Algebraic analysis and the use of indeterminate coefficients by Etienne Bézout (1730-1783)

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The name of Étienne Bézout is well-known in mathematics, but we are only now able to throw some light on his mathematical career and his exact achievements (see [1]). Bézout (1730–1783), recruited to the Paris Academy of Sciences in 1758, began working on algebraic analysis in 1762 and presented an important work on elimination on February 1, 1764. However, a few months later, he was appointed examiner for the Naval Officers Schools and put in charge of reforming mathematical studies; the Artillery School was added to his load in 1768. Spending most of his time on the road, visiting six Officers Schools all around France, he restricted his research interest to one topic only, algebraic analysis, essentially the theory of equations; however, he also wrote mathematical textbooks which remained best-sellers for about a century. The closure of the Artillery school in 1773, by order of the king, allowed Bézout to turn again to more advanced projects, in particular his most famous work, the *General Theory of Algebraic Equations*, published in 1779. It contains one of his most famous results, the theorem which bears his name in modern algebraic geometry. Here, we will briefly follow him from his early work on elimination to his synthetic treatise on algebraic equations, and study more closely his specific use of indeterminate coefficients to find the degree of the resultant.

Bézout's first works The fundamental question in the 1760s was to solve equations of degree equal to or higher than 5. His early works on this topic in 1762 and 1765 ([2], [4]) convinced Bézout that elimination was the most important step towards computing roots (we know today that he was wrong).

To give a simple example of elimination, let us consider $\left\{\begin{array}{c} ax^2+bx+c=0 \\ a'x^2+b'x+c'=0 \end{array}\right.$, with the condition $ab'-ba'\neq 0$, where a, b, c, a', b', c', can be either real numbers or polynomials in, say, y. Eliminating x in this system gives the existence condition for solutions, $(ab'-ba')(c'b-b'c)=(a'c-ac')^2$, which is called the resultant (or eliminant) of the two equations. This is either a numerical relation, if x is the only unknown, or a polynomial equation in y, if a, b, c, a', b', c', are themselves polynomial functions of y. In order to find this resultant in general, Bézout used a method of indeterminate coefficients.

The Method of Indeterminate Coefficients This method, already used and explained in 1637 by Descartes in *La Géométrie*, was thus described in the *Encyclopédie* by d'Alembert:

> This method consists in assuming the unknown equal to a quantity in which there are coefficients supposedly known, and which are designated by letters; we substitute this unknown in the equation, and, [...] *we determine in this way, the indeterminate coefficients.*

But Bézout did *not* use the method of indeterminate coefficients in his works on elimination, neither in 1764 nor in 1779, in order to compute the indeterminate coefficients, nor to search for a function matching certain conditions, nor to give an a priori form for certain components of the problem, as his predecessors did. He used indeterminate coefficients as a *tool* to build the resultant: he did not compute their value, but counted them to find their optimal number and explored through them the conditions of existence of solutions – these conditions are expressed in the resultant.

For instance, let us consider the 1764 memoir [3]. Let

$$\begin{cases} Ax^m + Bx^{m-1} + Cx^{m-2} + Dx^{m-3} + Ex^{m-4} + ... + V = 0 \\ A'x^{m'} + B'x^{m'-1} + C'x^{m'-2} + D'x^{m'-3} + E'x^{m'-4} + ... + V' = 0 \end{cases}$$

be 2 equations where $A, B, \ldots, A', B', \ldots$, are polynomial functions of y, with respective degrees $p, p+1, p+2, \ldots$, and $p', p'+1, p'+2, \ldots$. Bézout multiplied each of them respectively by (L) and (L') given by:

$$\begin{cases} (L) \quad Mx^n + Nx^{n-1} + Px^{n-2} + Qx^{n-3} + Rx^{n-4} + ... + T \\ (L') \quad M'x^{n'} + N'x^{n'-1} + P'x^{n'-2} + Q'x^{n'-3} + R'x^{n'-4} + ... + T' \end{cases}$$

Equating to zero every power of x in the sum of the two products, he obtained the conditions $m + n = m' + n'$, and $m + n + 1 = n + 1 + n' + 1$, and a linear system in $M, N, \ldots, M', N', \ldots$.

Thus $n' = m - 1$ and $n = m' - 1$, and the condition of existence of a non-trivial solution of the system is that its determinant must be 0. The polynomial equation given by the vanishing of the determinant is the resultant of the system. Computing the terms of the determinant and geometric series of $p, p', \ldots$, Bézout found that its degree is $G = mm' + mp' + m'p$.

We thus see here, as said before, that Bézout did not compute the indeterminate coefficients $M, N, \ldots$, but only established an existence condition. He seems to have been the first to conceive the resultant in such a manner.

Using the same conception, he found in the same paper a new method for two equations of the same degree. Later, he generalized this method and published it in his algebra textbook of 1766, [5] – a good testimony to the close link he perceived between research and teaching at that moment (see [1]). This method was to become the now widely-used method of the Bezoutian. However, already in 1764, Bézout had recognized that he could not handle systems more complicated than 2 equations with 2 variables, without first finding the degree of the resultant. This research gave rise to the first part of his 1779 book, [6].

The degree of the Resultant for complete equations A complete equation is one where no power is missing in the polynomial expression. Let us use Bézout's notations and denote n complete equations with n unknowns by $(u....n)^t = 0$, $(u....n)^{t_1} = 0$, $(u....n)^{t_2} = 0$, $t_2 \geq ... \geq t_{n-1}$. Through the last $(n-1)$ equations, Bézout obtained the values of x^{t_1}, y^{t_2}, z^{t_3}, $\ldots$, and put these values in a polynomial $Q = (u....n)^T$ with indeterminate coefficients and in the product $Q(u....n)^t = (u....n)^{T+t} = 0$. All the terms in $x, y, z, \ldots$ in the product have to be eliminated, in order to keep only one variable u. Thus, one has to compute the number A of terms left in the equation product $Q(u....n)^t = (u....n)^{T+t} = 0$ after the substitution of x^{t_1}, y^{t_2}, z^{t_3}, $\ldots$; and the number B of terms left in $Q = (u....n)^T$, after the same substitutions. These terms can be used to eliminate the terms of $(u....n)^{T+t}$ that contain x, y, z, etc. Bézout denoted by $N(u....n)^T$ the number of terms in a complete polynomial with degree T and n unknowns, and computed it by induction: $N(u....n)^T = \frac{(T+1)(T+2)....(T+n)}{1.2.3....n}$.

He obtained

$$
\begin{aligned}
A &= d^{n-1}\left[N(u....n)^{T+t}\right]\begin{pmatrix} T+t \\ -t_{1,} - t_{2,\ldots} - t_{n-1} \end{pmatrix} \\
B &= d^{n-1}\left[N(u....n)^{T}\right]\begin{pmatrix} T \\ -t_{1,} - t_{2,\ldots} - t_{n-1} \end{pmatrix}
\end{aligned}
$$

where d^{n-1} is the $(n-1)$-finite difference. If D is the degree of the resultant, the number of u-terms is $D+1$. So $(A-D-1)$ terms have to be eliminated in $(u....n)^{T+t} = 0$, and there are $(B-1)$ indeterminate coefficients in Q. Thus, one should have $A-1 = B-D-1$, which finally gives the degree of the resultant as the product of the degrees: $D = tt_1t_2...t_{n-1}$.

Conclusion We have sketched here some aspects of Bézout's original methods in algebraic analysis, both in the use of indeterminate coeeficients and in the conception of the resultant. In the preface of [6] Bézout wrote:

> The analysis of infinitesimals has drawn off all the interest and all the toil, and the algebraic analysis of finite quantities seems to have been looked at only as a field in which either there remained nothing further to be done, or else, that whatever was left to do would have proved fruitless speculation (...). We hope that this work may prove the occasion of great progress in analysis, by turning the talent and the cleverness of analysts of our time towards that important field.

Indeed, his last theorem has proved to be a key result in algebraic geometry and, thanks in particular to computer science, his Bezoutian has found many new applications in the last decades. In view of Bézout's multiple activities, it is only fitting that this last improvement was proposed in a textbook, underlining how one might find a path to develop new results in a pedagogical context.

References

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