C.F.Gauss and the beginnings of invariant theory

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There is a direct link between the work of Gauss and Einstein's general relativity theory, this link is constituted by invariant theory. Invariant theory is twofold, one part lies within algebra, one part within differential geometry; the first one is called invariant theory, the second the theory of differential forms. Both have something to do with forms, either algebraic forms, or, equivalenty, differential forms.

Gauss was not the first who investigated forms, but he was the first who established a new discipline, the theory of forms. More than half of Gauss' *Disquisitiones arithmeticae* are devoted to the theory of forms, it contained mainly the theory of binary and ternary forms, i.e., transformations, similarities, classifications and compositions of forms ([1], I, 120-379). However, Gauss is not the founder of invariant theory, this theory is due to the so-called British school. The main contributors in the beginning were George Boole, Arthur Cayley and James Joseph Sylvester. Invariant theory became one of the most flourishing mathematical disciplines in the second half of the 19th century and was still an important discipline in the beginning of the 20th century.

Gauss's theory of surfaces In 1820 Gauss started with surveying, viz., with the triangulation of the Kingdom of Hannover. As a mathematical result Gauss published in 1828 his surface theory: *General investigations of curved surfaces.* There he presented his main theorem, the so called "theorema egregium", where he used the word "invariata":"If a curved surface is developed upon any other surface whatsoever, the measure of curvature in each point remains invariant" ([1], IV, 237). In the following paragraph Gauss redefined "surface" as depending on the linear element only. He also spoke of absolute and relative properties of surfaces. The absolute remain invariant, whatever the form into which the surface is bent, to these the theory of shortest lines belongs. In letters to Heinrich Christian Schumacher Gauss made a distinction between "Oberflächen" and "Flächen", this distinction can only be given in the German language ([2], III, 104f & IV 83f). Gauss's linear

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element was the first differential form and his curvature the first differential invariant in history:

$$ds^2 = Edp^2 + 2Fdpdq + Gdq^2$$

Beginnings of the theory of differential invariants There is no doubt that, intellectually, Bernhard Riemann was a pupil of Gauss, as far as geometry is concerned. In his famous speech in 1854 "On the Hypotheses which lie at the Bases of Geometry" Riemann introduced n-dimensional manifolds, while Gauss's surfaces had only been two-dimensional. And, for n-dimensional manifolds Riemann needed an n-dimensional linear element and a n-dimensional curvature. This famous speech and the mathematical theory, however, were only posthumously published, in 1868. After the publication of Riemann's "Hypotheses" there were several authors who presented further investigations of differential forms and differential invariants, as Elwin Bruno Christoffel, Rudolf Lipschitz, Gregorio Ricci, Tullio Levi-Cività and others. So the theory of differential invariants, founded on the theory of differential forms, was further developed and at last became the "Absolute Differential Calculus". In 1901 Gregorio Ricci together with his pupil Tullio Levi-Civitá published [3] where they presented the absolute differential calculus as an "algorithm" which revolutionized the so-called intrinsic geometry. In 1908 Edmund Wright published a book on the "Invariants of quadratic differential forms", where the whole theory including the methods of Christoffel, Lie and Maschke were presented. It is noteworthy that the terminology of the theory of differential forms was the same as the terminology within invariant theory, as can be seen e.g. in the terminology (e.g. covariance, contravariance). So, sooner or later, invariant thinking became very common, not only within mathematics but also within physics. It was this invariant thinking which was the source of Einstein's general relativity theory.

Einstein's relativity theory The young Einstein was an excellent physicist but a bad mathematician. In 1912 Einstein had a clear idea or vision how his new physical theory should look like, but he did not know enough mathematics to formulate the new mathematical theory. He got help from his friend Marcel Grossmann who as a mathematician was familiar with the newest developments in mathematics. Grossmann recommended to adopt the absolute differential calculus, presented by Ricci and Levi-Cività in 1901. In this theory, all mathematical tools for relativity theory could be found. In 1913 Einstein and Grossmann jointly published two papers where they introduced the line element in its general form.

In 1922 Einstein remembered:

> If all [accelerated] systems are equivalent, then Euclidean geometry cannot hold in all of them. To throw out geometry and keep [physical] laws is equivalent to describing thoughts without words. We must search for words before we can express thoughts. What must we search for at this point? This problem remained unsolvable to me until 1912, when I suddenly realized that Gauss's theory of surfaces holds the key for unlocking this mystery. I realized that Gauss's surface coordinates had a profound significance... ([4], 211)

It was Gauss's surface theory which played a key role for Einstein and it was the idea of invariance which became the main point. Einstein and Grossmann were the first who used the letter g for the description of the metric: g_{ik}. Of course, the letter g should remind the reader of gravitation but perhaps it should also remind the reader of Gauss and his surface theory.

Neither Gauss nor his followers, had thought of applicating their theories in physics, they had created a purely mathematical theory. Einstein, however, was in first line a physicist, for him the g_{ik} were not purely the metric of the metric tensor, but the metric was a consequence of the surrounding gravitational field. With Einstein the g_{ik} got a physical meaning which it did not have previously. Neither Gauss, Riemann, Ricci nor others did answer the question, why there existed a curvature of space, they only answered the question how space curvature looks like. They did not ask for physical reasons. It was Einstein who connected mathematics and physics in this way: Matter tells space how to create curvature, and space tells matter, how to move.

References

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