell}^m u\right)^*\right\| &\le \psi\left({r_\ell}^m u\right) && (3.14) \end{aligned}$$

for all $\lambda \in S^1 := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$, all $u \in {\mathcal{A}_1}^+ \cup \{i\}$ and all $x, x_1, \cdots, x_n \in \mathcal{A}$ and all $m \in \mathbb{N}_0$. Assume that

$$\begin{aligned} &\lim_{m\to\infty} {r_\ell}^{-m} \psi_1\left({r_\ell}^m u, x\right) = 0, \quad \textit{for all} \quad u \in {\mathcal{A}_1}^+ \cup \{i\}, \quad \textit{all } x \in \mathcal{A}, && (3.15)\\ &\lim_{m\to\infty} {r_\ell}^{-m} \psi\left({r_\ell}^m u\right) = 0, \quad \textit{for all} \quad u \in {\mathcal{A}_1}^+ \cup \{i\}. && (3.16) \end{aligned}$$

Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof. By Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$, defined by $H(x) := \lim_{m\to\infty} {r_\ell}^{-m} h\left({r_\ell}^m x\right)$, satisfying the functional inequality

$$\|h(x) - H(x)\| \le \sum_{j=0}^{\infty} \frac{1}{|r_\ell|^{j+1}} \phi(0, \cdots, 0, \underbrace{{r_\ell}^j x}_{\ell-th}, 0, \cdots, 0)$$

for all $x \in \mathcal{A}$.

By (3.14) and (3.16), we have $H(u^*) = H(u)^*$ for all $u \in {\mathcal{A}_1}^+ \cup \{i\}$, and so

$$\begin{aligned} H(a^*) &= H\left(|a| \cdot \frac{a^*}{|a|}\right) = |a| H\left(\frac{a^*}{|a|}\right) = \left[|a| H\left(\frac{a}{|a|}\right)\right]^* \\ &= H(a)^* \end{aligned}$$

for all nonzero $a \in \mathcal{A}^+ \cup \{i\}$. Now, for any element $a \in A$, $a = a_1 + ia_2$, where $a_1, a_2 \in \mathcal{A}_{sa}$; furthermore, $a = a_1^+ - a_1^- + ia_2^+ - ia_2^-$, where a_1^+, a_1^-, a_2^+ and a_2^- are all positive elements (see [2, Lemma 38.8]). Since H is $\mathbb{C}$-linear, we figure out

$$\begin{aligned} H(a^*) &= H\left((a_1^+ - a_1^- + ia_2^+ - ia_2^-)^*\right) \\ &= H(a_1^{+*}) - H(a_1^{-*}) + H((ia_2^+)^*) - H((ia_2^-)^*) \\ &= H(a_1^+)^* - H(a_1^-)^* - iH(a_2^+)^* + iH(a_2^-)^* \\ &= \left[H(a_1^+ - a_1^- + ia_2^+ - ia_2^-)\right]^* = H(a)^* \end{aligned}$$

for all $a \in \mathcal{A}$.

Using (3.13) and (3.15) we get $H(ux) = H(u)h(x)$ for all $u \in {\mathcal{A}_1}^+ \cup \{i\}$ and all $x \in \mathcal{A}$, and so $H(ax) = H(a)h(x)$ for all $a \in \mathcal{A}^+ \cup \{i\}$ and all $x \in \mathcal{A}$ because

$$\begin{aligned} H(ax) &= H\left(|a| \frac{a}{|a|} \cdot x\right) = |a| H\left(\frac{a}{|a|} \cdot x\right) \\ &= |a| H\left(\frac{a}{|a|}\right) \cdot h(x) = H(a)h(x), \quad \forall a \in \mathcal{A}^+. \end{aligned} \tag{3.17}$$

Now, for any element $a \in A$, $a = a_1^+ - a_1^- + ia_2^+ - ia_2^-$, where a_1^+, a_1^-, a_2^+ and a_2^- are positive elements (see [2, Lemma 38.8]). Thus we calculate by (3.17) and the additivity of H

$$\begin{aligned} H(ax) &= H\Big(a_1^+ x - a_1^- x + ia_2^+ x - ia_2^- x\Big) \\ &= H(a_1^+ x) - H(a_1^- x) + iH(a_2^+ x) - iH(a_2^- x) \\ &= \Big(H(a_1^+) - H(a_1^-) + iH(a_2^+) - iH(a_2^-)\Big) h(x) \\ &= H(a)h(x) \end{aligned} \tag{3.18}$$

for all $a, x \in \mathcal{A}$. By (3.18) and the additivity of H, one has

$$\begin{aligned} H(ax) &= {r_\ell}^{-m} H\left({r_\ell}^m ax\right) = {r_\ell}^{-m} H\left(a {r_\ell}^m x\right) \\ &= {r_\ell}^{-m} H(a) h\left({r_\ell}^m x\right) = H(a) {r_\ell}^{-m} h\left({r_\ell}^m x\right), \end{aligned}$$

which yields by taking the limit as $m \to \infty$

$$H(ax) = H(a)H(x) \tag{3.19}$$

for all $a, x \in \mathcal{A}$.

It follows from (3.18) and (3.19) that for a given u_0 subject to (3.6)

$$H(u_0)H(x) = H(u_0 x) = H(u_0)h(x)$$

for all $x \in \mathcal{A}$. Since $H(u_0) = \lim_{m\to\infty} {r_\ell}^{-m} h\left({r_\ell}^m u_0\right) \in \mathcal{A}_{in}$, we see that $H(x) = h(x)$ for all $x \in \mathcal{A}$. Hence the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism, as desired. ■

Theorem 3.3. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping with $h(0) = 0$ satisfying (2.2), (3.2) and (3.3) such that*

$$\|D_\lambda h(x_1, \cdots, x_n)\| \le \phi(x_1, \cdots, x_n) \tag{3.20}$$

holds for $\lambda = 1, i$. Assume that the conditions (3.4), (3.5) and (3.6) are satisfied, and that h is measurable or $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$. Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof. Fix $\lambda = 1$ in (3.20). By the same reasoning as in the proof of Theorem 2.1, there exists a unique additive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying the inequality (2.6). By the assumption that h is measurable or $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, the mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear, that is, $H(tx) = tH(x)$ for all $t \in \mathbb{R}$ and all $x \in \mathcal{A}$ [5, 13]. Put $\lambda = i$ in (3.20). Then applying the same argument to (2.8) as in the proof of Theorem 2.1, we obtain that

$$D_i H(0, \cdots, 0, \underbrace{x}_{\ell-th}, 0, \cdots, 0) = 0,$$

or $H(ix) = iH(x)$, and so for any $\mu = s + it \in \mathbb{C}$

$$\begin{aligned} H(\mu x) &= H(sx + itx) = H(sx) + H(itx) = sH(x) + itH(x) \\ &= (s + it)H(x) = \mu H(x) \end{aligned}$$

for all $x \in \mathcal{A}$. Hence the mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as the proof of Theorem 3.1. ■

Theorem 3.4. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping with $h(0) = 0$ satisfying (2.2), (3.6), (3.13) and (3.14) such that*

$$\|D_\lambda h(x_1, \cdots, x_n)\| \le \phi(x_1, \cdots, x_n) \tag{3.21}$$

holds for $\lambda = 1, i$. Assume that the equations (3.15), (3.16) are satisfied, and that h is measurable or $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$. Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof. The proof is the similar to that of Theorem 3.3. ■

Theorem 3.5. *Let $\mathcal{B}$ be a unital C^*-algebra in which the norm is multiplicative. Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping with $h(0) = 0$ for which there exist a constant $\delta \geq 0$ and a mapping $\phi : \mathcal{A}^n \to \mathbb{R}^+$ satisfying (2.2) ((2.3), respectively), such that*

$$\begin{aligned}
&\left\| h\left(\sum_{i=1}^{n} r_i \lambda x_i\right) - \sum_{i=1}^{n} r_i \lambda h(x_i)\right\| \leq \phi(x_1, \cdots, x_n), \\
&\|h(xy) - h(x)h(y)\| \leq \delta, \\
&\|h\left({r_\ell}^m u^*\right) - h\left({r_\ell}^m u\right)^*\| \leq \phi\left({r_\ell}^m u, \cdots, {r_\ell}^m u\right) \\
&\left(\left\|h\left({r_\ell}^{-m} u^*\right) - h\left({r_\ell}^{-m} u\right)^*\right\| \leq \phi\left({r_\ell}^{-m} u, \cdots, {r_\ell}^{-m} u\right), \text{ respectively,}\right)
\end{aligned} \tag{3.22}$$

for all $\lambda \in S^1 := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$, all $u \in U(\mathcal{A})$, all $x, y, x_1, \cdots, x_n \in \mathcal{A}$ and all $m \in \mathbb{N}_0$. Assume that

$$\begin{aligned}
&\lim_{m\to\infty} {r_\ell}^{-m} h\left({r_\ell}^m u_0\right) \in \mathcal{A}_{in}, \quad \text{for some} \quad u_0 \in \mathcal{A} \\
&\left(\lim_{m\to\infty} {r_\ell}^{m} h\left({r_\ell}^{-m} u_0\right) \in \mathcal{A}_{in}, \quad \text{for some} \quad u_0 \in \mathcal{A}, \text{ respectively}\right).
\end{aligned}$$

Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof. It follows from (3.22) that the mapping h either is bounded or satisfies the equation $h(xy) = h(x)h(y)$ [1, 5]. Utilizing Theorem 3.1 with $\psi_1 := 0$, we have the desired result. ∎

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