The qualitative structure of projective varieties: geometric, arithmetic and complex hyperbolic aspects

Fréderic Campana

Introduction

Most mathematical theories aim at classifiying objects in a certain *category*.

Such classifications usually proceed in two steps:

(1) Definition of *irreducible*[1]objects.

(2) Decomposition, often unique, of arbitrary objects as *(twisted) products* of irreducible objects.

Examples: Prime numbers and factorisation of integers, Finite simple groups and Jordan-Hölder sequences.

• A third example: complex (finite-dimensional) Lie algebras L. These are functorially decomposed in two steps: First as unique extensions: $0 \to R \to L \to S \to 0$ of a semi-simple S by a solvable R.

Then R is canonically decomposed as a sequence of extensions of Abelian algebras, by means of its derived series.

Here extensions play the role of *twisted* products, Solvable (resp. Abelian) and semi-simple Algebras are the irreducible objects in the first (resp. second) step.

The qualitative structure of an arbitrary L can then be understood from the antithetic properties of its "components" R and S, the properties of R being further deduced from its "decomposition" as a tower of abelian algebras.

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[1]or primitive, indecomposable, etc...

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• The aim of the present talk [2] is to describe a formally similar decomposition for the category of complex projective manifolds.

This analogy rests on the following dictionary:

Lie Algebras	**Projective Varieties**
Semi-simple	hyperbolic
Solvable	Special
Abelian	Spherical, or flat
Extension	Fibration
Kernel	General Fibre
Quotient	Orbifold Base

• Let us explain very briefly the rough meaning of the terms in the rightmost column:

A *fibration* $f : X \to Y$ is a surjective map[3] with connected fibres.

Its *general fibres* F are smooth, but not necessarily pairwise isomorphic as complex manifolds, although they are diffeomorphic, and even real analytically isomorphic[4]. Usually, they "degenerate" to singular fibres located over a complex codimension one subset D of the base Y.

Then X may be *roughly* seen as a twisted product of (the deformation class of) F by its *orbifold* base.

This orbifold base (which is the main new ingredient of the constructions described here) is a *virtual ramified covering* of Y.

This virtual ramified cover (see remark 1, in §4, and §6 for more details) branches exactly over the locus of so-called *multiple fibres*, with order prescribed by the multiplicities, and is constructed so as to *virtually* eliminate them.

In the often occuring case where f has no multiple fibres, the orbifold base is simply the actual base Y.

• We shall thus define and describe non-technically:

(1) The 3 *pure* geometries: hyperbolic, flat and spherical.

(2) The special geometry. It will turn out (see theorem 3 in §5) that it should be the *orbifold combination* of the spherical and flat geometries.

(3) A decomposition theorem, for any X, by means of a single fibration (called the *core* of X) splitting X into its antithetical parts: special (the fibres), and hyperbolic (the orbifold base). See theorem 2 in §5.

Conjecturally, this fibration splits X arithmetically, too (and also according to other properties, as well).

In the last §7, we have given technically precise definitions.

[2] held in Gent, May 2005, on the occasion of the BeNeLuxFr mathematical meeting.

[3] meromorphic; for simplification assumed to be holomorphic, here.

[4] one says that the complex structure "deforms".

1 Projective varieties

The complex projective manifolds play a central role in various areas of mathematics (symplectic and differential geometry, algebraic topology, partial differential equations, algebraic groups, arithmetics) and physics.

We shall denote with X a compact connected complex analytic manifold which is *projective*, that is: admits a holomorphic embedding $h : X \to \mathbb{P}^N$ in some complex projective space $\mathbb{P}^N$. We let n be the complex dimension of X, half of its real dimension. When $n = 1$ (resp. 2), we say that X is a curve (resp. a surface).

There are many compact complex manifolds which are not projective: this can be due to topological obstructions (Hopf surfaces), but also to deformations of complex structures (complex tori).

• Algebraic and Analytic Structures.

By a result of Chow, the image $h(X) \subset \mathbb{P}^N$ of X is defined by homogeneous polynomial equations : $h(X) := \{x = (x_0 : x_1 : \cdots : x_N) \in \mathbb{P}^N | P_j(x) = 0, \forall j = 1, \ldots, k\}$, the $P_j(x)'s, j = 1, 2, \ldots, k$ being nonzero homogeneous polynomials on $\mathbb{C}^{N+1}$.

When X is defined by a single homogeneous equation of degree $d \geq 1$, we say that X is a (smooth) hypersurface of degree d, and write $X = H_d$[5]. When $d = 1, X \equiv \mathbb{P}^n$, when $d = 2$, we say that X is an (n-dimensional) quadric, and a cubic, quartic, quintic,$\cdots$, when $d = 3, 4, 5, \cdots$.

Then X becomes an algebraic variety. The algebraic structure being independent of h, by Serre's *GAGA principle*: the global analytic objects on X(defined by holomorphic or meromorphic functions) coincide with the algebraic objects (defined by polynomials or rational functions). Thus complex projective manifolds can be studied both by analytic (or "transcendental"), and algebraic methods. Some results being known only by one of the two methods.

One remarkable example of results obtained by algebraic methods (with a strong arithmetic flavour), and entirely inaccessible by known analytic or characteristic zero algebraic methods is the construction in 1981 by S. Mori of *rational* curves on projective manifolds with *non-semipositive canonical bundle*. The proof proceeds by reduction modulo p (all suitable primes), using the fact that the iterated Frobenius automorphism produce covers of curves of arbitrarily large geometric degrees without increasing the genus. The rational curves so produced have now become the main tool for the study of manifolds of *spherical geometry* mentioned above.

When X is described by equations, its intrinsic geometric properties are not apparent at all. The immediate invariants, such as number and degrees of equations, are extrinsic (depend on the embedding) and usually give few or no insight on the structure of X, except for the rough impression that the complexity of the structure of X increases with the degrees (but even this is not always true (rational normal curve of degree d in $\mathbb{P}^d$): the embedding may be complicated even if X is simple).

[5]These are not biholomorphic when $n \geq 3$, but form one deformation family.

Centuries (or rather two and half millenia) of experience led to extract *one single invariant* which seems to governs the entire geometry of X. And not only its qualitative *geometry* (what is understood by this is particularly difficult to describe precisely, and we shall not attempt to do it), but also conjectural aspects, at first sight totally unrelated: its arithmetics and its complex hyperbolicity properties. This single invariant is the canonical bundle K_X and its positivity or negativity properties (in rough terms, made precise below).

• Arithmetic Structure.

If $P_j(x) \in k[x], j = 1, \ldots, k$, k a number field (ie: a finite extension of $\mathbb{Q}$, generated by an algebraic number α)[6] , X gets an *arithmetic structure*. Its object of study is the set $X(k)$ of points of X with all coordinates in k (the so-called *k-rational points of X*).

This set $X(k)$ depends considerably on the coefficients of the equations defining X (even when then remain in k), and its determination is a problem of arithmetic nature[7].

But things change drastically if we allow finite extensions k'/k of k: X is *potentially dense* if $X(k')$ is dense in X for some finite extension k'/k. This property seems to be of geometric nature. We shall give below a conjectural *geometric* characterisation of potentially dense projective manifolds.

Simplest example of potentially dense X with $X(k) = \emptyset$: the conic: $x^2+y^2+z^2 = 0$, defined over $k = \mathbb{Q}$. Then $X(\mathbb{Q}) = \emptyset$, while $X(\mathbb{Q}[i])$ is dense.

Another less trivial example is given by the Fermat curves X_d: it was shown by Euler that $X_3(\mathbb{Q}) = \{(1 : -1 : 0), (1 : 0 : 1), (0 : 1 : 1)\}$, however X_3 is potentially dense. By contrast, X_d is not potentially dense for $d \geq 4$. See §2 below.

• Complex hyperbolicity.

If V is any complex connected manifold the Kobayashi pseudometric d_V is a metric (in general degenerated in the sense that $d_V(x, y)$ may vanish for two distinct points on V) defined as the largest pseudometric d on X such that for any holomorphic map $h : \mathbb{D} \to X$, and any $u, v \in \mathbb{D}$, one has: $d_{\mathbb{D}}(u, v) \geq d(h(u), h(v))$, where $d_{\mathbb{D}}$ is the Poincaré's metric on the unit disc $\mathbb{D}$. (The two meanings for $d_{\mathbb{D}}$ are shown then to coincide). Any holomorphic $f : V \to W$ is then distance decreasing: $f^*(d_W) \leq d_V$.

A fundamental (although trivial) fact is: $d_{\mathbb{C}} \equiv 0$: because $\mathbb{D}$ can be mapped to $\mathbb{C}$ by $u \to n.(u+a)$, for any $n > 0, a \in \mathbb{C}$, which shows that $d_{\mathbb{C}}(z, z') \leq n^{-1}.|z'-z|, \forall n > 0$.

Thus $d_V(a, b) = 0$ if $a, b \in V$ lie in some entire curve $h(\mathbb{C})$, where $h : \mathbb{C} \to X$ is any holomorphic map. If two arbitrary points in V can be joined by an entire curve $h : \mathbb{C} \to V$, then $d_V \equiv 0$. (The converse for $V = X$ projective is not known, but

[6] But one could take any finitely generated extension of $\mathbb{Q}$, and this would apply to any projective X.

[7] Fermat's last theorem is the statement that $X_d(\mathbb{Q}) = \{(1 : -1 : 0), (1 : 0 : 1), (0 : 1 : 1)\}$ for any $d > 2$, $X_d \subset \mathbb{P}^2$ the Fermat curve defined by $x^d + y^d - z^d = 0$.

quite adequately provides a seemingly good intuitive feeling for this pseudometric, in the complex projective case, at least). For example: $d_{\mathbb{P}^N} \equiv 0$, and $d_T \equiv 0$, if T is a complex torus $\mathbb{C}^n/\Lambda$, with $\Lambda \cong \mathbb{Z}^{\oplus 2n}$ a lattice in $\mathbb{C}^n$.

In the opposite direction, if $d_V > 0$ (meaning that $d_V(a,b) > 0$ for any two distinct points $a, b \in X$, one says that V is *complex hyperbolic*. The distance d_V then defines the same topology as the metric topology on V. This property has been investigated by S. Kobayashi, who showed that holomorphic maps to complex hyperbolic manifolds behave like maps to bouded domains of $\mathbb{C}^n$ (which are particular examples, as well as their compact quotients, all projective).

A theorem of Brody asserts that X, compact complex, is complex hyperbolic if and only if all holomorphic maps from $\mathbb{C}$ to X are constant. In particular, a complex hyperbolic projective manifold X does not contain any rational curve (ie: a curve isomorphic to $\mathbb{P}^1$).

The Kobayashi pseudometric, transcendentally defined, and qualitatively determined only in very rare cases (such as curves, see below), will be given below a *conjectural* simple description of algebro-geometric nature in arbitrairy dimension n, by means of the above *core*.

• This description, and its arithmetic analogue, are inspired by, and extend, the conjectures of S. Lang, who formulated them for manifolds of *general type*. They establish an equivalence between geometry (positivity of K_X), arithmetics, and complex hyperbolicity.

To give an example: it is conjectured that X being *special* (in the precise sense defined below) is equivalent to being potentially dense, and also to have $d_X \equiv 0$. This last condition may also mean that two arbirary points are joined by an entire curve on X).

2 Curves

Let X be a curve (projective complex). Differentiably, X is an orientable surface X_∞, ie: a sphere with g handles, for some unique $g(X) = g$, where $g \geq 0$ is an integer called the *genus* of X. The 3 pure geometries are determined by the genus.

One has: $g = 0 : X_\infty = S^2$, $g = 1 : X_\infty = S^1 \times S^1$, $g \geq 2 : X_\infty = \sharp_1^g(S^1 \times S^1)$

g	X	$deg(K_X)$	$\pi_1(X)$	d_X	$X(k')$
0	$\mathbb{P}^1$	< 0	$\{1\}$	$\equiv 0$	dense
1	$\mathbb{C}/\Lambda$	0	$\Lambda \cong \mathbb{Z}^2$	$\equiv 0$	dense
≥ 2	$\mathbb{D}/\Gamma$	> 0	Γ=Fuchsian group	$> 0 (Liouville's theorem)$	finite

• The degree of the canonical bundle K_X: $deg(K_X) := Z - P$; where Z is the number of zeros, P is the number of poles of w, and w is any nonzero differential meromorphic 1-form on X, written $w(z) = f(z)dz$, f meromorphic nonzero, in any local complex coordinate z.

For example, if $X = \mathbb{P}^1$, then $w(z) = dz = -du/u^2$, has no zero, and a double pole at infinity ($u = 0$). The degree of $K_{\mathbb{P}^1}$ is thus -2.

If $X = \mathbb{C}/\Lambda$ is an elliptic curve, the translation invariant form dz on $\mathbb{C}$ descends to a 1-form of X without zero and pole. Thus degree of $K_{\mathbb{C}/\Lambda}$ is 0.

More generally, if $X \subset \mathbb{P}^2$ is a curve (hypersurface) of degree d, one shows that $deg(K_X) = d(d-3)$. Because the degree of K_X is also equal to $2g-2$, the genus of X is given by $g = (d-1)(d-2)/2$. In particular, we recover the fact that $g = 0$ for $d = 1, 2$, and $g = 1$ for $d = 3$. And that $g = 3, 6, 10$ for $d = 4, 5, 6$.

Z The sign of the canonical bundle is the *opposite* of the sign of the Ricci curvature (of an hermitian metric of constant curvature).

• Arithmetic Structure. Potentially dense curves (see Section 1) are thus exactly those with genus 0 or 1. For $g = 0$, this is easy. But for elliptic curves, one has already to produce points with algebraic coordinates of infinite order. The set $X(k)$ for k a number field is then an abelian group (for the addition law, once an origin has been fixed) shown to be of finite rank by Mordell-Weil (this rank was introduced by Poincaré, who seemed to have considered its finiteness as self-evident).

When $g(X) \geq 2$, $X(k)$ is a finite set, for any number field over which X is defined. This is Mordell's conjecture, showed by G. Faltings in 1983.

• Complex hyperbolicity. Again, the situation is entirely similar: the curve X is complex hyperbolic if and only if $g(X) \geq 2$. This is an immediate consequence of the Poincaré-Koebe uniformisation, and Liouville's theorem.

• The trichotomy. We see the appearance of three geometries, clearly distinct at the level of fundamental group, arithmetics and Kobayashi pseudometric.

g	$deg(K_X)$	Geometry	
0	< 0	Spherical	Special
1	0	Flat	Special
≥ 2	> 0	Hyperbolic	

The first two geometries $K_X < 0, K_X \equiv 0$, respectively termed here *spherical* and *flat*, differ in a relatively minor way, and define in dimension 1 the *special* geometry, antithetical of the third one ($K_X > 0$), termed *hyperbolic*.

3 The three pure geometries: numerical version

• Intersection numbers.

The *sign* of K_X will be used to define the 3 pure geometries in higher dimensions also.

The canonical bundle $K_X := det(\Omega^1_X)$ is the (complex) line bundle on X whose meromorphic sections are the global *meromorphic volume forms* w, written in local complex coordinates $z = (z_1, \ldots, z_n)$: $w(z) = f(z).(dz_1 \wedge \cdots \wedge dz_n)$, $f(z)$ meromorphic.

For any projective complex curve C traced on X, define: $K_X.C := Z - P$, where Z (resp. P) is the number of zeros (resp. poles) of w restricted to C, (provided w does not vanish identically on C, and C is not contained in the locus of poles of w (for a given C, there are lots of such $w's$, and the resulting number does not depend on the choice of w)).

In general, the restriction to C of w is not a 1-form on C, and so $K_X.C \neq deg(K_C)$.

The difference is the degree of the normal bundle to C in X, analogue of the second fundamental form of Riemannian geometry.

• The sign of the canonical bundle.

The 3 pure geometries are then defined by the fact that the sign of $K_X.C$ be constant, independent on the curve C traced on X.

This condition is, of course, satisfied if X is a curve (since $X = C$ is the only choice!), but turns out to be very restrictive if $n \geq 2$.

We thus obtain the 3 possible signs [8], generalising directly the case of curves:

Sign of K_X	*Definition*	*Examples*
$K_X < 0$: Spherical	$K_X.C < 0$	$\mathbb{P}^n$; (G/P); $H_d, d \leq n+1$
$K_X \equiv 0$: Flat	$K_X.C = 0$	$(\mathbb{C}^n/\Lambda)$; H_{n+2}
$K_X > 0$: Hyperbolic	$K_X.C > 0$	$(\mathbb{B}/\Gamma)$; Hermit.Loc.Symmetric Spaces; $H_d, d \geq n+3$

Notations: H_d is a smooth n-dimensional hypersurface of degree $d > 0$ in $\mathbb{P}^{n+1}$, defined by one homogeneous polynomial equation of degree d.

And (G/P) is a rational homogeneous manifold (such as $\mathbb{P}^n$, a Quadric, or a Grassmannian).

$\mathbb{B}$ is the unit ball in $\mathbb{C}^n$.

• Known and expected similarities with curves are summarised in the following tableau:

Sign of K_X	*Fundamental group*	d_X	$X(k')$
$K_X < 0$ (spherical)	$\{1\}$	$\equiv 0$	dense ?
$K_X \equiv 0$ (flat)	Virtually abelian	$\equiv 0$?	dense?
$K_X > 0$ (hyperbolic)	No structure form.	*gen.* > 0? (Lang)	*gen.finite*? (Lang)

Notation: On the last line, the symbols: *gen.* > 0 and *gen.finite* mean that the property holds on the complement $U := X - S$ of some suitable proper algebraic subset S of X. The symbol ? means that the property is conjectured.

Fundamental groups of arbitrary hyperbolic manifolds are completely unknown. No general pattern emerges from the known examples, although there are many restrictions, mainly derived from Hodge theory.

[8] Omitting a condition on the top intersection number of K_X, conjecturally always satisfied, and easy to check in practice

More on the Spherical geometry.

This is the simplest geometry, the results on the first line of the above table (when $K_X < 0$) are obtained using the characteristic $p > 0$ methods introduced by S. Mori, and more precisely using the following:

Theorem 1. *If $K_X < 0$, then X is rationally connected (RC, for short).*

Definition 1. *We say X is rationally connected if, for any $x, y \in X$, there exists a holomorphic map $h : \mathbb{P}^1 \to X$ such that $h(0) = x$, and $h(\infty) = y$.*

For example, $\mathbb{P}^n$ is RC, since two points are connected by a projective line. Thus RC manifolds (and so manifolds with $K < 0$) contain many lines, which let them look like $\mathbb{P}^n$ (up to a certain limited extent), and strongly influence their geometry.

Proposition 1. *Assume X is RC. Then:*

1. $\pi_1(X) = \{1\}$.
2. $d_X \equiv 0$.

(The second assertion is obvious).

We see here how the hypothesis of negativity of K_X translates into a strong geometric information (the rational connectedness) which dictates essentially all the qualitative geometry of X. Even more is true: Kollàr-Miyaoka-Mori could prove in 1992 the finiteness of the number of deformation families of spherical[9] manifolds in every dimension n, using quantitative versions of the rational connectedness statement above.

The proof of the rational connectedness theorem roughly goes as follows (to produce the initial step: *one* rational curve): take any curve C on X. Then deform it remaining isomorphic to itself through one base point $a \in C$ fixed. The existence of non-trivial deformations requires two properties: $K_X.C < 0$ (here our assumption comes in: it may be seen as saying, in much weaker, that there are infinitesimal vector fields along C pointing in some direction). But actual deformations will exist only after reduction modulo p. Then a degeneration of the deformed C will contain a rational curve R through a. One has still to lift to characteristic zero. Which can be made because the rational curve R can be splitted (*bent and broken*) as long as $(-K_X).R > (n + 1)$, which is optimal (lines on $\mathbb{P}^n$).

• When $n \geq 3$, the geometry of spherical manifolds can however be very different from the geometry of $\mathbb{P}^n$: they do not need to be *rational* (ie: birational to $\mathbb{P}^n$), or even maybe to be *unirational*: dominated meromorphically by $\mathbb{P}^n$. This last question is open.

On the arithmetic side, it has been shown by Harris-Tschinkel that quartics in $\mathbb{P}^4$ are potentially dense, which is very interesting, because these are known to be nonrational (Fano-Iskovskih-Manin), and are even expected to be non-unirational.

[9] Their usual name is *Fano* manifolds, introduced by V. Iskovskih.

4 Special manifolds

Our approach to the special geometry will be based on the following one-dimensional:

• **Observation:** If $f : X \to Y$ is a holomorphic *surjective* map between projective curves X and Y, then $deg(K_X) \geq deg(K_Y)$. (Because f is then a ramified cover of Y of a certain degree $d \geq 1$. If w is a (generic) nonzero meromorphic 1-form on Y, with Z zeros and P poles, then $f^*(w)$ is such on X. And it has at least $d.Z$ zeros, and at most $d.P$ poles on X).

This reflects a general principle: the positivity of K_Y restricts the existence of (meromorphic) maps to Y.

In particular: if X is a special curve, there is no surjective holomorphic map from X onto a hyperbolic curve.

Definition 2. *X is said to be special if there exists no fibration $f : X \to Y$ with base orbifold hyperbolic, and with $dim(Y) > 0$.*

(A precise definition of the orbifold base is given in in §. 7.)

Example 1.

1. If $K_X > 0$, then X is not special (if $n > 0$): take $f = id_X : X \to Y = X$.

2. If $K_X < 0$, or if $K_X \equiv 0$, then X is special (more difficult when $n \geq 2$: the proof depends on an orbifold version of particular case of conjecture $C_{n,m}$).

3. Conversely, any special X can be canonically decomposed[10] *as a tower of fibrations with fibres having either $K_X < 0$, or $K_X \equiv 0$, in a suitable orbifold birational sense. This decomposition should permit to reduce the conjectures below to the two particular cases $K < 0$ and $K \equiv 0$.*

Remark 1. *The orbifold base of $f : X \to Y$ was said to be a virtual ramified covering $\rho : Y' \to Y$ branching over the locus of so-called multiple fibres of f. See §. 7 for details.*

This role played by the orbifold base consists in replacing everywhere K_Y by $K_Y + \Delta(f)$ (which increases the positivity of K_Y), with $\Delta(f)$ being the ramification divisor of the (virtual, not existing, in general) $\rho : Y' \to Y$ which would by base change, eliminate the multiple fibres of f.

The special manifolds should be viewed as higher dimensional analogues of rational and elliptic curves.

Conjecture 1.

1. *If X is special, then $\pi_1(X)$ is virtually abelian.*

2. *X is special if and only if $d_X \equiv 0$.*

3. *X is special if and only if X is potentially dense (replacing number fields by fields finitely generated over $\mathbb{Q}$)*

4. *If X is special, so are any of its deformations and specialisations.*

[10]conditionally, under an orbifold version of $C_{n,m}$ conjecture

Some (of the few) known cases:

The property **1.** is known when $n = 2$, also when $\pi_1(X)$ has a faithful representation in some $Gl(N, \mathbb{C})$, and also when $K_X < 0$ or $K_X \equiv 0$. In fact, it can be shown that a surface is special if and only if it is not hyperbolic, and has virtually abelian fundamental group. This last result does not extend to $n \geq 3$.

Except for very few cases, **2.** is known only when $n \leq 2$, except maybe when X is a hyperbolic surface (in which case it is equivalent to Lang's conjecture), and also for submanifolds of complex tori. It is known in particular for $K3$ surfaces (eg: quartics in $\mathbb{P}^3$).

A partial result in this direction is the following: assume that there exists a meromorphic map: $h : \mathbb{C}^n \to X$ which has rank n at some point of $\mathbb{C}^n$. Then X is special. The link with $d_X$is that d_X then vanishes on the closure of $h(\mathbb{C}^n)$, which contains a nonempty open subset of X.

Except for some very particular and isolated cases, **3.** is known for curves, complex tori, again $K3$ surfaces (provided their Néron-Severi group is not cyclic), and also some specific elliptic surfaces with canonical dimension $\kappa = 1$ (see below).

. Remark The condition $d_X \equiv 0$ can be geometrically interpreted as follows if one accepts that $d_X(a, b) = 0$ if and only if a and b are joined by some entire curve on X. Then $d_X \equiv 0$ means that two arbitrary points can be joined by such an entire curve. This is a transcendental analogue of rational connectedness. The example of simple Abelian varieties shows that this property might not be seen algebraically.

5 Decomposition: the core

Theorem 2. *For any X, there exists a unique fibration*
$c_X : X \to C(X)$, called the core of X, such that:
1. Its general fibres are special, and
2. Its orbifold base is hyperbolic.

This is analogous to Levi-Malčev (semi-simple by solvable) for Lie algebras. The core splits the geometry of X into its two antithetical components: special (the fibres), and hyperbolic (the orbifold base). And X is special if and only if $C(X)$ is a point, while X is hyperbolic if and only if $C(X) = X$.

Examples.

• For curves, the core is either constant (if $g = 0, 1$), or the identity map (if $g \geq 2$).

• When $n = 2$, there are several cases: either X is special, or $dim(C(X) = 1$, or $dim(C(X)) = 2$, which happens if and only if X is hyperbolic. The second case $dim(C(X)) = 1$ happens if and only if (after a finite unramified cover), X admits an elliptic fibration $J : X \to C$ over a curve C with $g(C) \geq 2$.

• The core is functorial for surjective holomorphic maps $f : X \to Y$; ie: it induces functorially a map $c_f : C(X) \to C(Y)$ such that $c_Y \circ f = c_f \circ c_X$. If $u : X' \to X$ is a finite unramified cover, then $c_u : C(X') \to C(X)$ is finite (ramified, in general). This fact is surprisingly hard to show.

The core is expected to split X also at the arithmetic and complex hyperbolic level, which yields the following conjectural description of the Kobayashi pseudometric, and of the distribution of k-rational points of any X.

Conjecture 2. *There exists a proper algebraic subset $S \subset C(X)$ such that for any k (finitely generated over $\mathbb{Q}$), $c_X(X(k)) \cap U$ is finite, if $U := X - S'$, and $S' := c_X^{-1}(S)$.*

Conjecture 3. *There exists a (unique) pseudometric δ on $C(X)$ such that $d_X = c_X^*(\delta)$, and $\delta > 0$ on U (the same as in conjecture 2).*

Moreover, δ is the (naturally defined) Kobayashi pseudometric of the base orbifold of c_X.

This conjecture combines the conjecture 1 and an orbifold version of Lang's conjectures. One can formulate these conjectures more precisely.

Remark Again if one assumes that $d_X(a, b) = 0$ if and only if some entire curve on X joins a and b, one gets a more geometric picture of the complex hyperbolicity conjecture above: any entire curve on X is contained either in some fibre of c_X, or in $S' := c_X^{-1}(S)$.

6 The decomposition of the core

Finally, we have the following conditional decomposition of the core; assuming that the orbifold conjecture $C_{n,m}^{orb}$ holds[11]:

Theorem 3. *$c_X = (J \circ r)^n$ is canonically composed of fibrations J (resp. r) with fibres flat (resp. spherical), in a suitable orbifold sense.*

In particular, special geometry is the orbifold combination of flat and spherical geometries.

This second result should permit to inductively reduce the proofs of properties of special manifolds (such as the ones conjectured above) to their orbifold versions in the flat and spherical cases.

CAUTION: The above two theorems 2 and 3 are proved, not for the pure *numerical* geometries as defined in Section 3, but for weaker birational variants (derived from the asymptotic behaviour of the pluricanonical linear systems). However, the aim of the so-called *Log Minimal Model Program* is (or should be) to prove that the two notions actually coincide, after suitable birational transformations.

On the other hand, the results presented here are, more generally, valid for compact Kähler manifolds. The conjectures stated (except those concerned with arithmetics) can be maintained for this larger class.

[11] See section 7.6 below for its statement.

7 Base orbifolds

We shall give some more precise definitions of the notions introduced, assuming from the reader some notions of complex geometry (Kodaira Dimension,for example). We start with:

7.1 A simple motivating example

Let E, C be an elliptic (resp. a hyperelliptic) curve.

Let $t : E \to E$ (resp. $h : C \to C$) be a translation of order 2 (resp. the hyperelliptic involution).

Thus $C/ < h > \cong \mathbb{P}^1$, and the natural quotient $v : C \to C/ < h >$ ramifies at order 2 above $N = 2(g(C) + 1)$ points $p_1, \ldots, p_N \in \mathbb{P}^1$.

Thus, $t \times h : X' := E \times C \to X'$ is a fixed-point free involution.

Let $u : X' \to X := X'/ < t \times h >$ be the (étale, of degree 2) quotient map.

We thus get a commutative diagram:

$$\begin{array}{ccc} E \times C = X' & \xrightarrow{u} & X \\ \downarrow{\scriptstyle J'} & & \downarrow{\scriptstyle J} \\ C & \xrightarrow{v} & \mathbb{P}^1 \end{array}$$

Notice that J', J are the natural projections, but also the Moishezon-Iitaka fibrations of X', X respectively.

Now u is étale, so X' and X possess all the same qualitative properties (arithmetic, Kobayashi pseudometric, fundamental group). However, X' has a quotient of general type (C), while X has no such quotient.

Or, better said, we should consider $\mathbb{P}^1$ here as being of general type, by looking at X only.

• One object remains, which keeps track of the construction made: the multiple fibres of J. These are double fibres lying exactly above the points $p_1, \ldots, p_N$.

If we "enrich" the canonical bundle of $\mathbb{P}^1$ by the *orbifold divisor*

$$\Delta(J) := \sum_1^N (1 - 1/2).p_j,$$

we get: $v^*(K_{\mathbb{P}^1} + \Delta(J)) = K_C$.

In other words, due to the multiple fibres of J, we should consider that the canonical bundle of $\mathbb{P}^1$ has been increased by the ramification term $\Delta(J)$. This is the process of virtual elimination of multiple fibres that we shall now generalise.

7.2 Orbifold base of a fibration

• Orbifolds

Let Y be projective smooth, connected.

An *orbifold divisor* (on Y) is any $\Delta := \sum_J (1 - 1/m_j).D_j$, J finite, $m_j > 0$ integers, D_j distinct prime divisors. Notice: if $m_j = 1$, D_j does not appear.

Orbifold:=pair (Y/Δ). Canonical bundle: $K_{(Y/\Delta)} := K_Y + \Delta$ (on Y).

Canonical (or Kodaira) dimension: $\kappa(Y/\Delta) := \kappa(Y, K_Y + \Delta)$ (lies between $dim(Y)$ and $\kappa(Y)$).

• Orbifold Base of a fibration

$f : X \to Y$ a fibration onto, connected, regular, Y, X smooth connected.

• Let $D \subset Y$ be a prime divisor. $f^*(D) = \sum_H m_h.D_h + R$, where: $f(D_h) = D, \forall h$, while $f(R)$ has codimension 2 or more in Y.

Define: $m(f, D) := inf\{m_h, h \in H\}$.

This differs from the classical multiplicity $m^*(f, D)$, in which inf is replaced by gcd.

Notice: $m(f, D) = 1$ for all but finitely many $D's$.

Define: $\Delta(f) := \sum_D (1 - 1/m(f, D)).D$.

The sum is finite, and defines an orbifold divisor on Y.

The *orbifold base* of f is $(Y/\Delta(f))$.

Main invariant: $\kappa(Y/\Delta(f)) := \kappa(Y, K_Y + \Delta(f))$.

Remark: When, in addition, X is equipped with an orbifold divisor Δ_X, we also define an orbifold divisor $\Delta(f, \Delta_X))$ on Y in such a way that if $\Delta_X = \Delta(g)$, for a fibration $g : Z \to X$, then $\Delta(fg) = \Delta(f, \Delta(g))$ (in nice situations at least).

7.3 Birational equivalence

• The birational equivalence of fibrations f, f' is the one generated by commutative diagrams:

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{v} & Y \end{array}$$

in which u, v are birational.

We write: $f \sim f'$.

• In this situation: $\kappa(Y'/\Delta(f')) \leq \kappa(Y/\Delta(f))$.

Strict inequality may occur (but only when $\kappa(Y) = -\infty$).

f' can be constructed explicitely such that: $f' \sim f$, and $\kappa(fY'/\Delta(f'))$ is minimum: flatten f by birational base change, and smoothen the main component of the fibre product.

• We define: $\kappa(f) := inf\{\kappa(f'), \forall f' \sim f\}$.

This is now a birational invariant, defined for $f : X \to Y$ meromorphic, and X, Y arbitrary irreducible varieties.

• We say that f is a *fibration of general type* if $\kappa(f) = dim(Y) > 0$. These will play a central role in our constructions.

7.4 Fibrations of general type and Bogomolov Sheaves

A **Bogomolov Sheaf** on X is a rank-1 coherent subsheaf $L \subset \Omega^p_X, p > 0$, such that $\kappa(X, L) = p$.

(This is the maximum possible, after Bogomolov's theorem). One defines $\kappa(X, L)$ by making L free on some birational model of X.

Let $Bog(X)$ the (possibly empty) set of Bogomolov sheaves on X.

Theorem 4. *There exists a natural bijection between $Bog(X)$ and (equivalence classes of) meromorphic fibrations $f : X \to Y$ which are of general type.*

Idea (from f to $Bog(X)$: if $f : X \to Y$ is of general type, with $p := dim(Y) > 0$, define L_f to be the saturation of $f^*(K_Y) \subset \Omega^p_X$. It will contain $f^*(K_Y) + \lceil f^*(\Delta(f)) \rceil$, and will so belong to $Bog(X)$.

• This correspondance works for the inf-multiplicities, but *not* for the classical gcd-multiplicities, and is one of the main reasons for the introduction of the former ones).

7.5 Orbifold additivity

Theorem 5. *Let $f : X \to Y$ be a regular fibration; X, Y smooth. Let Δ_X be an orbifold divisor on X. Assume some snc conditions, and that: $\kappa(Y/\Delta(f, \Delta_X)) = dim(Y)$.*

Then: $\kappa(X/\Delta_X) = dim(Y) + \kappa(X_y/(\Delta_X)_{|X_y})$, for general $y \in Y$.

This (together with its proof) is an orbifold extension of the famous result of E. Viehweg, in which $\Delta_X = \emptyset$, and Y is of general type. But its range of application is much larger, since it allows cases when $\kappa(X_y)$, or $\kappa(Y)$ are $-\infty$.

If $\Delta_X = \emptyset$, we get:

Corollary 1. *Let $f : X \to Y$ be a fibration of general type. Then $\kappa(X) = dim(Y) + \kappa(X_y)$, for $y \in Y$ general.*

In particular, if $\kappa(X) = 0$, then $dim(Y) = 0$.

Central here is the following, which motivated Theorem 2:

Corollary 2. *Let $f : Z \to X$ and $g : X \to Y$ be (meromorphic) fibrations. Assume that:*

1. $gf : Z \to Y$ is of general type.
2. $f_y : Z_y \to X_y$ is of general type for $y \in Y$ general.
Then: $f : Z \to X$ is of general type.

Idea: Apply (on suitable models) Theorem 5 to $\Delta_X := \Delta(f)$, considering the diagram:

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ & \underset{gf}{\searrow} & \downarrow g \\ & & Y \end{array}$$

7.6 The orbifold conjecture C$_{n,m}$:

This is the statement of theorem 5, but without assuming that $\kappa(Y/\Delta(f, \Delta_X)) = dim(Y)$, and with the conclusion replaced by: $\kappa(X/\Delta_X) = \kappa(Y/\Delta(f, \Delta_X)) + \kappa(X_y/(\Delta_X)_{|X_y})$, for general $y \in Y$.

Without orbifolds, it was formulated by S. Iitaka, and remains open in general. Theorem 2 of §5 depends on it (but might be approached also from the geometry of orbifold rational curves).

7.7 Special manifolds II.

Definition: A (connected projective) manifold (or variety) X is said to be **special** if there is no meromorphic fibration $f : X \to Y$ of general type (or, equivalently, if $Bog(X) = \emptyset$).

This property is birational.

Examples:

• If $X$is of general type, then X is not special ($f = id_X$).
• If X is RC, then X is special ($\kappa(X, L) = -\infty$ for any rank one $L \subset \Omega^p_X, p > 0$).
• If $\kappa(X) = 0$, then X is special (corollary 1 of theorem 2).
• **Caution!** Being special does not depend on the Canonical ("Kodaira")-dimension alone. For example, for any $n > 0$, and $\kappa \in \{-\infty, 0, 1, \ldots, (n-1)\}$, there exists X special, n-dimensional, with $\kappa(X) = \kappa$.
• If X is special, so is Y if there exists a dominating rational $f : X \to Y$. So is also any finite étale cover X' of X.
• If any two generic points of X, **smooth**, can be connected by a connected chain of special subvarieties, then X is special. (The cone over a curve of general type gives a couterexample in the singular case).

• If X is special, its Albanese map is surjective, has connected fibres, and no multiple fibre in codimension one. (When $\kappa(X) = 0$, this slightly strengthens a result of Y. Kawamata).

7.8 The core II.

Recall the statement:

Theorem 6. *For any projective (connected) manifold X, there exists a unique (meromorphic, almost holomorphic) fibration $c_X : X \to C(X)$, called* **the core** *of X, such that:*

1. its (general) fibres are special.

2. it is either of general type, or constant (this if and only if X itself is special).

• **Idea** of proof:

The *existence* is a consequence of orbifold additivity:

Take $f : X \to Y$ be of general type with $dim(Y)$ maximum. This is the core of X. Indeed: if not, the fibres of f are not special, and one can construct (using Chow Schemes) a factorisation $f = gh$ of f, by fibrations $h : X \to Z$ and $g : Z \to Y$ such that: $dim(Z) > dim(Y)$, and the restriction: $h_y : X_y \to Z_y$ is of general type for $y \in Y$ general. But then corollary 2 of orbifold additivity shows that $h : X \to Z$ is of general type, contradicting the maximality of $dim(Y)$.

The *uniqueness* is a consequence of:

Lemma 1. *Let $f : X \to Y$ and $g : X \to Z$ be (meromorphic) fibrations. Assume that f has general fibres special, and that g is of general type. There exists a (unique) fibration $h : Y \to Z$ such that: $g = hf$.*

References

Ann. Inst. Fourier 54 (2004), 499-665, where references for the results mentioned in the text are given. See also math.AG/0110051 for function field versions.

Université Nancy 1. BP 239
F-54506. Vandoeuvre-les-Nancy. Cédex.
campana@iecn.u-nancy.fr