A Note on Article 36 in Gauss's *Disquisitiones*

A Ramificated Story in the Margin of the Re-Writing of Section II

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Even 150 years after Carl Friedrich Gauss's death, his magnum opus, the *Disquisitiones Arithmeticae* ([5], I = DA), has lost nothing of its fascination. Gauss and his work have been described as the occurence of a comet in a clear sky, inaugurating a new era of mathematics, of number theory in particular. The *Disquisitiones*, however, did not fall from the sky, and many links with not only previous number-theoretical works but also with near forgotten mathematicians as Hindenburg or with a tradition of German textbooks can be found (see [1], part 3). This opens up the often dense if not cryptic text of the *Disquisitiones* and helps to gain a richer understanding of the environment in which Gauss's treatise was written und thus of its proper innovations. This article will illustrate this point by focusing on a small example, DA's article 36, showing its rooting in current discussions in German mathematics before 1800.

In the DA, Article 36 follows Gauss's treatment of the Chinese Remainder Problem and considers the system (I) $[A, B, C$ being relatively prime]:

$$\begin{aligned} X &\equiv a \pmod{A} \\ X &\equiv b \pmod{B} \\ X &\equiv c \pmod{C} \end{aligned}$$

Solving this system of congruences for $a = 1, b = 0, c = 0$; then for $a = 0, b = 1, c = 0$ and for $a = 0, b = 0, c = 1$ generates the 3 solutions for X: α, β, γ. The formula to calculate all solutions to (I) for a given triple (a, b, c) is: $X \equiv \alpha a + \beta b + \gamma c$ (mod. ABC).

And then follows an example, one of the few instances of an everyday problem in the DA, the calculation of year's number in a Julian period:

> This formula is useful for the chronological problem, where the Julian year is asked, when given the indiction [a], golden number [b] and solar cycle [c]. Here $A = 15$,

> $B = 19$, $C = 28$ ... the year is then the minimal residue of $6916a + 4200b + 4845c$ (DA, art. 36)

The use of the variables indiction, sun cycle and golden number for solving this calender problem goes back to Clavius and their tabulation has a long standing tradition of being included in handbooks of chronology. Gauss had already treated the calculation of these variables in his Easter-formula, published 1800 ([5], VI, 73-79). Article 36 is the correlate of this work: The Easter-formulae calculate sun cycle, indiction, and golden number explicitly, and replace the usual look-up tables; the DA-example on the contrary analytically proves a simple formula to calculate the year in a Julian period, given the 3 variables.

The simple formula Gauss deduces in DA's article 36 was not new. Due to the fact that from 1777 onwards only, the German protestant and catholic states agreed to have a common rule for the calculation of Easter, this topic was quite intensively studied and debated in the period 1750-1800 among German mathematicians. Following a suggestion of J.H. Lambert, J. Bernouilli was the first to indicate the constant numbers 6916, 4200 and 4845, an analytical justification was given by the Göttingen professor Kästner in [7], II, 437-441. Apart from Kästner's long proof through successive substitution, the Leipziger professor C.F. Hindenburg also provided a deduction of the formula in an article on Diophantine problems ([6], 301-302). The earliest solution, however, although never mentioned in Kästner or Hindenburg, is in a 1735-paper by the young Euler [4], that constitutes an early attempt to homogenise the treatment of classic "Rechenbuch" remainder problems, not unlike Gauss's own Section II in the DA (see [2]).

Another framework ambitioning such a homogenisation is Hindenburg's, who solves the problems with his "Complexionen" (complexions) and "Ordnungszahlen" (ordering numbers), i.e., enumerated combinations of cycles, as e.g. in this example (combination of cycles with length 2, 3 and 4):

(1) 1, 1, 1	(5) 1, 2, 1	(9) 1, 3, 1
(2) 2, 2, 2	(6) 2, 3, 2	(10) 2, 1, 2
(3) 1, 3, 3	(7) 1, 1, 3	(11) 1, 2, 3
(4) 2, 1, 4	(8) 2, 2, 4	(12) 2, 3, 4

The 1st column ranges over the natural numbers modulo 2, the 2nd column modulo 3 and so on. To solve system (I) (and hence calculate the constants in the Julian year formula), Hindenburg respectively sets his 2nd and 3d; his 1st and 3d; his 1st and 2nd cycle to zero (actually to the maximum number, equal to the length of the cycle) - the equivalent of Gauss's and Euler's trick.

Article 36 thus offers a glimpse into the diverse traditions Gauss absorbed during his education (for more details, [1], part 3.3), but is also notable for its algorithmic optimisation, a salient feature in Gauss's work in general, and in the writing process of the DA in particular (see [3]).

This last aspect of the problem resurfaces in the computer era. Both Hindenburg's system and Gauss's formula were largely forgotten by that time, but in 1955 A. Svoboda and M. Valàch rediscovered Hindenburg's complexions, but called it modular number systems, in 1958 Gartner found a variant of Gauss's and Hindenburg's formula for conversion of modular number representations into decimal representation, which Szabó proved to be the best possible general algorithm in 1961 ([8], 284-292, algorithm 291).

References

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