A mathematical framework for Dirac's calculus

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Abstract

The observation and the discussion of the physical reality of phenomena, leads to bring out concepts which have to be described in a non ambiguous mathematical language. Concerning Dirac's calculus we shall introduce, besides the usual definitions for the concepts of point, number, function etc ... , additional concepts for the physical point, the physical equalities, physical infinities and infinitesimals ... etc ... In particular we introduce a new equality $\stackrel{D}{=}$, called Dirac-equality, which differs as well from the classical equality as from the weak equalities introduced in various theories of generalized functions. All these definitions are based on a definition in the language of Relative Set Theory, see [15], of the metaconcepts of improperness used by P.A.M. Dirac in [Di], when he claimed " *Strictly of course,* $\delta(x)$ *is not a proper function of* x*, ... , ...* $\delta'(x)$*,* $\delta''(x)$*.... are even more discontinuous and less proper than* $\delta(x)$ *itself*". We defined this way a concept of observed derivative which extends the usual one to a large class of *discontinuous* possibly non-standard functions. All the multiplications of improper or very improper functions, including the delta-functions and their observed derivatives, are obviously allowed. Now the problem of the multiplication is replaced by another one: under which conditions is the Dirac-equality of two functions preserved by a multiplication term by term?

1 Basic language and definitions

We will indicate first how to define the basic vocabulary in order to develop consistently a discourse similar to that of physicists. More precisely, we want to introduce the words *improper*, *very improper* etc .... and elements of vocabulary which play the same role in reasoning that the constants dx, dy, etc ... and which generalize them by specifying degrees in infinitesimality . All this can be defined precisely

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in the language of Relative Set Theory, **RST**. This language is built on two binary predicates : The usual predicate $\ll \cdot \ \in \ \cdot \gg$ and a new predicate denoted $\ll \cdot \ \mathcal{SR} \ \cdot \gg$.
Let us summarize some rudiments concerning **RST**. (See [15], for further details.)

(a) $\ll x \ \mathcal{SR} \ \ y \gg$ is to be read : x *is standard relatively to* y.

(b) Any set defined with uniqueness in the classical mathematics is standard relatively to any other set : we summarize saying that it is *standard*.

(c) The collection of sets is totally preordered by the predicate $\ll \cdot \mathcal{SR} \cdot \gg$. There are classes of equistandardness. The class of equistandardness of a is the collection $\alpha = [a]$ of x such that ($x \ \mathcal{SR} \ a \ \wedge \ a \ \mathcal{SR} \ x$). We say that x is ${}^{[a]}$standard, or ${}^{\alpha}$standard if $x \ \mathcal{SR} \ a$. Then we use the denotation ${}^{[a]}st(x)$ or ${}^{\alpha}st(x)$. Each class of equistandardness has infinitely many members, there are infinitely many classes.

The theory **RST** is a conservative extension of the usual theory **ZFC**, the E.Nelson's **IST** (see [12]) is a subtheory of **RST**.
E.Gordon also introduced in [7] a binary relation of relative standardness, but his relation is defined inside IST.

1.1 Definition in RST of the levels of improperness.

Let us define finitely many level of improperness **1, 2, ... w**, where **w** is an integer we can choose as large as we need (for example $\mathbf{w} = 100$).
For that we will fix distinct classes of equistandardness

$$\alpha_0 = [0], \ \alpha_1 = [a_1], \alpha_2 = [a_2], \ .... \ \alpha_{\mathbf{w}} = [a_{\mathbf{w}}],$$

such that the first class α_0 is the standard one and

$$a_1 \ \mathcal{SR} \ a_2, \ a_2 \ \mathcal{SR} \ a_3, \ \cdots \ a_{\mathbf{w}-1} \ \mathcal{SR} \ a_{\mathbf{w}}.$$

If $1 \leqq n \leqq \mathbf{w}$, we say that a number or a function X is ^{n}improper or that X has the level n of improperness, and we denote ${}^{n}imp(X)$, if

$${}^{\alpha_n}st(X) \ \& \ \neg \ [{}^{\alpha_{n-1}}st(X)].$$

In order to extend the former definition we will write

$${}^{0}imp(X) \iff st(X), \quad {}^{\mathbf{w}+1}imp(X) \iff \neg[{}^{\alpha_{\mathbf{w}}}st(X)].$$

We denote $\eta(X)$ the level of improperness of X.

Definition of a finite sequence of infinitesimalities.

We use the abbreviations :

$\forall^\alpha x\ \Phi(x)$ for $\forall x\ {}^\alpha st(x) \ \Rightarrow\ \Phi(x)$, $\quad \exists^\alpha x\ \Phi(x)$ for $\exists x\ (\ {}^\alpha st(x)\ \&\ \Phi(x)\ )$.

We define binary relations $\cdot \sim\ \cdot$ and $\cdot \overset{n}{\sim} \cdot$ on $\mathbb{R}$ $(1 < n \leqq \mathbf{w}+1)$ by :

$x \sim y \ \Leftrightarrow\ \forall^{st}\varepsilon > 0\ (\ |x-y| < \varepsilon\ ), \quad x \overset{n}{\sim} y \ \Leftrightarrow\ \forall^{\alpha_{n-1}}\varepsilon > 0\ (\ |x-y| < \varepsilon\ )$.

Definition of a finite sequence of infinitesimal positive increments.

We fix once and for all $\quad h = h_1,\ h_2,\ \cdots\ h_n\ \cdots\ h_{\mathbf{w}}$ $\quad$ such that

$h \sim 0$ and h ^{1}improper ... $h_n \overset{n}{\sim} 0$ and h_n ^{n}impropre r.

A number $x \in \mathbb{R}$ will be said ^{n}*limited* (*limited*, if $n = 1$) if there exists a ${}^{n-1}$improper $\lambda \in \mathbb{R}$ such that $|x| \leqq \lambda$.

Of course, the axioms of **RST** legitimate the assertion, in the language of **RST**, that numbers satisfying the preceding definitions exist.
We have the following obvious properties.
Proposition 1

(a) Each relation $\cdot \sim \cdot, \cdot \overset{2}{\sim} \cdot, \ldots , \cdot \overset{n}{\sim} \cdot \ldots$ is an equivalence relation.

(b) ($x \overset{n}{\sim} y$ and a ^{n}limited)$\Rightarrow ax \overset{n}{\sim} ay$.

(c) If $n > 1$, $x \overset{n}{\sim} y \Rightarrow x \overset{n-1}{\sim} y$.

1.2 Relationship between RST and Gordon's levels of relative standardness defined in IST.

IST is a sub-theory of **RST** because if we define in **RST** the unary relation $\mathbf{ST}(x)$ by $\mathbf{ST}(x) \Leftrightarrow \forall a\ (x\ \mathcal{SR}\ a)$, then **RST**'s statements, obtained by replacing, in the formulation of **IST**'s axioms, *st* by **ST** are theorems of **RST**.

The definition of Gordon's binary relation of relative standardness, denoted "$\cdot st\ \cdot$", is as follows

$$x\ st\ y \ \Leftrightarrow\ \exists^{st}\varphi\ (Ffin(\varphi)\ \&\ y \in dom\ \varphi\ \&\ x \in \varphi(y)\ ).$$

$Ffin(\varphi)$ means : φ is a function, and for any y in the domain $dom(\varphi)$ of φ, $\varphi(y)$ is finite.

This definition generates a limitation, which has been pointed out by Gordon: There exists a number n (non standard) and an $x \in [0, 1]$ such that each y which is infinitely close to x is not ${}^{[n]}$standard. In particular no ${}^{[n]}$shadow of x, namely a ${}^{[n]}$standard ${}^{[n]}x$ which is ${}^{[n]}$infinitely close of x, exists in $[0, 1]$. This implies that the relativized principle of standardization does not hold, contrary to what happens in **RST**. The relativized principle of standardization is very useful, the general principles of transfer idealization and choice we have proved in [16] are linked to the existence of such a possibility of standardization. Now a lot of applications can be treated with Gordon's predicate.

2 Derivations of classically non derivable functions

We need first to develop the concept of point. Of course, none of these definitions completely captures the intuitive notion.

2.1 Various conceptual symbolical patterns for the physical point.

Let a be a point and n a level of improperness such that $\eta(a) \leqq n$.

We put $]a[_n =]a - h_n, a + h_n[$. If $n = \eta(a)$ we use the simplified notations $]a[$ for $]a[_n$.

If a is a number and n a level of improperness, which have no relation a priori with $\eta(a)$, we put $\wr a \wr_n = \{x \in \mathbb{R} : \ x \overset{n}{\sim} a\}$ and we use the simplified notation $\wr a \wr$ if $n = 0$.

Remarks.

(1) The symbol $\wr a \wr_n$ represents a *collection* which is not a "*true set*" in the relative set theory.

(2) In **RST**, it is not possible to define uniformly the open point $]a[$ for any a of $\mathbb{R}$ by means of a function $h : \mathbb{R} \longrightarrow \mathbb{R}$ which associates to any a an infinitesimal strictly less standard than a itself. The reason is that if such an h existed then an element $a \in \mathbb{R}$ would exist such that ($h \ \mathcal{SR} \ a$) because, if $a \ \mathcal{SR} \ h$ for any a in $\mathbb{R}$, $\mathbb{R}$ would be finite (see [15], Theorem 2) . Then the axiom of transfer would give $h(a) \ \mathcal{SR} \ a$ and this is contradictory.

If $x \in \mathbb{R}$ we shall use the terms :

"*Point*" to indicate the number x,
"*open point*" to indicate $]x[$,
"*analysed point*" to name $\wr x \wr$.

2.2 The collection $\mathcal{R}$.

The set $\mathcal{R}$ below is our basic space of functions. This set could seem very particular. In reality the possibility for a function $u \in \mathcal{R}$ to be improper or very improper makes it very general: In particular, $\mathcal{R}$ contains exact representatives for any standard or improper distribution .... and more ...

Its description requires the next definition.

Definition 1. *A subset F of $\mathbb{R}$ is said Locally standard-finite if, for any limited x and y, $[x,y]\cap F$ is finite and its cardinality is standard.*

The axioms of **RST** enable to prove that if F is a locally standard-finite subset of $\mathbb{R}$ then a standard subset ${}^oF$ of $\mathbb{R}$ exists such that, for any limited $x,\ y\in\mathbb{R}$:

(a) ${}^oF\cap[x,y]$ is a standard finite set,

(b) for any $t\in F\cap[x,y]$ there exists a standard element ${}^ot\in{}^oF$, called the *shadow* of t, such that $t\sim\ {}^ot$.

The set ${}^oF$ is called the *shadow* of F.

Definition 2. *We say that a function f $\mathbb{R}\longrightarrow\mathbb{R}$ is regular if*

(a) *it have a level of improperness among the levels $\{1,\ 2,,\mathbf{w}\}$ fixed above,*

(b) *f is $\mathcal{C}^\infty$ at any point except the points of a locally standard-finite possibly empty set $F(f)$,*

(c) *f has all order left and right derivatives at any point of $F(f)$.*

We denote $\mathcal{R}$ the collection of regular functions. In this paper, if $f,\ g\in\mathcal{R}$ we put $f=g$ if there exits a finite set F (possibly non standard) such that for any $t\in\mathbb{R}\setminus F,\quad f(t)=g(t)$.

Basic examples. In these examples, for any subset A of $\mathbb{R}$, χ_A denotes the characteristic function of A.

1. *The Heaviside function H_a at the point a* defined by $H_a=\chi_{[a,+\infty[}$ is regular and $F(H_a)=\{a\}$.

2. If we simplify H_0 in H, $f=H\circ\sin$ is regular and $F(f)=\pi\cdot\mathbb{Z}$

3. Let $a\in\mathbb{R}$ with the level $\eta(a)$ of improperness, $0\leqq\eta(a)<\mathbf{w}$. We define the *principal evaluation* δ_a of the Dirac-function at the point a by

$$\delta_a=\frac{1}{2h_{\eta(a)}}\chi_{]a[}.\text{ We will simplify }\delta_0\text{ in }\delta.$$

δ_a is ${}^{\eta(a)+1}$improper, regular and $F(\delta_a)=\{a-h_{\eta(a)},\ a+h_{-\eta(a)}\}$.

4. We shall need also "hyperevaluations" of the Dirac-function.
If $0\leqq\eta(a)<n\leqq\mathbf{w}$, we put $\overset{n}{\delta}_a=\frac{1}{h_n}\chi_{]a[_n}$.
This implies that $\overset{1}{\delta}_a=\delta_a$. We shall use the simpler notations $\widetilde{\delta}_a$ rather than $\overset{\eta(a)+1}{\delta_a}$, and $\widetilde{\delta}$ for $\overset{2}{\delta}$.

5. *The functions P and Z*

$$P(x)=1-\chi_{]0[},\qquad Z(x)=\chi_{]-h,0[}-\chi_{]0,h[}.$$

6. *Principal evaluation of the function* $\dfrac{1}{x}$.

$$\underline{\underline{\frac{1}{x}}} = \frac{P(x)}{x} - \frac{1}{h}\,Z(x)$$

Principal evaluation of the logarithm

$$Ln(x) = P\ ln(x)$$

Other evaluations in $\mathcal{R}$ of the Dirac-function and $\dfrac{1}{x}$ will be considered elsewhere.

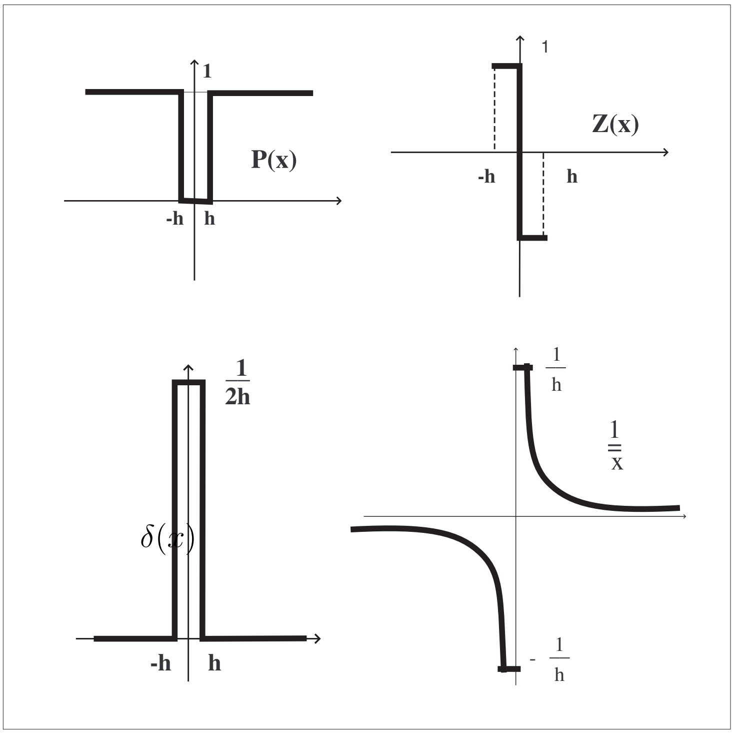

FIGURE 1. Graphical Representations of some improper evaluations

Definition 3. *Let* $f \in \mathcal{R}$ *be a* ${}^{n-1}$*improper function, with* $1 \leqq n \leqq \mathbf{w}$. *We call observed derivative of* f, *the function* $f^{]'[}$ *defined by*

$$f^{]'[}(x) = \frac{f(x+h_n) - f(x-h_n)}{2h_n}\ .$$

For any finite integer k such that $n + k \leqq \mathbf{w}$, we put

$$f^{]''[} = (f^{]'[})^{]'[}, \cdots, f^{]k[} = (f^{]k-1[})^{]'[}.$$

We also define in a natural way, for any ${}^{n-1}$improper function $f \in \mathcal{R}$, and any k such that $0 \leqq n < k \leqq \mathbf{w}$, the ^{k}*hyper-observed derivative*

$$f^{]'[_k}(x) = \frac{f(x+h_k) - f(x-h_k)}{2h_k}.$$

It will be noticed that the index k in $f^{]'[_k}(x)$ indicates the level of improperness of the hyper-observed derivative. It would be possible to also define concepts of hyper-observed second derivative, hyper-observed third derivative etc.....

Examples :

(1) For any standard $a \in \mathbb{R}$, H_a is standard, ^{0}improper, so
$H_a^{]'[} = \dfrac{1}{2h}\ [\ H_{a+h} - H_{a-h}\ ] = \delta_a,\ H_a^{]'[_2} = \dfrac{1}{2h_2}\ [\ H_{a+h_2} - H_{a-h_2}\ ] = \widetilde{\delta}_a,$

(2) δ_a is ^{1}improper hence
$\delta_a^{]'[} = \dfrac{1}{2h_2}\ [\ \delta_{a+h_2} - \delta_{a-h_2}\ ],\ \delta_a^{]'[_3} = \dfrac{1}{2h_3}\ [\ \delta_{a+h_3} - \delta_{a-h_3}\ ].$ $\delta_a^{]'[}$ is ^{2}improper, $\delta_a^{]'[_3}$ is ^{3}improper.

Figure 2 below represents $\delta^{]'[}$. It is only a synoptic representation : we consider that we are really unable to see precisely inside the point.

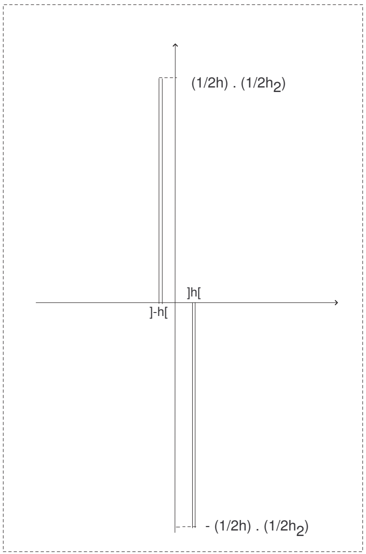

FIGURE 2. The function $\delta^{]'[}(x)$

If we denote, for any function $f \in \mathcal{R}$ with the level $n-1$ of improperness,

$$f_+(x) = f(x+h_n) \text{ and } \quad f_-(x) = f(x-h_n).$$

then the observed derivatives have the obvious following properties. For any $f,\ g \in \mathcal{R}$ *with the same level of improperness*

$$(f+g)^{]'[} = f^{]'[} + g^{]'[}, \qquad (1)$$

$$(f\ g)^{]'[} = f^{]'[}\ g_+ + f_-\ g^{]'[}. \qquad (2)$$

These formulae are false if f and g don't have the same degree of improperness, for example

$$(H + H_h)^{]'[} = \widetilde{\delta} + \delta_h,$$

$$H^{]'[} + H_h^{]'[} = \delta + \delta_h.$$

The second formula is not the exact formula of Leibnitz,‘

$$(f\ g)^{]'[} = f^{]'[}\ g + f\ g^{]'[}.$$

However, physicists sometimes freely use formula (1), the formula of Leibnitz as well as the formula of integration by parts even in the presence of jumps, and this does not seems to lead to any concrete *physical contradiction*.

The introduction, besides the concepts above, for points functions and derivatives, of a concept of equality adapted to the physicist's discourse, will lighten the mystery.

2.3 Dirac-equalities.

Dirac considers that any improper function whose value is zero outside of the origin and such that the integral is egal to 1, is identical to delta. Starting from this idea, we will define a relation between elements of $\mathcal{R}$ which, according to our opinion, is more significant than the classical weak equality of distributions. Its definition uses the below definite notion of *reiterated primitives.*

If $f \in \mathcal{R}$, we denote

$$\;_1\!\int_a^x f(s)\ ds = \int_a^x f(s)\ ds,\ \;_2\!\int_a^x f(s)\ ds = \int_a^x dt\ \int_a^t f(s)\ ds,$$
$$\ldots,\ \;_{n+1}\!\int_a^x f(s)\ ds = \int_a^x dt\ \;_1\!\int_a^t f(s)\ ds\ \ldots$$

The next proposition gives useful values and infinitesimal approximations of some reiterated primitives. Let us denote $y << a << x$ if $y < a < x$, $x \notin \wr a \wr$ and $y \notin \wr a \wr$. Then we can state.

Proposition 1. *for any* $y,\ a,\ x$ *such that* $y << a << x$*, for any standard* $k \in \mathbb{N}^\star$*, for any* $n > \eta(a)$

$$(1)\;_k\!\int_y^x H_a = \frac{(x-a)^k}{k!},\quad (2)\;_k\!\int_y^x \overset{n}{\delta}_a \ \overset{n}{\simeq}\ \frac{(x-a)^{k-1}}{(k-1)!},$$

$$(3)\;_k\!\int_y^x (\overset{n}{\delta}_a)^2 \ \overset{n}{\simeq}\ \frac{(x-a)^{k-1}}{2h_n\ (k-1)!},\quad (4)\ \int_y^x (t-a)\overset{n}{\delta}_a(t)\ dt = 0,$$

$$(5)\;_2\!\int_y^x (t-a)\ \overset{n}{\delta}_a(t)\ dt = -\frac{h_n^2}{3},\qquad \textit{if } k > 1\ (6)\;_k\!\int_y^x (\overset{n}{\delta}_a)^{]'[}\ \overset{n}{\simeq}\ \frac{(x-a)^{k-2}}{(k-2)!}.$$

Proof. These results are obtained through elementary calculations. The proof of (1) is obvious. Let us prove (2). We will remark first that

$$\overset{n}{\delta}_a = \frac{1}{2h_n}[H_{a-h_n} - H_{a+h_n}].$$

An application of (1) gives

$$\;_k\!\int_y^x \overset{n}{\delta}_a = \frac{(x-(a-h_n))^k - (x-(a+h_n))^k}{2h_n k!} =$$

$$\frac{(x-(a-h_n))^{k-1}+(x-(a-h_n))^{k-2}(x-(a+h_n))+\cdots+(x-(a+h_n))^{k-1}}{k!}.$$

If we remark that each of the k terms between the brackets is ^{n}infinitely close to $(x-a)^{k-1}$, we obtain (2). The proofs of (3), (4) and (5) are let to the reader. In order to prove (6) let us remark first that

$$(\overset{n}{\delta}_a)^{]'[} = \frac{1}{2h_n}\left[\overset{n+1}{\delta}_{a-h_n} - \overset{n+1}{\delta}_{a+h_n}\right].$$

The sequel of the proof makes use of (2) and the binomial formula.
We generalize the definition of analyzed points by writing

$$\wr +\infty \wr = \{x \in \mathbb{R} \ : \ x \sim +\infty\}, \wr -\infty \wr = \{x \in \mathbb{R} \ : \ x \sim -\infty\}.$$

For any locally standard-finite set F, we denote

$$\wr F \wr = \bigcup_{x \in F \cup \{-\infty,+\infty\}} \wr x \wr .$$

For any set F we put

$$\Big]F\Big[_n = \bigcup_{x\in F}]x-h_n, x+h_n[.$$

Definition 4. *Let f and g be elements of $\mathcal{R}$ and let n be a standard integer. We say that f is Dirac-equal to g without integration and denote $f =_0^D g$, or more precisely $f =_0^D g \quad (F)$, if there exists a locally standard-finite set F such that for any standard order of derivation k (in the classical sense) for any $x \in \mathbb{R}$, $x \notin \wr F \wr \Rightarrow f^{(k)}(x) \sim g^{(k)}(x)$.*

We say that f is Dirac-equal to g up to n integration and denote it $f =_n^D g$, or more precisely $f =_n^D g \quad (F)$, if a locally standard-finite set F exists such that for any $a,\ b \in \mathbb{R}$,

$$a,\ b \notin \ \wr F \wr \Rightarrow \ {}_k\!\int_a^b f(x)\,dx \sim {}_k\!\int_a^b g(x)\,dx \text{ for any } k \in \{\ 1,\ \ldots,\ n\ \}.$$

We say that f is Dirac-equal to g, and denote $f =^D g$, if for any standard $n \in \mathbb{N}$, $f =_n^D g$.

Theorem. *For any standard integer n*

(a) For any f, g, h in $\mathcal{R}$: $f =_n^D g$ and $g =_n^D h \Rightarrow f =_n^D h$
(b) For any f, f_1, g, g_1 in $\mathcal{R}$:

$$(\ f =_n^D f_1 \text{ and } g =_n^D g_1\ ) \Rightarrow f+g =_n^D f_1+g_1$$

Proof. The proof is a simple check.

A general result similar to (b) relative to the *usual* product of functions does not exists.

Counterexamples :

1 - $H\delta =^D \frac{1}{2}\,\delta$ but $H(H\delta) = H\delta =^D \frac{1}{2}\,\delta \not\mathrel{\underline{=}}^D \frac{1}{4}\,\delta =^D H\,(\frac{1}{2}\,\delta)$.

2 - If f is constant, $f = \frac{1}{2h}$, $g = H_{-h} - H_h$ then $g =^D 0$ but $fg = \delta \not\mathrel{\underline{=}}^D 0$.

However, we shall see useful particular cases in section 3, theorem 8 and 9.
The next proposition states some useful relations.
Proposition 2. *Let $a \in \mathbb{R}$ be such that $\eta(a) \leqq \mathbf{w}$:*
(1) *If $\eta(a) < n \leqq m \leqq \mathbf{w}$, then for any $\lambda \in \mathbb{R}$ such that $\eta(\lambda) \leqq n-1$,*

$$\lambda\overset{n}{\delta}_a \mathrel{\underline{=}}^D \lambda\overset{m}{\delta}_a. \quad \textit{In particular,} \quad \delta_a \mathrel{\underline{=}}^D \widetilde{\delta}_a.$$

(2) $H_a\,\delta_a \mathrel{\underline{=}}^D \frac{1}{2}\delta_a$, (3) $\delta_a\widetilde{\delta}_a =^D \delta_a^2$, (4) $\frac{1}{x}\,\delta_h =^D_1 2\delta^2$ *but* $\frac{1}{x}\,\delta_h \neq^D 2\delta^2$.
(5) $\delta_a^{]'[} =^D_0 0$, (6) $\int_{-\infty}^x \delta_a^{]'[} =^D \delta_a$, (7) $\delta_a^2 \neq^D \widetilde{\delta}_a^2$, (8) $\frac{1}{\overline{x}}\,\delta \mathrel{\underline{=}}^D - \frac{1}{2}\,\delta^{]'[}$,
(9) $\delta_a\,\delta_{a+} =^D_1 \frac{1}{2}\delta_a^2$ *but* $\delta_a\delta_{a+} \neq^D \frac{1}{2}\delta_a^2$, (10) $\frac{1}{2h}\,Z \mathrel{\underline{=}}^D 0$, (11) $\frac{1}{2h}Z^{]'[} \mathrel{\underline{=}}^D 0$,
(12) $\Big(\frac{1}{2h}\Big)^2\,Z \mathrel{\underline{=}}^D \frac{1}{4}\delta^{]'[}$, (13) $H\delta^2 =^D_1 \frac{1}{2}\delta^2$ *but* $H\delta^2 \neq^D \frac{1}{2}\delta^2$,
(14) $\frac{1}{2h}(\,\widetilde{\delta} - \delta_h\,) \mathrel{\underline{=}}^D \frac{1}{2}\,\delta^{]'[}$, (15) $\frac{1}{\overline{x}}\,\delta^{]'[} \mathrel{\underline{=}}^D -4\,\delta^2\widetilde{\delta} - \frac{1}{2}\,\delta^{]''[} =^D_1 -4\,\delta^3 - \frac{1}{2}\,\delta^{]''[}$,
(16) $\Big(\frac{1}{\overline{x}}\Big)^{]'[}\delta \mathrel{\underline{=}}^D 4\,\delta^2\,\widetilde{\delta} =^D_1 4\,\delta^3$.
Proof. (1) The functions $\lambda\overset{n}{\delta}_a \mathrel{\underline{=}}^D \lambda\overset{n}{\delta}_a$ and $\lambda\overset{n}{\delta}_a \mathrel{\underline{=}}^D \lambda\overset{m}{\delta}_a$ have the value 0 outside $\wr\, a\, \wr$. It remains to be shown that the reiterated integrals are infinitesimally close one to the other. In order to prove it let us fix x and y such that $x \notin\, \wr\, a\, \wr$ and $y \notin\, \wr\, a\, \wr$. From the former proposition we get

$${}_k\!\int_y^x \overset{n}{\delta}_a \overset{n}{\sim} \frac{(x-a)^{k-1}}{(k-1)!} \overset{m}{\sim} {}_k\!\int_y^x \overset{m}{\delta}_a \text{ hence } {}_k\!\int_y^x \overset{n}{\delta}_a \overset{n}{\sim} {}_k\!\int_y^x \overset{m}{\delta}_a.$$

Multiplying term by term by the ^{n}standard number λ we obtain

$$\lambda\, {}_k\!\int_y^x \overset{n}{\delta}_a = {}_k\!\int_y^x \lambda\overset{n}{\delta}_a \overset{n}{\sim} {}_k\!\int_y^x \lambda\overset{m}{\delta}_a = \lambda\, {}_k\!\int_y^x \overset{m}{\delta}_a.$$

(2) and (3) are obvious. Let us prove (4). let us fix x and y such that $x \notin\, \wr\, a\, \wr$, $y \notin\, \wr\, a\, \wr$ and $y < a < x$. Let us denote $\varphi(t) = \int_y^t \frac{1}{s}\delta_h(s)\,ds$.
$\frac{1}{s}\delta_h(s) = 0$ if $s \notin [h]$ and, if $s \in [h]$

$$\frac{1}{2h_2(h+h_2)} \leqq \frac{1}{s}\delta_h(s) \leqq \frac{1}{2h_2(h-h_2)}.$$

Taking the integral we obtain

$$\frac{1}{h+h_2} \leqq \varphi(x) = \int_{[h]} \frac{1}{s}\delta_h(s)\,ds \leqq \frac{1}{h-h_2}.$$

As $\dfrac{1}{h+h_2} \overset{2}{\sim} \dfrac{1}{h} \overset{2}{\sim} \dfrac{1}{h-h_2}$ and $\displaystyle\int_y^x \delta^2 = \frac{1}{2h}$, we obtain $\dfrac{1}{x}\delta_h =_1^D 2\delta^2$.
This implies

$$_2\int_y^x \frac{1}{t}\delta_h(t)\,dt = \int_y^x \varphi(t)\,dt \leqq \frac{x-(h-h_2)}{h-h_2}.$$

Now, it follows from $x \notin \wr 0 \wr$ and $h_2 \overset{2}{\sim} 0$ that $\dfrac{x-(h-h_2)}{h-h_2} \overset{2}{\sim} \dfrac{x}{h} - 1$.
On the other hand, $\displaystyle_2\int_y^x \delta^2 \sim \frac{x}{2h}$. Hence $\displaystyle_2\int_y^x \frac{1}{t}\delta_h(t)\,dt \not\sim_2 \int_y^x 2\delta^2(t)\,dt$ and this implies that $\frac{1}{x}\delta_n \neq^D 2\delta^2$.

We let to the reader the proofs of (5), (6)and (7). Let us prove (8).
We deduce from the formula $\dfrac{1}{\overline{\overline{x}}}\,\delta = \dfrac{1}{2h^2}\left[-H_{-h} + 2H - H_h\right]$ that

$$_k\int_y^x \frac{1}{\overline{\overline{x}}}\,\delta = \frac{-(x+h)^k + 2x^k - (x-h)^k}{2h^2 k!} \sim -\frac{1}{2}\frac{x^{k-2}}{(k-2)!} \sim_k \int_y^x -\frac{1}{2}\delta]'[.$$

Let us prove (9) now. We have $\delta\delta_h =_0^D \frac{1}{2}\delta^2$ because both $\delta\delta_h$ and $\frac{1}{2}\delta^2$ have the value zero outside of $\wr 0 \wr$. Let now x and y be elements of $\mathbb{R} \setminus \wr 0 \wr$. If $y < 0 < x$ then an easy calculation gives

$$_n\int_y^x \delta\delta_h = \frac{(x-(h-\widetilde{h})^n - (x-h)^n}{4hh_2 n!}, \quad _n\int_y^x \frac{1}{2}\delta^2 = \frac{(x+h)^n - (x-h)^n}{4h^2 n!}.$$

So we obtain :

with $n = 1$: $\quad \displaystyle_1\int_y^x \delta\delta_h =_1 \int_y^x \frac{1}{2}\delta^2 = \frac{1}{h},$

with $n = 2$: $\quad \displaystyle_2\int_y^x \delta\delta_h = \frac{\frac{1}{2h}x - 1}{4}, \qquad _2\int_y^x \frac{1}{2}\delta^2 = \frac{x}{8h}$. Hence

$$\delta\delta_h =_1^D \frac{1}{2}\delta^2, \text{ but } \quad \delta\delta_h \neq^D \frac{1}{2}\delta^2 \text{ because } \delta\delta_h \neq_2^D \frac{1}{2}\delta^2.$$

the formulas (10) to (16) are left to the reader.

Let us consider now the following functions, which usually play the role of Dirac-functions in physics.

$$\delta 1(x) = \frac{1}{2h}e^{-\frac{|x|}{h}},\ \delta 2(x) = \frac{1}{\pi}\,\frac{h}{x^2+h^2},\ \delta 3(x) = \frac{1}{h\sqrt{\pi}}\ e^{-\frac{x^2}{h^2}},\ \delta 4(x) = \frac{sin(\frac{x}{h})}{\pi x}.$$

Then we have

Proposition 3. *$\delta 1 =^D \delta 2 =^D \delta 3 =^D \delta$ but $\delta 4 \neq^D \delta$.*
If $a \sim \pm\infty$ and $\xi_a = \frac{1}{2h}\,(H_{a_-} - H_{a_+})$ then $\xi_a =^D 0$.

Proof. It is an immediate check

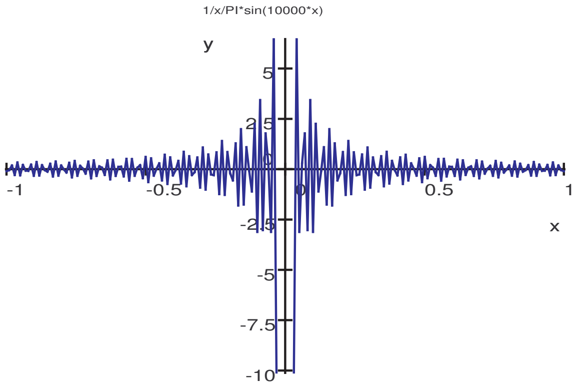

FIGURE 3. $\dfrac{1}{\pi}\,\dfrac{sin(\frac{x}{h})}{x},\ h = 10^{-4}$

Remark.
1. Although it is an infinitesimal approximation (in the space of distributions) of the Dirac-distribution, the last function, $\delta 4(x)$, cannot be considered as a Dirac-function, even if its integral is 1, because $\delta 4$ is not infinitesimal outside of $\wr 0 \wr$.
2. It would be possible to extend the definition of $\mathcal{R}$, permitting non standard cardinalities for the sets $F(f)$. Then relations of Dirac-hyperequalities should be defined.

3 Basic theorems.

Theorem 1. *For any $f,\ g \in \mathcal{R}$ with f continuous, and any level of improperness $n > \eta((f,g))$:*

(a) $f^{]'[_n} g \overset{D}{=} f'g$.

(b) if $\eta(f) \leqq \eta(g)$ then $fg^{]'[} \overset{D}{=} fg^{]'[_n}$.

Proof. We remark that f' is defined except on a finite set.
(a) Let us prove that $f^{]'[_n} g \overset{D}{=}_0 f'g$. Let k be a standard order of derivation. For any $x \notin \wr F(f) \cup F(g) \wr$, $(f^{]'[_n} g)^k(x) = \sum_{p=0}^{p=k} \mathbf{C}_k^p (f^{]'[_n})^{(p)}(x) g^{(k-p)}(x)$
and for each p $(f^{]'[_n})^{(p)}(x) = (f^{(p)})^{]'[_n}(x) \overset{n}{\sim} (f^p)'(x) = (f')^{(p)}(x)$ so, each $g^{(k-p)}(x)$ being ^{n}limited, we have

$$(f^{]'[_n} g)^k(x) \overset{n}{\sim} \sum_{p=0}^{p=k} C_k^p (f')^{(p)}(x) g^{(k-p)}(x) = (f'g)^{(k)}(x).$$

This prove $f^{]'[} g \overset{D}{=}_0 f'g$. Let us compare the reiterated integrals.
For any limited $x,\ y$,

$$\int_x^y f^{]'[_n}(t) g(t)\ dt = \int_x^y \frac{f(x+h_n) - f(x-h_n)}{2h_n}\ g(t)\ dt.$$

If x does not belong to any $]a[_n$ with $a \in F(f)$ then

$$\frac{f(x+h_n) - f(x-h_n)}{2h_n} = f'(x + \theta h_n) \text{ with } \theta \in]-1, 1[.$$

Now, by the definition of $\mathcal{R}$, f' is uniformly continuous over each interval of continuity of f, so

$$f'(x+\theta h_n) \overset{n}{\sim} f'(x),\ f'(x+\theta h_n)g(x) \overset{n}{\sim} f'(x)g(x)$$

because g is ^{n}limited, and $\int_{x'}^{y'} f'g \sim \int_{x'}^{y'} f^{]'[_n}g$ whatever $x',\ y'$ such that $[x',y'] \subset ([x,y] \setminus]F(f)[_n$.

If $a \in F(f)$, then $f'g$ and $f^{]'[_n}g$ are both ^{n}limited on $]a[_n$ so the integrals $\int_{x"}^{y"} f'g$ and $\int_{x"}^{y"} f^{]'[_n}g$ are infinitesimals for any $x", y" \in]a[_n,\ a \in F(f)$, because $|y" - x"|$ is ^{n}infinitesimal.

We conclude that $\int_x^y f'g \sim \int_x^y f^{]'[_n}g$. Computing the iterated primitives up to a standard rank k, we get ${}_k\!\int_x^y f'g \sim {}_k\!\int_x^y f^{]'[_n}g$.

Hence $f'g \overset{D}{=} f^{]'[_n}g$.

(b) Let us expand $g \in \mathcal{R}$ in the form $g = u + \sum\limits_{a_i \in G} \alpha_i H_{a_i}$, with locally standard-finite G and continuous $u \in \mathcal{R}$. An easy proof, making use of the axiom of transfer, see [15], shows that $\eta(u) \leqq \eta(g) < n$ and for any $i \in G,\ \eta(a_i) \leqq \eta(g) < n,\ \eta(\alpha_i) \leqq \eta(g) < n$.

As G is locally standard-finite then for any limited x and y the integrals ${}_k\!\int_y^x fg^{]'[}$ and ${}_k\!\int_y^x fg^{]'[_n}$ are respectively equal to

$${}_k\!\int_y^x fu^{]'[_{\eta(g)+1}} + \sum_{a_i \in G_{x,y}} {}_k\!\int_y^x f.(\alpha_i H_{a_i})^{]'[_{\eta(g)+1}}$$

$$\text{and } {}_k\!\int_y^x fu^{]'[_n} + \sum_{a_i \in G_{x,y}} {}_k\!\int_y^x f[\alpha_i H_{a_i}]^{]'[_n}.$$

This sum being standard-finite, we only have to prove

$$fu^{]'[_{\eta(g)+1}} \overset{D}{=} fu^{]'[_n} \quad \text{and} \quad f.(\alpha_i H_{a_i})^{]'[_{\eta(g)+1}} \overset{D}{=} f.(\alpha_i H_{a_i})^{]'[_n}.$$

The first relation is an application of (a). Item (a) applies because

$$n \geqq \eta(g)+1 > \eta(g) \geq \eta(u), \text{ and } \eta(g) \geqq \eta(f) \ \Rightarrow \eta(g)+1 > \eta(u,f).$$

The last inequality is a consequence of the axiom of transfer.

The second Dirac-equality follows from

$$f.(\alpha_i H_{a_i})^{]'[_{\eta(g)+1}} \overset{D}{=} f(a_i)\alpha_i \overset{\eta(g)+1}{\delta}_{a_i} \overset{D}{=} f(a_i)\alpha_i \overset{n}{\delta}_{a_i} \overset{D}{=} f(\alpha_i H_{a_i})^{]'[_n},$$

the verification of which is easy.

In order to prove the necessity of the hypothesis, we have to produce counterexamples.

Counterexamples.

(1) With $f = H$ and $g = H$ we have, $f' = 0 \;\Rightarrow f'g = 0$. $f^{]'[} = \delta$, $f^{]'[}g =^{D} \dfrac{1}{2}\delta$. So $f'g \neq^{D} f^{]'[}g$. The missing assumption is the continuity of f

(2) Let be $f(x) = \begin{cases} 0 \quad if\ x\ \leq \frac{h}{2} \\ \frac{1}{\widetilde{h}}\, x - \ \frac{h}{2\widetilde{h}}\ \ if\, x \in [\frac{h}{2}\,,\ \frac{h}{2} + \widetilde{h}] \\ 1\ if\ x\ \geq\ \frac{h}{2} + \widetilde{h} \end{cases}$ and $g = H$.

Then f is continuous, $fH^{]'[_2} = f\widetilde{\delta} = 0$ and $fH^{]'[} = f\delta =^{D} \dfrac{1}{4}\delta$. In this example the problem comes from the relation $\eta(f) > \eta(g)$.

(3) Concerning (b), let $f(x) = \delta^3$ and $g(x) = x^3$. An easy computation yields $g^{]'[}(x) = 3x^2 + h^2$, so $f(x)g^{]'[}(x) - f(x)g'(x) = h^2\delta^3$. Now the computation of the first integral prove that $h^2\delta^3 \not=^{D} 0$. We should obtain the Dirac-equality replacing $g^{]'[}$ by $g^{]'[_2}$.

Corollary. *For any $f \in \mathcal{R}$ and any standard integer $n \geq \eta(f)$,*

$$f^{]'[} =^{D} f^{]'[_n} \quad and \quad f^{]'[} =^{D} f' \quad if\, f\ is\ continuous.$$

Theorem 2. *For any $f \in \mathcal{R}$ and $y,\ x \notin\ \wr F(f) \wr$,*

$$\int_y^x f^{]'[}(t)\ dt \sim f(x) - f(y).$$

Proof. Any function of $\mathcal{R}$ decomposes in a sum of elementary functions. It is enough to prove the theorem for these elementary functions. The result is obvious if f is a continuous function. If $f = \alpha H_a$ with limited a then $f^{]'[} = \alpha\overset{n}{\delta_a}$ with $n = Max\{\eta(\alpha), \eta(a)\}{+}1$. For any $y,\ x$

$$\int_y^x f^{]'[} = \begin{cases} \alpha & \text{if } y << a << x, \\ 0 & \text{if } y, x << a, \text{ or } y, x >> a \end{cases} \quad = \quad f(x) - f(y).$$

Theorem 3. *For any $f,\ g \in \mathcal{R}$, whatever their level of improperness :*

$$(f+g)^{]'[} =^{D} f^{]'[} + g^{]'[}$$

Proof. Let $n = Max\{\eta(f), \eta(g), \eta(f{+}g)\}{+}1$. Then by the definition of ^{n}hyperobserved derivatives we have $(f+g)^{]'[_n} = f^{]'[_n} + g^{]'[_n}$. Now, the corollary above and properties (a), (b) gives,

$$(f+g)^{]'[} =^{D} (f+g)^{]'[_n} = f^{]'[_n} + g^{]'[_n} =^{D} f^{]'[} + d^{]'[}.$$

Theorem 4. *For any $f,\ g \in \mathcal{R}$, whatever their level of improperness if $n \geq \eta((f,g)) + 1$:*

$$(fg)^{]'[_n} =^D f\ g^{]'[_n} + g\ f^{]'[_n}$$

Proof. It is enough to show it for elementary functions .

If $f = \alpha H_a$ and $g = \beta H_b$.

If $a < b$.

Then n is strictly larger than $\eta(a),\ \eta(b),\ \eta(\alpha)$ and $\eta(\beta)$ (Transfer). So we have

$$f^{]'[_n} = \alpha \overset{n}{\delta}_a, \quad g^{]'[_n} = \beta \overset{n}{\delta}_b, \quad (fg))^{]'[_n} = (\alpha\beta H_b)^{]'[_n} = \alpha\beta \overset{n}{\delta}_b$$

$$]a[_n \cap]b[_n = \emptyset \ \Rightarrow \begin{cases} f g^{]'[_n} = \alpha\beta H_a \overset{n}{\delta}_b = \alpha\beta \overset{n}{\delta}_b \text{ and} \\ f^{]'[_n} g = \alpha\beta \overset{n}{\delta}_a H_b = 0 \end{cases}$$

Hence the formula is true.

If $a = b$

$$(fg)^{]'[_n} = \alpha\beta \overset{n}{\delta}_a, \quad f^{]'[_n} g = g^{]'[_n} f = \alpha\beta H_a \overset{n}{\delta}_a =^D \frac{\alpha\beta}{2} \overset{n}{\delta}_a$$

If f is continuous and $g = H_a$.

Then $(fH_a)^{]'[_n}(x) = f(x - h_n)\overset{n}{\delta}_a(x) + f^{]'[_n}(x) H_{a+h_n}(x)$. The facts that f is continuous and that the jumps of the successive derivatives have a level of improperness strictly less than n allow us to prove, through an easy calculation that $f_{h_n} \overset{n}{\delta}_a =^D f \overset{n}{\delta}_a$ and $f^{]'[_n} H_{a+h_n} =^D f^{]'[_n} H_a$.

The proof is immediate also if both f and g are continuous elements of $\mathcal{R}$.

Corollary 1 *If f and g have the same level of improperness then*

$$(fg)^{]'[} =^D f g^{]'[} + g f^{]'[}.$$

There is nothing to prove.

If f and g have distinct levels of improperness there are counterexamples.

Counterexamples.

(1) Let f and g be respectively H and $H_{\widetilde{h}}$.
Then $f.g = H_{\widetilde{h}}$, $(fg)^{]'[} = \delta_{\widetilde{h}}$, $fg^{]'[} = H\delta_{\widetilde{h}} = \delta_{\widetilde{h}}$. An easy calculation gives $f^{]'[}g = \delta H_{\widetilde{h}} =^D \frac{1}{2}\delta$. so $(fg)^{]'[} \neq^D fg^{]'[} + gf^{]'[}$.

(2) This counterexample aims at proving that, even if f and g are generated, starting from the standard elements of $\mathcal{R}$, by observed derivations, then the rule of corollary 1 does not generally apply. Let us take $f = H$ and $g = \delta^{]'[}$. We obtain directly $(H\,\delta^{]'[})^{]'[} = H\,\delta^{]''[}$, and the Leibniz rule would give $(H\,\delta^{]'[})^{]'[} = \delta\delta^{]'[} + H\,\delta^{]"[}$. Now, we can verify directly that $\delta\,\delta^{]'[} \neq^D 0$, or wait until theorem 6 in order to use the arguments that $\delta\,\delta^{]'[} =^D \dfrac{1}{2}(\delta^2)^{]'[} \neq^D 0$ for $\delta^2 \neq^D 0$.

Corollary 2. *For any $f,\ g \in \mathcal{R}$, if f is continuous and $\eta(f) \leqq \eta(g)$ then*

$$(fg)^{]'[} =^D f g^{]'[} + f'g.$$

Proof $(fg)^{]'[} = (fg)^{]'[_{\eta(g)}} = f g^{]'[_{\eta(g)+1}} + f^{]'[_{\eta(g)+1}}$ (Theorem 4). Now theorem 1 gives $f^{]'[_{\eta(g)+1}} g =^D f'g$. This end the proof.

Example :
Let $x_+^q = H.x^q$, q being a standard positive integer. Then for any standard integer n,

$$(x_+^q\, \delta^{]n[})^{]'[} =^D q\, x_+^{q-1}\, \delta^{]n[} + x_+^q\, \delta^{]n+1[}.$$

Remark.

Let be $f \in \mathcal{R}$ with $f(t) \neq 0$ for all t. Then we can verify the formula

$$(\frac{1}{f})^{]'[} = -\frac{f^{]'[}}{(f_-)(f_+)}.$$

However, $(\dfrac{1}{f})^{]'[}$ is not generaly Dirac-equal to $-\dfrac{f^{]'[}}{f^2}$. In order to prove it, let us consider $f = 1 + H$. We have $f^{]'[} = \delta$, $\dfrac{1}{f} = 1 - \dfrac{1}{2}\,H$, $\dfrac{1}{f^2} = 1 - \dfrac{3}{4}\,H$,

$$(\frac{1}{f})^{]'[} = -\frac{1}{2}\,\delta,\ -\frac{f^{]'[}}{f^2} = -\delta\ (1 - \frac{3}{4}\,H) =^D -\frac{5}{8}\,\delta.$$

Now, corollary 1 allows us to write from the equality $\dfrac{1}{f}\ f = 1$,

$$(\frac{1}{f})^{]'[}\ f =^D -\frac{1}{f}\ f^{]'[}.$$

The difficulty lies in the illegality of the multiplication by $\frac{1}{f}$ of the two members of a Dirac-equality.

Demonstrations of theorems 5 and 6 below, derive directly from the definition of the relations $=_k^D$ and $=^D$.

Theorem 5. *For any $f, g \in \mathcal{R}$ and any standard $k \in \mathbb{N}^\star$,*

$$(\ f =_k^D g \quad (F) \text{ and } a \notin \wr F \wr\ ) \ \Rightarrow (x \to \int_a^x f) \ =_{k-1}^D\ (x \to \int_a^x g).$$

Theorem 6 *For any $f, g \in \mathcal{R}$, and any $k \in \mathbb{N}$,*

$$f =_k^D g \ \Rightarrow\ f^{]'[} =_{k+1}^D g^{]'[}.$$

Theorem 7. *For any $f,\ \varphi \in \mathcal{R}$, if φ is $\mathcal{C}^\infty$ on $\mathbb{R}$, if $\varphi^{-1}(F(f))$ and $\varphi^{-1}({}^oF(f))$ are locally standard-finite then $f\circ\varphi \in \mathcal{R}$ and for any level $n > \eta((f,\varphi))$* of improperness

$$(f\circ\varphi)^{]'[} \stackrel{D}{=} (f^{]'[_n}\circ\varphi)\times\varphi'.$$

To avoid complicated denotations, we shall suppose that φ is standard. The general result is obtained through a classical shift of the levels of improperness.

Lemma. *If φ is continuous on $\mathbb{R}$:*

(a) *For any limited $a \in \mathbb{R}$ and any locally standard-finite and <u>standard</u> $F_0 \subset \mathbb{R}$, $[-a,a] \bigcap \varphi^{-1}(\wr F_0 \wr) \ \subset\ \wr \varphi^{-1}(F_0)\wr$.*

(b) *For any locally standard-finite $F \subset \mathbb{R}\ :\ \wr\varphi^{-1}(F)\wr \ \subset\ \wr\varphi^{-1}({}^oF)\wr$.*

Proof. Let $x \in [-a,a] \bigcap \varphi^{-1}(\wr F_0\wr)$ then there exits $u_0 \in F_0$ such that $\varphi(x) \sim u_0$. From x limited and φ standard we deduce that x has a shadow ${}^ox$ and $\varphi(x)$ is limited. Also u_0 is limited. Now from F_0 standard and locally standard finite we deduce that u_0 is standard. From the continuity of φ we get $\varphi(x) \sim \varphi({}^ox)$. Hence

$$\varphi({}^ox) = u_0, \quad {}^ox \in \varphi^{-1}(F_0), \quad x \in \wr\varphi^{-1}(F_0)\wr.$$

This prove the inclusion of (a). Let us prove (b).
If $x \in \wr\varphi^{-1}(F)\wr$ then there exists a such that $x \sim a$ and $\varphi(a) \in F$. We have $\varphi({}^oa) = {}^o\varphi(a) \in^o F$. From $x \sim {}^oa$, and ${}^oa \in \varphi^{-1}({}^oF)$ we derive $x \in \wr\varphi^{-1}({}^oF)\wr$. This ends the proof of (b).

Proof of Theorem 7.

Case where f is continuous.

Let us prove that $(f\circ\varphi)^{]'[} \stackrel{D}{=} (f^{]'[_n}\circ\varphi)\varphi' \quad (\ \varphi^{-1}({}^oF(f))\ )$.
It makes sense because $\varphi^{-1}({}^oF(f))$ is locally standard-finite.
The function f being in $\mathcal{R}$ its derivatives exists on $\mathbb{R}\setminus F$. Hence, $(f\circ\varphi)^{(p)}$ exists on $\varphi^{-1}(F(f))$ at any order p . The function $f\circ\varphi$ is continuous so we have, from the corollary of theorem 1

$$(f'\circ\varphi)\times\varphi' = (f\circ\varphi)' \stackrel{D}{=} (f\circ\varphi)^{]'[} \quad (\ \varphi^{-1}(F(f))\ ).$$

As $\wr\varphi^{-1}(F)\wr \ \subset\ \wr\varphi^{-1}({}^oF)\wr$ (lemma, (b))

$$(f'\circ\varphi)\times\varphi' = (f\circ\varphi)' \stackrel{D}{=} (f\circ\varphi)^{]'[} \quad (\ \varphi^{-1}({}^oF(f))\ ).$$

It remains to prove that

$$(f'\circ\varphi)\times\varphi' \stackrel{D}{=} (f^{]'[_n}\circ\varphi)\times\varphi' \quad (\ \varphi^{-1}({}^oF(f))\ ).$$

(a) Proof of : $(f'\circ\varphi)\times\varphi' \stackrel{D}{=}_0 (f^{]'[_n}\circ\varphi)\times\varphi' \quad (\ \varphi^{-1}({}^oF(f))\ )$.

Let be $x \notin \wr \varphi^{-1}({}^oF(f)) \wr$ then , item (b) of lemma, $x \notin \wr \varphi^{-1}(F(f)) \wr$. Consequently $\varphi(x) \notin F(f)$ and $(f')^{(k)}(\varphi(x)) \sim (f^{]'[_n})^{(k)}(\varphi(x))$ for any standard k. Multiplication by limited products of $\varphi^{(p)}(x)$, member by member conserve the relation $\sim$. We remark that $(f^{]'[_n})^{(p)} = (f^{(p)})^{]'[_n}$ for any p. We deduce from these facts that for any limited k,

$$((f' \circ \varphi)(x)\varphi'(x))^{(k)} \sim ((f^{]'[_n} \circ \varphi)(x)\varphi'(x))^{(k)}.$$

Proof of : $\forall^{standard} k \in \mathbb{N}\ (f' \circ \varphi) \times \varphi' =^D_k (f^{]'[_n} \circ \varphi) \times \varphi' \quad (\ \varphi^{-1}({}^oF(f))\ )$.

It is enough to prove that for any $x \notin \wr \varphi^{-1}({}^oF(f)) \wr$ and any limited y

$$\int_x^y (f' \circ \varphi)(t) \times \varphi'(t)\ dt \sim \int_x^y (f^{]'[_n} \circ \varphi)(t) \times \varphi'(t)\ dt$$

even if y is in $\wr \varphi^{-1}({}^oF(f)) \wr$. We have to consider mainly the cases where $[x, y] \cap \wr \varphi^{-1}({}^oF(f)) \wr \subset \wr y \wr_n$ because $[x, y]$ is an union of standard finitely many such intervals.
If $y \notin \wr \varphi^{-1}({}^oF(f)) \wr_n$ then $f'(s) \overset{n}{\sim} f^{]'[_n}(s)$ for all $s \in [\varphi(x), \varphi(y)]$ so

$$\int_x^y (f' \circ \varphi)(t) \times \varphi'(t)\ dt = \int_{\varphi(x)}^{\varphi(y)} f'(s)\ ds \overset{n}{\sim} \int_{\varphi(x)}^{\varphi(y)} f^{]'[_n}(s)\ ds$$
$$= \int_x^y (f^{]'[_n} \circ \varphi)(t) \times \varphi'(t)\ dt.$$

If $y \in \wr \varphi^{-1}({}^oF(f)) \wr_n$ then for any ^{n}improper integer $m > \dfrac{1}{|y-x|}$, $[x, y - \dfrac{1}{m}] \subset \mathbb{R} \setminus \wr \varphi^{-1}({}^oF(f)) \wr_n$. Such an integer exists because $x \notin \wr \varphi^{-1}({}^oF(f)) \wr$, $\Rightarrow x \not\sim y$. This implies that for any any ^{n}improper $\ m \in \mathbb{N}$,

$$m > \frac{1}{|y-x|} \implies \int_x^{y-\frac{1}{m}} (f' \circ \varphi)(t) \times \varphi'(t)\ dt \overset{n}{\sim} \int_x^{y-\frac{1}{m}} (f^{]'[_n} \circ \varphi)(t) \times \varphi'(t)\ dt.$$

Now a so-called *principle of permanence* of non standard mathematics allows us to say that this infinitesimal equivalence remains true for some ^{n}infinitely large integer $\overline{m}$.

Both $\displaystyle\int_{y-\frac{1}{\overline{\overline{m}}}}^y (f' \circ \varphi)(t) \times \varphi'(t)\ dt$ and $\displaystyle\int_{y-\frac{1}{\overline{\overline{m}}}}^y (f^{]'[_n} \circ \varphi)(t) \times \varphi'(t)\ dt$ are ^{n}infinitesimal because the length of the interval $[y - \dfrac{1}{\overline{m}}, y]$ is ^{n}infinitesimal and the function $(f' \circ \varphi)(t) \times \varphi'(t)$ as well as $(f^{]'[_n} \circ \varphi)(t) \times \varphi'(t)$ is ^{n}limited.

Case where $f = \mu\ H$, and φ has only one (isolated) zero a.

Let be $n \geq \eta((a, \varphi))$.

If f is locally increasing or decreasing, let α and β such that $\varphi(\alpha) = -h_n$ and $\varphi(\beta) = h_n$. Then we have

If φ is locally increasing

$$f \circ \varphi = \mu\ H \circ \varphi = \mu\ H_a$$
$$\mu\ H_{-h_n} \circ \varphi = \mu\ H_\alpha$$
$$\mu\ H_{h_n} \circ \varphi = \mu\ H_\beta$$

If φ is locally decreasing

$$\begin{aligned} f\circ\varphi = \mu\ H\circ\varphi &= \mu\,(1-H_a)\\ \mu\ H_{-h_n}\circ\varphi &= \mu\,(1-H_\alpha)\\ \mu\ H_{h_n}\circ\varphi &= \mu\,(1-H_\beta) \end{aligned}$$

If φ is locally increasing

$$(f\circ\varphi)^{]'[} = (\mu\ H_a)^{]'[} = \mu\ \delta_a$$

$$(f^{]'[_n}\circ\varphi)\times\varphi' = \frac{\mu\,(H_{-h_n}\circ\varphi - H_{h_n}\circ\varphi)\times\varphi'}{2h_n} = \frac{\mu(H_\alpha - H_\beta)}{2h_n}\times\varphi'$$

Now, $\dfrac{1}{2h_n}\ \mu\,(H_\alpha - H_\beta)\times\varphi'$ is Dirac-equal to μ,δ_a for it is zero outside of $\wr\, a\,\wr$, it have a constant sign, and if $y << a << x$ then

$$\int_y^x \frac{1}{2h_n}\ \mu\ (H_\alpha - H_\beta)\times\varphi') = \frac{1}{2h_n}\mu, \int_\alpha^\beta \varphi' = \frac{1}{2h_n}\ \mu(\varphi(\beta)-\varphi(\alpha))$$

$= \mu\,\dfrac{1}{2h_n}\cdot 2h_n = \mu$. Hence: $(f\circ\varphi)^{]'[} \overset{D}{=} (f^{]'[_n}\circ\varphi)\times\varphi'$.
The proof is quite similar if φ is locally decreasing.

If a is a local extremum then $f\circ\varphi$ is constant, its value is μ or 0 so $(f\circ\varphi)^{]'[}$ is 0. So is $(f^{]'[_n}\circ\varphi)\times\varphi'$, because $H_{-h_n}\circ\varphi = H_{h_n}\circ\varphi = 1$ or 0.
The reader is now able to proceed alone to the more general case.

If $\eta(f)\geq\eta(\varphi)$, the formula of the theorem becomes

$$(f\circ\varphi)^{]'[} \overset{D}{=} (f^{]'[}\circ\varphi)\varphi'.$$

If $\eta(\varphi) > \eta(f)$ the formula $(f\circ\varphi)^{]'[} \overset{D}{=} (f^{]'[}\circ\varphi)\varphi'$ could be false.

Counterexample.

$f = H$ and $\varphi(x) = hx^2$ give :
$(f\circ\varphi)^{]'[} = 0$. For any locally standard-finite $F\subset\mathbb{R}$, there exists x such that $-x, 2x \notin\ \wr\, F\,\wr$, $-x << 0 << 2x \leqq 1$. $(f^{]'[}\circ\varphi)\varphi'(x) = (H^{]'[}(hx^2))\times 2h\,x = \delta(hx^2)\ 2hx \neq^D 0$, because $\displaystyle\int_{-x}^{2x}\delta(ht^2)\times 2h\,t\ dt = \frac{1}{2h}\,h\left[t^2\right]_{-x}^{2x} = \frac{3\,x^2}{2} \not\sim 0$.

Corollary. *for any $f\in\mathcal{R}$ and any derivable standard φ such that φ' have isolated zeros then $f\circ\varphi\in\mathcal{R}$ and*

$$(f\circ\varphi)^{]'[} \overset{D}{=} (f^{]'[}\circ\varphi)\varphi'.$$

Proof. The reason is that under the assumption on φ, $\varphi^{-1}(F(f))$ and $\varphi^{-1}({}^oF(f))$ are locally standard-finite. Let us prove it. Let $[x,\ y]$ be a standard compact interval then $\varphi([x,y])$ is a standard compact interval too so, it contains a finite-standard

number of element of $F(f)$.
If $\varphi^{-1}(F(f)) \cap [x,\ y]$ or $\varphi^{-1}({}^oF(f))$ contains non finite-standard many elements then $c \in F(f)$ exists such that $card(\varphi^{-1}(\{c\}))$ is not standard-finite. This implies that the standard set $Z = (\varphi')^{-1}(\{0\})) \cap [x, y]$ is infinite and contradicts the hypothesis that the zero of φ' are isolated. Hence $card(\varphi^{-1}(F(f)) \cap [x, y])$ and $\varphi^{-1}({}^oF(f))$ are standard-finite.

Examples.

1. $H(x^2) = 1$ gives $H(x^2)^{]'[} = 0$. With de denotations of theorem 7 we have $F(f) ={}^o F(f) = \varphi^{-1}({}^oF(f)) = \{0\}$ so $H(x^2)^{]'[} \stackrel{D}{=} \delta(x^2) \times x^2$. Hence $\delta(x^2) \times x^2 \stackrel{D}{=} 0$, which can be obtained also through a direct calculation.

2. $\delta(x^2)^{]'[} \stackrel{D}{=} \delta^{]'[}(x^2) \times x^2$.

Open problem. What happens if it is only supposed that $\varphi \in \mathcal{R}$. Under which conditions is the formula $(f \circ \varphi)^{]'[} \stackrel{D}{=} (f^{]'[} \circ \varphi)\varphi^{]'[}$ true?

The next theorems, 8 and 9, approach the problem of the conservation of the Dirac-equality after a multiplication term by term.

Theorem 8. *For any $f,\ g \in \mathcal{R}$ and any standard integer n*

$$f \stackrel{D}{=} g \quad \Rightarrow \quad x^n\, f \stackrel{D}{=} x^n\, g.$$

Proof. It is enough to prove that $f \stackrel{D}{=} g \quad \Rightarrow \quad x\, f \stackrel{D}{=} x\, g$, and one obtains the theorem through an induction. Let us prove inductively that the property $P(n)$: $\forall\ f,\ g \in \mathcal{R} \quad (\ f =_n^D g \ \Rightarrow\ x\, f =_n^D x\, g\ )$,
is satisfied for any standard n. $P(0)$ is obvious. The reason is that for any $x \notin \wr F(f) \wr \cup \wr F(g) \wr$, and any limited order $k \geqq 0$ of derivation, $(xf(x))^{(k)} = xf^{(k)}(x) + kf^{(k-1)}(x)$. The numbers x and k being limited, the relations $f^{(k)}(x) \sim g^{(k)}(x)$ and $f^{(k-1)}(x) \sim g^{(k-1)}(x)$ (if $k > 0$) involve $xf^{(k)}(x) \sim xg^{(k)}(x)$ and, if $k > 0$, $kf^{(k-1)}(x) \sim kg^{(k-1)}(x)$. Hence $(xf(x))^{(k)} \sim (xg(x))^{(k)}$.
Let us suppose $P(n)$ for a fixed standard n. Let $f,\ g$ be elements of $\mathcal{R}$ Let us fix $y \notin \wr F(f) \wr \cup \wr F(g) \wr$. Let us put $F(x) = \int_y^x f(t)\ dt$ and $G(x) = \int_y^x g(t)\ dt$.
Theorem 5 gives $F \stackrel{D}{=} G$, so $F =_n^D G$.
$P(n)$ being true, we obtain $x\, F(n) =_n^D x\, G(x)$, and by theorem 6,
$(x\, F(n))^{]'[} =_{n+1}^D (x\, G(x))^{]'[}$. From the corollary 2 of theorem 4 we get
$x\, f(x) + F(x) =_{n+1}^D x\, g(x) + G(x)$. As $F =_{n+1}^D G$, we obtain
$x\, f(x) =_{n+1}^D x\, g(x)$. Hence $P(n)$ is true for any standard n.

Of course, in theorem 8 we can replace x^n by $(x-a)^n$. We shall prove now a similar theorem where x^n is replaced by a non polynomial function.

Theorem 9. *Let $f,\ g,\ u \in \mathcal{R}$ with $f \overset{D}{=} g\ (F)$ and u standard. Let be $G = F \cup F(u)$. If for any $a \in G$ there exists a standard integer n_a and $y_a,\ x_a \in \mathbb{R}$ such that,*

$$[y_a, x_a] \cap \wr G \wr = \wr a \wr \ \text{ and } \int_{y_a}^{x_a} |(t-a)^{n_a}(f(t)-g(t))|\ dt \sim 0,$$

then if u has n_a continuous derivatives at each $a \in G$: $u\, f \overset{D}{=} u\, g$.

Proof. $f^{(p)}(t) \sim g^{(p)}(t)$ for any $t \notin \wr G \wr$ and any limited p. Hence u being standard, we have $(uf)^{(k)}(t) \sim (ug)^{(k)}(t)$ for any $t \notin \wr G \wr$ and any limited k. Hence $u\, f =_0^D u\, g$. To end the proof it is enough to show that ${}_k\int_{y_a}^{x_a} u(t)\, f(t)\ dt \underset{k}{\sim} {}_k\int_{y_a}^{x_a} u(t)\, g(t)\ dt$ for any limited $a \in G$ and any standard k. Obviously, y_a and x_a can be chosen limited. Now, by the hypothesis on u, we have an expansion of the form

$$u(t) = \sum_{i=0}^{n-1} (t-a)^i\ \frac{u^{(i)}(a)}{i!} + (t-a)^{n_a}\ \frac{u^{(n_a)}(a+\theta(t-a))}{n_a!},\ \ \theta \in [0,1].$$

As $u^{(n_a)}$ is continuous on $[y_a, x_a]$ there is a standard constant C such that, for any $x \in [y_a, x_a]$,

$$\left| \int_{y_a}^{x} (t-a)^{n_a}\ \frac{u^{(n_a)}(a+\theta(t-a))}{n_a!}\ (f(t)-g(t))\ dt \ \right| \leqq$$

$$C \int_{y_a}^{x_a} |(t-a)^{n_a}(f(t)-g(t)|\ dt \sim 0.$$

Hence : $\displaystyle\int_{y_a}^{x} (t-a)^{n_a}\ \frac{u^{(n_a)}(a+\theta(t-a))}{n_a!}\ (f(t)-g(t))\ dt\ \sim 0.$

This implies for the k-iterated integrals, with standard k :

$$\textbf{(1)} \qquad {}_k\int_{y_a}^{x_a} (t-a)^{n_a}\ \frac{u^{(n_a)}(a+\theta(t-a))}{n_a!}\ (f(t)-g(t))\ dt\ \sim 0.$$

Now, theorem 8 gives, for any $i \in \{0, ..., n_a - 1\}$ and any standard k

$$\textbf{(2)} \qquad {}_k\int_{y_a}^{x_a} (t-a)^i\, u^{(i)}(t)\, (f(t)-g(t))\ dt\ \sim 0.$$

The conjunction of **(1)** and **(2)** gives ${}_k\displaystyle\int_{y_a}^{x_a} u(t)\, f(t)\ dt \underset{k}{\sim} {}_k\int_{y_a}^{x_a} u(t)\, g(t)\ dt.$

Example of an application of theorem 9.

Let be $f = 4\,(1-2H)\,\delta^2,\ \ g = \delta^{]'[}$ and $u(x) = e^x$ ($F(u) = \emptyset$).

Let us admit $f \overset{D}{=} g\ (F)$ with $F = \{0\}$, whose checking is not very difficult. Let us prove that for any x_0, y_0 such that $y_0 << 0 << x_0$, $\displaystyle\int_{y_0}^{x_0} |t^3\,(f(t)-g(t))|\ dt \sim 0.$

On the one hand $\displaystyle\int_{y_0}^{x_0} |t^3 f(t)|\ dt = \int_{-h}^{h} |t^3 f(t)|\ dt = \frac{1}{h^2}\left[\frac{t^4}{4}\right]_{-h}^{h} \sim 0,$
on the other hand

$\displaystyle\int_{y_0}^{x_0} |t^3 g(t)|\, dt = \int_{-h-\widetilde{h}}^{-h+\widetilde{h}} |t^3 g(t)|\, dt + \int_{h-\widetilde{h}}^{h+\widetilde{h}} |t^3 g(t)|\, dt =$
$\displaystyle\frac{1}{4h\widetilde{h}} \times 2\widetilde{h} \times (-h+\theta_1\widetilde{h})^3 + \frac{1}{4h\widetilde{h}} \times 2\widetilde{h} \times (h+\theta_2\widetilde{h})^3,\ \theta_1, \theta_1 \in [-1,1].$
This equality implies that $\displaystyle\int_{y_0}^{x_0} |t^3 g(t)|\, dt \sim 0.$
From $\displaystyle\int_{y_0}^{x_0} |t^3 f(t)|\, dt \sim 0 \sim \int_{y_0}^{x_0} |t^3 g(t)|\, dt$ we deduce that

$$\int_{y_0}^{x_0} |t^3(f(t) - g(t))|\, dt \sim 0.$$

Hence theorem 9 applies and we can conclude that $u\, f \stackrel{D}{=} u\, g$.

4 Applications.

1 - From $x\delta =^D 0$ and theorem 6 we obtain $(x\ \delta)^{]'[} \stackrel{D}{=} 0 \stackrel{D}{=} x\delta^{]'[} + \delta$ so, $x\ \delta^{]'[} =^D -\delta$.

An easy induction give : For any standard integers p and q

$$\textbf{(1)} \qquad x^p\, \delta^{]q[} =^D \begin{cases} 0 \ if \ p > q \\ (-1)^p\, \dfrac{q!}{(q-p)!}\, \delta^{]q-p[},\ if\ q \geq p \end{cases}$$

The statement of this result is similar to a classical result in distribution theory. The principal difference is that our relations involve Dirac equality, and not equality of distributions .

The next formula has no equivalent in distribution theory because the products $\delta\, \delta^{]'[}$ and δ^2 are not defined.

2 - From the easily verifiable relation $x\delta^2 \stackrel{D}{=} 0$ we get $(x\delta^2)^{]'[} \stackrel{D}{=} 0$ by an application of theorem 6 , which can be developed in

$$\delta^2 + 2x\delta\delta^{]'[} \stackrel{D}{=} 0\ , \quad 2x\delta\delta^{]'[} \stackrel{D}{=} -\delta^2. \qquad (\mathbf{2})$$

3 - If $x_+^p(t)$ is the function with value 0 for negative t and t^p for positive t (in particular $x_+^0 = H$) then for any standard integer p

$$x_+^p\, \delta^{]p[} \stackrel{D}{=} \frac{(-1)^p\, p!}{2}\, \delta. \qquad (\mathbf{3}_p)$$

The formula is true if $p = 0$ because $x_+^0\, \delta^{]0[} = H\, \delta \stackrel{D}{=} \dfrac{1}{2}\, \delta$. Let us suppose it is true for a fixed standard integer p. Multiplying by x the two members of $x_+^p\, \delta^{]p[} \stackrel{D}{=} \dfrac{(-1)^p\, p!}{2}\, \delta$ we obtain, through theorem 8,

$$x_+^{p+1}\, \delta^{]p[} \stackrel{D}{=} \frac{(-1)^p\, p!}{2}\, x\, \delta \stackrel{D}{=} 0.$$

Taking the derivatives we get thanks to theorems 1 (see example) and corollary 2 of theorem 4 we have

$$(p+1)\, x_+^p\, \delta^{]p[} + x_+^{p+1}\, \delta^{]p+1[} \stackrel{D}{=} 0$$

which gives

$$x_+^{p+1}\, \delta^{]p+1[} \stackrel{D}{=} \frac{(-1)^{p+1}\,(p+1)!}{2}\, \delta$$

so the relation is true for any p.

The fourth example corresponds to a recent (1999) formulation by Damyanov [Da] in Colombeau's theory of generalized functions, with a weak equality.

4 - with x_+ defined above, p standard integer

$$\frac{(-1)^p}{p!}\, x_+^p\, \delta^{]p+1[} + \delta^2 \stackrel{D}{=} \frac{p+1}{2}\, \delta^{]'[}. \qquad \mathbf{(4)}$$

The result is true if $p=0$, because $(H\,\delta)^{]'[} \stackrel{D}{=} \widetilde{\delta}\,\delta + H\,\delta^{]'[}$, $\quad x_+^0 = H$, $(H\,\delta)^{]'[} \stackrel{D}{=} \frac{1}{2}\delta^{]'[}$ and $\delta\,\widetilde{\delta} \stackrel{D}{=} \delta^2$.

Let us suppose that the formula is true with the integer p. From the formula of example $(\mathbf{3}_{p+1})$, we obtain through a derivation

$$(-1)^{p+1}\, \frac{(p+1)!}{2}\, \delta^{]'[} \stackrel{D}{=} (p+1)\, x_+^p\, \delta^{]p+1[} + x^{p+1}\, \delta^{]p+2[}.$$

If we apply the hypothesis of induction we obtain

$$\frac{(-1)^{p+1}}{(p+1)!}\, x_+^{p+1}\, \delta^{]p+2[} + \delta^2 \stackrel{D}{=} \frac{p+2}{2}\, \delta^{]'[},$$

so the formula is true for $p+1$ and consequently, for any standard p.

Final remarks

1- Many theorems exist which aim at representing or "approaching" *standard* distributions by nonstandard functions : see for example $[Gr, I, K, Rob, (S-L), T]$. Among all these approaches Kinoshita's one, continued later by J.P. Grenier the formulation of which are close to numerical analysis, presents a particular interest, and can be linked to the work presented in this paper. Now, it will be first necessary to forget reference to the theory of distributions and to study Kinoshita-Grenier's generalized functions for themselves, referring to the physical world.

2 - The relations which we established are not true for any evaluation of the delta function, of the functions $\frac{1}{x}$, *ln* x etc ... . For example the relation $\frac{1}{x}\,\delta \stackrel{D}{=} \frac{1}{2}\,\delta^{]'[}$ is strongly dependant on the choice of the evaluations. This choice should be decided by the physical observation.

However, from theorems 6 and 8 we can prove that the formula **(1)** is still true for any other choice of the delta function.

The formula **($\mathbf{3}_p$)** remains true if we replace the principal evaluation δ of the Dirac function by any even function ζ such that $\zeta =^D \delta$. Now there is a change in **(4)**, δ^2 should be changed, not in ζ^2 but in $\zeta\,\widetilde{\zeta}$, **(2)** is true for any delta function ζ such that $x\,\zeta^2 =^D 0$. Otherwise, if ζ is even then $x\zeta^2$ is odd so

$$x\,\zeta^2 =_1^D 0, \qquad (x\,\zeta^2)^{]'[} =_2^D 0, \text{ and } 2\,x\,\zeta^2 =_2^D -\zeta^2.$$

References

[1] J.F. Colombeau, Elementary introduction to new generalized functions, North-Holland Mathematics Studies. 113, Amsterdam, 1985.

[2] B. Damyanov, Multiplication of Schwartz distributions and Colombeau Generalized functions, Journal of Appl. Anal., 5 (1999), 2, 249-60.

[3] A. Delcroix, Fonctions généralisées et anlyse non standard. [J] Monatsh. Math, 123, n2 (1997) 127-134.

[4] P.A.M. Dirac, The physical interpretation of the Quantum Dynamics, Proc. of the Royal Society, London, section A, 113 (1926-27), 621-641.

[5] Yu.V. Egorov, A contribution to the theory of generalized functions, Russian Math. Surveys 45;5 (1990), 1-49.

[6] S.J.L. Van Eijndhoven - J. De Graaf, A Mathematical Introduction to Dirac's Formalism, North-Holland Mathematical Library, vol 36 (1986).

[7] E.I. Gordon, Relatively standard element in Nelson's internal set theory, Sib. Math. J. 30, n 1 (173) (1989), 86-95.

[8] J.P. Grenier, Représentation discrète des distributions standard, Osaka J. of Math. 32 (1995), 799-815.

[9] C.Impens Standard and nonstandard polynomial approximation. J. Math. Anal. Appl. 171, n 2 (1992) 361-376.

[10] M. Kinoshita, Non-standard representations of distributions Osaka J. Math.,I 25 (1988), 805-824 and II 27 (1990), 843-861.

[11] J. Mikusinski, Bull. Acad. Pol. Ser. Sci. Math. Astron. Phys., 43 (1966), 511,13.

[12] E. Nelson, Internal set theory : a new approach to nonstandard analysis. Bull. A.M.S. 83 (1977), 1165-1198.

[13] M. Oberguggenberger, Product of distributions, Journal fr Mathematic. Band 365 (1984).

[14] M. Oberguggenberger, T. Todorov, An embedding of Schwartz distributions in the algebra of asymptotic functions; Int. J. Math. and Math.Sci. Vol 21 n 3 (1998) 417-428.

[15] Y. Péraire, Théorie relative des ensembles internes, Osaka J.Math. 29 (1992), 267-297.

[16] Y. Péraire, Some extensions of the principles of idealization transfer and choice in the relative internal set theory, Arch. Math. Logic (1995) 34 : 269-277.

[17] C.Raju, Product and compositions with the Dirac delta function,J.Phys. A: Marh. Gen. 15 (1982), 2, 381-96.

[18] A. Robinson, Non Standard Analysis, North-Holland (1974).

[19] E. Rosinger, Distributions and nonlinear partial differential equations, Lecture Notes in Math., Springer-Verlag (1978).

[20] K.D. Stroyan, W.A.J. Luxemburg, Introduction to the theory of infinitesimals New-York Academic Press (1976) 299-305.

[21] T. Todorov, A nonstandard delta function, Proc. of the A.M.S. (1990) Vol. 110, n4.

[22] L.Schwartz, Sur l'impossibilité de la multiplicatiion des distributions, C.R. Acad. Sci. Paris, 239 (1954), 847-848.

[23] L.Schwartz, Théorie des distributions, Hermann-Paris (1966).

[24] C. Zuily, Distributions et équations aux dérivées partielles, Hermann, collection Méthodes, (1994).

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