Integrability of homogeneous polynomials on the unit ball

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Abstract

We construct some measure Θ^α such that if $0 < \alpha \leq 2n-2$, $\beta = n - \frac{2+\alpha}{2}$ and E is a circular set of type G_δ such that $E \subset \partial\mathbb{B}^n$ and $\Theta^\alpha(E) = 0$ then there exists $f \in \mathbb{O}(\mathbb{B}^n) \cap L^2(\mathbb{B}^n)$ such that

$$E = E^\beta(f) := \left\{ z \in \partial B^n : \int_{\mathbb{D}z} |f|^2 \chi_\beta d\mathfrak{L}^2 = \infty \right\}$$

where $\chi_s : \mathbb{B}^n \ni z \longrightarrow \chi_s(z) = (1 - \|z\|^2)^s$ and $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$.

1 Introduction

In the paper [6] a natural number K and a sequence $\{p_n\}_{n=0}^\infty$ of homogeneous polynomials in $\mathbb{C}^d$ was constructed so that $|p_n(z)| \leq 2$ and $\sum_{j=Km}^{K(m+1)-1} |p_n(z)| \geq 0.5$ for all z belonging to the boundary of the unit ball $\partial\mathbb{B}^d$. In the paper [1] we introduced some additional arguments in such a way that for any circular set $E \subset \partial\mathbb{B}^d$ of type G_δ and F_σ we could construct a holomorphic function f on the unit ball $\mathbb{B}^d$ such that $E^2_{\mathbb{B}^d}(f) = E$.

Let $\chi_s : \mathbb{B}^n \ni z \longrightarrow \chi_s(z) = (1 - \|z\|^2)^s$. In the paper [3, Lemma 2.6, Theorem 2.7] we showed that there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}z} |f|^2 \chi_{n-1} d\mathfrak{L}^2 \leq C \int_{\mathbb{B}^n} |f|^2 d\mathfrak{L}^{2n}$$

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for a holomorphic, square integrable function f. In particular $E^{n-1}(f)=\emptyset$. Due to the above inequality the following question can be posed: what additional conditions have to be fulfilled for the set E of type G_δ from $\partial\mathbb{B}^n$ so that there exists a holomorphic function f square integrable such that, for some $0<s<n-1$, $E=E^s(f):=\left\{z\in\partial B^n:\ \int_{\mathbb{D}z}|f|^2\chi_s d\mathfrak{L}^2=\infty\right\}$. In this paper we investigate this question.

1.1 Geometric notions.

Let X be a metric space with a pseudometric ρ. Assume that topology of X is given by countable base of open sets.

The set $E\subset X$ is ρ complete iff $\rho(z,w)>0$ for $z\in E$ and $w\in X\setminus E$. If $D,T\subset X$ then we denote $\rho(D,T):=\inf_{z\in D,w\in T}\rho(z,w)$.

We say that τ is a premeasure on X iff $0\le\tau(D)\le\infty$ for $D\subset X$. Moreover μ is a measure defined from premeasure τ on (X,ρ) iff

$$\begin{aligned} d_\rho(E) &:= \sup_{z,w\in E}\rho(z,w),\\ \mu_\delta(E) &:= \inf\left\{\sum_{i\in\mathbb{N}}\tau(E_i):\ E\subset\bigcup_{i\in\mathbb{N}}E_i,\ d_\rho(E_i)\le 2\delta, E_i=\overline{E_i}\subset X\right\},\\ \mu(E) &:= \sup_{\delta>0}\mu_\delta(E)\end{aligned}$$

for $E\subset X$.

If ρ is a norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ then we write symbol Y in place of Y_ρ.

Observe that if H^α is a measure from $h^\alpha(\circ)=(d(\circ))^\alpha$ on $\mathbb{R}^n$, then H^α is a Hausdorff measure. We also denote $\mathfrak{L}^n$ -n-dimensional Lebesgue measure on $\mathbb{R}^n$.

We denote $K_\rho(D,\varepsilon):=\{z\in X:\inf_{w\in D}\rho(z,w)<\varepsilon\}$ and $K_\rho(x,\varepsilon)=K_\rho(\{x\},\varepsilon)$ for $x\in X$. Now we define $s_{\rho\varepsilon}$ index of D as

$$s_{\rho\varepsilon}(D):=\inf\left\{s:\{x_i\}_{i=1}^s\subset D\subset\sum_{i=1}^s K_\rho(x_i,\varepsilon)\subset X\right\}.$$

We say that X is (n,ρ,η)-regular if there exist constants $\kappa_1,\kappa_2,\varepsilon_0>0$, measure η constructed from some premeasure so that $\kappa_1\varepsilon^n\le\eta\left(K_\rho(x,\varepsilon)\right)\le\kappa_2\varepsilon^n$ for $x\in X$ and $0<\varepsilon<\varepsilon_0$.

Now we can consider the following premeasure

$$\tau_\rho^\alpha(D) := \limsup_{\varepsilon\to 0}2^\alpha\varepsilon^\alpha s_{\rho\varepsilon}(D).$$

If additionally X is (n,ρ,η)-regular then we consider the premeasure

$$\nu_{\rho\mu}^\alpha(D):=\limsup_{\varepsilon\to 0}2^\alpha\varepsilon^{\alpha-n}\eta\left(K_\rho(D,\varepsilon)\right).$$

We also define measure Q_ρ^α from τ_ρ^α and $\Theta_{\rho\mu}^\alpha$ from $\nu_{\rho\mu}^\alpha$.

We use the pseudometric $\rho(z,w):=\sqrt{1-|\langle z,w\rangle|}$ and σ -$(2n-1)$-dimensional, natural measure on $\partial\mathbb{B}^n$.

Definition 1.1. Let $T \subset \partial\mathbb{B}^n$ and $C > 0$. If $A = \{\xi_1, ..., \xi_s\} \subset T$ and $\rho(\xi_i, \xi_j) > \beta$ for $i \neq j$ then we say that A is β-separated subset of T. Let us define homogeneous polynomials for the pair (C, T) as:

$$p_m(z) = p_{m,A}(z) = \sum_{\xi \in A} \langle z, \xi \rangle^m$$

where $A \subset T$, A is $\frac{C}{\sqrt{N}}$-separated subset of T and $N \leq m \leq 2N$.

2 Q_ρ^α and $\Theta_{\rho\mu}^\alpha$ measure

In this section we describe some basic properties of measures Q_ρ^α, $\Theta_{\rho\mu}^\alpha$. Let us define relation $y \in [x]$ iff $\rho(x, y) = 0$ and the metric space $X_\sim := \{[x] : x \in X\}$.

Lemma 2.1. *We have the following properties:*

1. *If D is a closed subset of $X_\sim$ then $H_\rho^\alpha(D) \leq \liminf_{\varepsilon \to 0} 2^\alpha \varepsilon^\alpha s_{\rho\varepsilon}(D) \leq \tau_\rho^\alpha(D)$.*
2. *If E is a Borel subset of $X_\sim$ then $H_\rho^\alpha(E) \leq Q_\rho^\alpha(E)$.*
3. *If E is a Borel subset of $X_\sim$ then E is H_ρ^α, Q_ρ^α and $\Theta_{\rho\mu}^\alpha$ measurable.*

Proof. Observe that $H_{\rho\varepsilon}^\alpha(D) \leq 2^\alpha \varepsilon^\alpha s_{\rho\varepsilon}(D)$ for $\varepsilon > 0$. Therefore property (1) is clear.

Let E be a Borel subset of $X_\sim$ such that $Q_\rho^\alpha(E) < \infty$. Let $\delta, \varepsilon > 0$. There exists a sequence $\{K_i\}_{i \in \mathbb{N}}$ of closed subsets of $X_\sim$ such that $E \subset \bigcup_{i \in \mathbb{N}} K_i$, $d_\rho(K_i) \leq 2\delta$ and $\sum_{i \in \mathbb{N}} \tau_\rho^\alpha(K_i) \leq Q_{\rho\delta}^\alpha(E) + \varepsilon$. We may estimate

$$H_\rho^\alpha(E) \leq H_\rho^\alpha\left(\bigcup_{i \in \mathbb{N}} K_i\right) \leq \sum_{i \in \mathbb{N}} H_\rho^\alpha(K_i) \leq \sum_{i \in \mathbb{N}} \tau_\rho^\alpha(K_i) \leq Q_{\rho\delta}^\alpha(E) + \varepsilon.$$

We conclude that $H_\rho^\alpha(E) \leq Q_\rho^\alpha(E)$.

Property (3) follows from [4, Theorem 19]. ■

Lemma 2.2. *Let X be (n, ρ, μ)-regular. There exists $\kappa_1, \kappa_2, \varepsilon_0 > 0$ such that:*

1. *If D is a closed subset of $X_\sim$ then $\kappa_1 s_{\rho\varepsilon}(D) \leq \varepsilon^{-n} \mu(K_\rho(D, \varepsilon)) \leq \kappa_2 s_{\rho\varepsilon}(D)$ for $0 < 3\varepsilon < \varepsilon_0$.*
2. *If $\{K_i\}_{i \in \mathbb{N}}$ is a sequence of closed subsets of $X_\sim$ such that $\rho(K_i, K_j) > 0$ for $i \neq j$ then $\nu_{\rho\mu}^n\left(\bigcup_{i \in \mathbb{N}} K_i\right) = \sum_{i \in \mathbb{N}} \nu_{\rho\mu}^n(K_i)$.*
3. *If D is a closed subset of $X_\sim$ then $\kappa_1 \tau_\rho^\alpha(D) \leq \nu_{\rho\mu}^\alpha(D) \leq \kappa_2 \tau_\rho^\alpha(D)$ for $\alpha > 0$.*
4. *If E is a Borel subset of $X_\sim$ then $\kappa_1 Q_\rho^\alpha(E) \leq \Theta_{\rho\mu}^\alpha(E) \leq \kappa_2 Q_\rho^\alpha(E)$ for $\alpha > 0$.*
5. *If E is a Borel subset of $X_\sim$ then $\Theta_{\rho\mu}^n(E) \leq \kappa_2 H_\rho^n(E)$.*

Proof. Due to X is (n,ρ,μ)-regular, there exists $\kappa_1,\kappa_2,\varepsilon_0>0$ such that $2^n\kappa_1\le\varepsilon^{-n}\mu(K_\rho(x,\varepsilon))\le 3^{-n}\kappa_2$ for $x\in X$ and $0<\varepsilon<\varepsilon_0$. We denote $s=s_{\rho\varepsilon}(D)$. Let r be a maximal natural number such that there exist points $x_1,...,x_r$ in D such that $\rho(x_i,x_j)\ge\varepsilon$ for $i\ne j$. Observe that $D\subset\bigcup_{i=1}^r K_\rho(x_i,\varepsilon)$. Therefore $s\le r$. Moreover $\bigcup_{i=1}^r K_\rho(x_i,\frac{\varepsilon}{2})\subset K_\rho(D,\varepsilon)$. If $s=\infty$ then $r=\infty$ and $\mu(K_\rho(D,\varepsilon))\ge\sum_{i=1}^\infty\mu(K_\rho(x_i,\frac{\varepsilon}{4}))=\infty$. Therefore we can assume that $s,r<\infty$.

There exist points $y_1,...,y_s$ such that $\{y_i\}_{i=1}^s\subset D\subset\bigcup_{i=1}^s K_\rho(y_i,\varepsilon)$. We define the sequence $i(1),...,i(t)$ such that $i(1)=1$ and $i(k+1)$ is a minimal index such that $\rho(y_{i(k+1)},y_{i(j)})>\varepsilon$ for $j=1,...,k$. Observe that $t\le s$. We prove that

$$D\subset\bigcup_{k=1}^t K_\rho(y_{i(k)},2\varepsilon).$$

Let $z\in D$. There exists $m\in\{1,...,s\}$ such that $z\in K_\rho(y_m,\varepsilon)$. There exists maximal $k\le t$ such that $i(k)\le m$. If $i(k)=m$ then $y\in K_\rho(y_{i(k)},2\varepsilon)$. If $i(k)<m$, then there exists an index $k_1\le k$ such that $\rho(y_m,y_{i(k_1)})\le\varepsilon$. In particular $\rho(z,y_{i(k_1)})\le\rho(z,y_m)+\rho(y_m,y_{i(k_1)})<2\varepsilon$. We conclude that $z\in K_\rho(y_{i(k_1)},2\varepsilon)$. Now we have

$$\bigcup_{k=1}^r K_\rho\left(x_k,\frac{\varepsilon}{2}\right)\subset K_\rho(D,\varepsilon)\subset\bigcup_{k=1}^t K_\rho(y_{i(k)},3\varepsilon).$$

Due to $\rho(x_i,x_j)\ge\varepsilon$ for $i\ne j$ we can estimate

$$\kappa_1 s\varepsilon^n\le\sum_{k=1}^r\mu\left(K_\rho\left(x_k,\frac{\varepsilon}{2}\right)\right)\le\mu\left(K_\rho(D,\varepsilon)\right)\le\sum_{k=1}^t\mu\left(K_\rho\left(y_{i(k)},3\varepsilon\right)\right)\le\kappa_2 s\varepsilon^n.$$

Now we prove (2). Observe that

$$\nu^n_{\rho\mu}(T)=\lim_{\varepsilon\to 0}\mu(K_\rho(T,\varepsilon))=\mu(\overline{T}).$$

Moreover

$$\sum_{j<i\Rightarrow\rho(T_i,T_j)>2\varepsilon}\mu\left(K_\rho\left(T_i,\varepsilon\right)\right)\ \le\ \mu\left(K_\rho\left(\bigcup_{i\in\mathbb{N}}T_i,\varepsilon\right)\right)\le\sum_{i\in\mathbb{N}}\mu\left(K_\rho\left(T_i,\varepsilon\right)\right).$$

In particular

$$\nu^n_{\rho\mu}\left(\bigcup_{i\in\mathbb{N}}T_i\right)=\sum_{i\in\mathbb{N}}\nu^n_{\rho\mu}\left(T_i\right).$$

The properties (3)-(4) are consequences of (1).

We prove (5). Let E be a Borel subset of $X_\sim$ such that $H^n_\rho(E)<\infty$. Let $\delta,\varepsilon>0$. There exists a sequence $\{K_i\}_{i\in\mathbb{N}}$ of closed subsets of $X_\sim$ such that $E\subset\bigcup_{i\in\mathbb{N}}K_i$, $r_i:=d_\rho(K_i)\le 2\delta$ and $\sum_{i\in\mathbb{N}}r_i^n\le H^n_{\rho\delta}(E)+\varepsilon$. There exists a sequence of points $\{x_i\}_{i\in\mathbb{N}}$ such that $K_i\subset K_\rho(x_i,2r_i)$. In particular for δ small enough we may estimate $\Theta^n_{\rho\mu(8\delta)}(E)\le\sum_{\in\mathbb{N}}\nu^n_{\rho\mu}(\overline{K_\rho(x_i,2r_i)})\le\sum_{\in\mathbb{N}}\mu(\overline{K_\rho(x_i,2r_i)})\le\sum_{i\in\mathbb{N}}3^{-n}\kappa_2 2^n r_i^n\le\kappa_2 H^n_{\rho\delta}(E)+\kappa_2\varepsilon$. Now we conclude that $\Theta^n_{\rho\mu}(E)\le\kappa_2 H^n_\rho(E)$. ■

Lemma 2.3. *Let* $0 < q < \frac{1}{2}$, $m \in \mathbb{N}$ *and* $\alpha_0 = \frac{-m\log 2}{\log q}$. *If* $E_0 := [0,1]^m \subset \mathbb{R}^m$, $E_{j+1} := ([0,q] \cup [1-q,1])\, E_j$ *and* $E = \bigcap_{j\in\mathbb{N}} E_j$ *then* $H^\alpha(E) = Q^\alpha(E) = 0$ *where* $\alpha_0 < \alpha$. *Moreover* $H^\alpha(E) = Q^\alpha(E) = \infty$ *for* $0 < \alpha < \alpha_0$ *and* $2^{-2m} \le H^{\alpha_0}(E) \le Q^{\alpha_0}(E) \le \sqrt{m^{\alpha_0}}q^{-\alpha_0}$. *Additionally* E *is* $(\alpha_0, \|\circ\|, Q^{\alpha_0})$ *regular.*

Proof. Let ε be such that $\sqrt{m}q^k < 2\varepsilon < \sqrt{m}q^{k-1}$ for some k. Since E_k can be covered by 2^{mk} cubes with the edge equal to q^k therefore we may estimate: $2^\alpha\varepsilon^\alpha s_\varepsilon(D) \le 2^\alpha\varepsilon^\alpha 2^{mk} \le \sqrt{m^\alpha}q^{-\alpha}\left(q^\alpha 2^m\right)^k$.

If $\alpha_0 < \alpha \le m$ then $2^m q^\alpha < 1$ and $\tau^\alpha(E) = \liminf_{\varepsilon\to 0} 2^\alpha\varepsilon^\alpha s_\varepsilon(D) = 0$. In particular $Q^\alpha(E) = 0$ for $\alpha_0 < \alpha \le m$.

Observe that $2^m q^{\alpha_0} = 1$. Moreover E_k is the sum of 2^{mk} disjoint cubes $I_1, ..., I_{2^{mk}}$ with the edges equal to q^k. Due to $2^{mk}\tau^{\alpha_0}(E\cap I_s) = \tau^{\alpha_0}(E\cap E_k) = \tau^{\alpha_0}(E) \le \sqrt{m^{\alpha_0}}q^{-\alpha_0}$ we conclude that $Q^{\alpha_0}(E) \le \sqrt{m^{\alpha_0}}q^{-\alpha_0}$.

Let U be an open subset of $\mathbb{R}^m$. Let $f_n(U)$ be a number of cubes from E_n which intersects U. Let $g_n(U) = 2^{-nm}f_n(U)$. Observe that $f_{n+1}(U) \le 2^m f_n(U)$ and $g_{n+1}(U) = 2^{-(n+1)m}f_{n+1}(U) \le 2^{-nm}f_n(U) = g_n(U)$. Let $g(U) = \lim_{n\to\infty} g_n(U)$. If $[0,1]^m \subset U$ then $g(U) = 1$. Moreover $g(U\cup V) \le g(U) + g(V)$.

Let I be an open cube with the edges equal to $r < q$. There exists $n \in \mathbb{N}$ such that $q^{n+1} \le r < q^n$. Observe that $f_n(I) \le 2^m$. In particular

$$g(I) \le 2^{-nm}f_n(I) \le 2^{-nm}2^m \le q^{\alpha_0 n}2^m \le q^{-\alpha_0}2^m r^{\alpha_0}.$$

Let $I_1, ..., I_s$ be a covering of E so that I_k is an open cube with the edges equal to r_k with $r_k < q$. We can estimate

$$\sum_{k=1}^{s} r_k^{\alpha_0} \ge q^{\alpha_0}2^{-m}\sum_{k=1}^{s} g(I_k) \ge q^{\alpha_0}2^{-m}g(\bigcup_{k=1}^{s} I_k) \ge q^{\alpha_0}2^{-m} = 2^{-2m}.$$

Therefore $2^{-2m} \le H^{\alpha_0}(E) \le Q^{\alpha_0}(E)$ and $\infty = H^\alpha(E) \le Q^\alpha(E)$ for $0 < \alpha < \alpha_0$.

Let $x \in E$ and $\varepsilon > 0$ be such that $0 < 2\varepsilon < q$. There exist $n, r \in \mathbb{N}$ such that $q^r < \varepsilon \le q^{r-1}$ and $q^{n+1} \le 2\varepsilon < q^n$. The set E_k is the sum of 2^{mk} disjoint, identical cubes $I_1^k, ..., I_{2^{mk}}^k$ with the edges equal to q^k. In particular $2^{nm}Q^{\alpha_0}(I_{i(k)}^n \cap E) = \sum_{i=1}^{2^{nm}} Q^{\alpha_0}(I_i^n \cap E) = Q^{\alpha_0}(E)$ for $k = 1, .., 2^{nm}$. Due to $f_n(K(x,\varepsilon)) \le 2^m$ we conclude that there exist $I_{i(1)}^n, ..., I_{i(s)}^n$ cubes such that $K(x,\varepsilon)\cap E \subset \bigcup_{k=1}^{s} I_{i(k)}$ and $s \le 2^m$. Moreover there exists k_0 such that $I_{k_0}^r \cap E \subset K(x,\varepsilon)\cap E$. We may estimate

$$q^{\alpha_0}\varepsilon^{\alpha_0} \le q^{\alpha_0 r} = 2^{-mr} \le \frac{Q^{\alpha_0}(K(x,\varepsilon)\cap E)}{Q^{\alpha_0}(E)} \le 2^{m-nm} = q^{\alpha_0(n-1)} \le q^{-2\alpha}2^{\alpha_0}\varepsilon^{\alpha_0}.$$

We conclude that E is $(\alpha_0, \|\circ\|, Q^{\alpha_0})$ regular. ∎

Lemma 2.4. *Assume that* $H_\rho^\alpha(U) = \infty$ *for* $0 < \alpha < m$ *and all the open* U *non empty subsets of* X. *There exists a set* $G \subset X$ *of type* G_δ *such that* $0 = H_\rho^\alpha(G) < Q_\rho^\alpha(G) = \infty$ *for* $0 < \alpha < m$.

Proof. Let $A = \{x_i\}_{i\in\mathbb{N}}$ be a countable and dense subset of X such that $x_{\lfloor i,j\rfloor} = x_{\lfloor i,1\rfloor}$ for all $i, j \in \mathbb{N}$. Let $U_i := \bigcup_{j=i}^{\infty} K_\rho\left(x_j, 2^{-j^2}\right)$ and $G = \bigcap_{i\in\mathbb{N}} U_i$. Let $\alpha > 0$ and $\delta, \varepsilon > 0$. Let j_0 be such that $\alpha(j^2-1) \ge j$, $2^{-j^2} < \delta$, $2^{-j+1} \le \varepsilon$ for $j \ge j_0$. We may estimate

$$H_{\rho\delta}^\alpha(G) \le \sum_{j=j_0} 2^{-\alpha j^2+\alpha} \le \sum_{j=j_0} 2^{-j} = 2^{-j_0+1} \le \varepsilon.$$

We now conclude that $H^{\alpha}_{\rho}(G) = 0$ for $\alpha > 0$.

Observe that $A \subset G$. Therefore $\overline{G} = X$. Suppose that $\tau^{\alpha_0}_{\rho}(G) < \infty$ for some $0 < \alpha_0 < m$. There exists a sequence $\{F_i\}_{i\in\mathbb{N}}$ of closed subsets of X such that $G \subset \bigcup_{i\in\mathbb{N}} F_i$ and $\sum_{i\in\mathbb{N}} \tau^{\alpha_0}_{\rho}(F_i) < \infty$. Moreover because G is of type G_δ there exists a sequence of closed sets $\{H_i\}_{i\in\mathbb{N}}$ with empty interiors such that $X \setminus G = \bigcup_{i\in\mathbb{N}} H_i$. Observe that:

$$X \subset X \setminus G \cup G \subset \bigcup_{i\in\mathbb{N}} H_i \cup \bigcup_{i\in\mathbb{N}} F_i.$$

Due to Bair's Theorem we conclude that there exists k such that interior of F_k is non empty. In particular due to Lemma 2.1 we conclude a contradiction $\infty = H^{\alpha_0}_{\rho}(F_k) \leq \tau^{\alpha_0}_{\rho}(F_k)$.

Therefore $Q^{\alpha}_{\rho}(G) = \infty$ for $0 < \alpha < m$. ■

Lemma 2.5. *There exists a compact E subset of $\mathbb{R}^m$ which is uncountable and $Q^{\alpha}(E) = \tau^{\alpha}(E) = 0$ for $\alpha > 0$.*

Proof. Let $E_0 := [0,1]^m \subset \mathbb{R}^m$, $E_{j+1} := \left([0, 4^{-j-1}] \cup [1 - 4^{-j-1}, 1]\right) E_j$ and $E = \bigcap_{j\in\mathbb{N}} E_j$. Let $\alpha > 0$ and $\sqrt{m}2^{-k(k+1)} < 2\varepsilon \leq \sqrt{m}2^{-(k-1)k}$. Since E_k has 2^{mk} cubes with the edges equal to $\prod_{j=1}^{k} 4^{-j} = 2^{-k(k+1)}$ therefore we may estimate: $2^{\alpha}\varepsilon^{\alpha} s_{\varepsilon}(E) \leq 2^{\alpha}\varepsilon^{\alpha} s_{\varepsilon}(E_k) \leq 2^{\alpha}\varepsilon^{\alpha} 2^{mk} \leq \sqrt{m^{\alpha}} 2^{mk-\alpha k(k-1)}$. Due to $\lim_{k\to\infty} mk - \alpha k(k-1) = -\infty$ we have $\tau^{\alpha}(E) = 0$. In particular $Q^{\alpha}(E) = 0$.

We prove that the set E is uncountable. Let U be an open set such that $U \cap E \neq \emptyset$. Observe that there exists $k \in \mathbb{N}$ such that $U \cap E_k \neq \emptyset$. Therefore there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset U \cap E$ such that $x_i \neq x_j$ for $i \neq j$. We may conclude that if $x \in E$ then $\{x\}$ is a not open subset of E. Suppose that E is countable and there exists a sequence $\{w_n\}_{n\in\mathbb{N}} = E$. Due to Bair's Theorem the interior of $\{w_{n_0}\}$ in E is not empty for some n_0. Therefore $\{w_{n_0}\}$ is an open subset of E which is impossible. ■

Lemma 2.6. *Let X be a metric space with pseudometric ρ and $\tilde{X}$ - metric space with the pseudometric $\tilde{\rho}$. Let $f : X \to \tilde{X}$ be a continuous function such that $c_1\rho(x,y) \leq \tilde{\rho}(f(x), f(y)) \leq c_2\rho(x,y)$ for $x, y \in X$ and some constants $c_1, c_2 > 0$. Then*

1. *$c_1^{\alpha}\tau^{\alpha}_{\rho}(D) \leq \tau^{\alpha}_{\tilde{\rho}}(f(D)) \leq c_2^{\alpha}\tau^{\alpha}_{\rho}(D)$ for $\alpha > 0$ and $D \subset X$.*
2. *$c_1^{\alpha}Q^{\alpha}_{\rho}(D) \leq Q^{\alpha}_{\tilde{\rho}}(f(D)) \leq c_2^{\alpha}Q^{\alpha}_{\rho}(D)$ for $\alpha > 0$ and $D \subset X$.*

Proof. Let $\{x_i\}_{i=1}^{s} \subset D$ be such that $D \subset \bigcup_{i=1}^{s} K_{\rho}(x_i, \varepsilon)$. Observe that $f(D) \subset \bigcup_{i=1}^{s} f(K_{\rho}(x_i, \varepsilon)) \subset \bigcup_{i=1}^{s} K_{\tilde{\rho}}(f(x_i), c_2\varepsilon)$. In particular $\tau^{\alpha}_{\tilde{\rho}}(f(D)) \leq c_2^{\alpha}\tau^{\alpha}_{\rho}(D)$.

Let $\{y_i\}_{i=1}^{s} \subset D$ be such that $f(D) \subset \bigcup_{i=1}^{s} K_{\tilde{\rho}}(f(y_i), c_1\varepsilon)$. Observe that $D \subset \bigcup_{i=1}^{s} f^{-1}\left(K_{\tilde{\rho}}(f(y_i), c_1\varepsilon)\right) \subset \bigcup_{i=1}^{s} K_{\rho}(y_i, \varepsilon)$. In particular $c_1^{\alpha}\tau^{\alpha}_{\rho}(D) \leq \tau^{\alpha}_{\tilde{\rho}}(f(D))$.

The property (2) follows directly from (1). ■

Lemma 2.7. *Let M be k-dimensional, C^1 class submanifold of $\mathbb{R}^m$. Then $Q^{\alpha}(M) = 0$ for $k < \alpha$ and $Q^{\alpha}(M) = \infty$ for $0 < \alpha < k$.*

Proof. Observe that M is a local graph of class C^1 function. Let $x \in M$. There exists an open, convex set U and C^1 function $f \in C^1(U)$ such that $\psi : \mathbb{R}^k \supset U \ni x \to (x, f(x)) \in M \subset \mathbb{R}^m$ and $x \in \psi(U) \subset M$. We can assume that f' is bounded on U. Observe that

$$\|x - y\| \leq \|\psi(x) - \psi(y)\| \leq \sqrt{1 + \|f'\|}\,\|x - y\|\,.$$

Due to $H^\alpha(U) = \infty$ for $0 < \alpha < k$ and Lemma 2.1 we may conclude that $Q^\alpha(U) = \infty$ for $0 < \alpha < k$. Now due to Lemma 2.6 we have $\infty = Q^\alpha(\psi(U)) \leq Q^\alpha(M)$ for $0 < \alpha < k$. Moreover due to Lemma 2.2 $Q^k(U) < \infty$ and therefore $Q^k(\psi(U)) < \infty$ and $Q^\alpha(M) = 0$ for $k < \alpha$. ∎

3 Homogeneous polynomials

In this section we consider $\rho(z,w) := \sqrt{1 - |\langle z, w\rangle|}$ and a natural $(2n-1)$-dimensional measure σ on $\partial\mathbb{B}^n$. Observe that $\partial\mathbb{B}^n$ is $(2n-2, \rho, \sigma)$-regular. In fact there exist constants κ_1, κ_2 such that $\kappa_1\varepsilon^{2n-2} \leq \sigma(K_\rho(x,\varepsilon)) \leq \kappa_2\varepsilon^{2n-2}$ for $x \in \partial\mathbb{B}^n$ and $0 < \varepsilon < 1$. In particular $\nu^\alpha_{\rho\sigma}(\partial\mathbb{B}^n) = 0$ for $\alpha > 2n - 2$.

Definition. Let us denote $\chi_s : \mathbb{B}^n \ni z \longrightarrow \chi_s(z) = (1 - \|z\|^2)^s$ and

$$E^s(f) := \left\{ z \in \partial B^n : \int_{\mathbb{D}z} |f|^2 \chi_s d\mathfrak{L}^2 = \infty \right\}.$$

Definition 3.1. Let $\alpha > 0$. A subset $A \subset \partial\mathbb{B}^n$ is called α-separated iff $\rho(z_1, z_2) > \alpha$ for different elements $z_1, z_2 \in A$.

Definition. Let a sequence of the pairs (i,j) be ordered according to the formula

$$\lfloor i,j \rfloor < \lfloor k,l \rfloor \Leftrightarrow \begin{cases} i + j < k + l & \text{gdy} \quad i + j \neq k + l \\ i < k & \text{gdy} \quad i + j = k + l \end{cases}.$$

Lemma 3.2. *Let $C > 2$. Assume that a set A is $\frac{C}{\sqrt{N}}$-separated. For $z \in \partial\mathbb{B}^n$ we define*

$$A_m(z) := \left\{ \xi \in A : \frac{mC}{2\sqrt{N}} \leq \rho(z,\xi) \leq \frac{(m+1)C}{2\sqrt{N}} \right\}.$$

Therefore for $m = 1, 2, ...$ a set $A_m(z)$ has up to $2^{n-1}(m+2)^{2n-2}$ elements. A set $A_0(z)$ has up to one element. Additionally $s \leq N^{n-1}$.

Proof. First part of the Lemma it is in fact the [6, Lemma 1]. To prove that $s \leq N^{n-1}$ we can estimate

$$s\frac{C^{2n-2}}{2^{2n-2}N^{n-1}} \leq \sum_{j=1}^{s} \sigma(K_\rho(\zeta_j; \frac{C}{2\sqrt{N}})) \leq 1$$

since the balls $B(\zeta_j; C/(2\sqrt{N}))$ are disjoint. Therefore we get $s \leq N^{n-1}$. ∎

Lemma 3.3. *[6, Lemma 2] If $A \subset \partial\mathbb{B}^n$ is $\alpha/\sqrt{N}$-separated then for each $\beta > \alpha$ there exists an integer $K = K(\alpha, \beta)$ such that A can be partitioned into K disjoint $\beta/\sqrt{N}$-separated sets.*

Proposition 3.4. *We can estimate $\left(1+\frac{1}{x}\right)^x < e < \left(1+\frac{1}{x}\right)^{x+1}$ for $x \geq 1$.*

Proof. For $0 < y < 1$ we have the following inequality $y - \frac{y^2}{2} \leq \ln(1+y) < y$. Let $f(x) = x \ln\left(1+\frac{1}{x}\right)$ and $g(x) = (x+1)\ln\left(1+\frac{1}{x}\right)$. We may estimate $f'(x) = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1} \geq \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{x+1} = \frac{x^2-x}{2x^3(x+1)} > 0$ for $x > 1$. Moreover $g'(x) = \ln\left(1+\frac{1}{x}\right) - \frac{1}{x} < 0$ for $x > 1$. In particular $f(x) < f(\infty) = 1 = g(\infty) < g(x)$ for $x \geq 1$. ∎

Theorem 3.5. *There exists a constant C_0 such that $C_0 > 2$ and for all $C > C_0$, $\delta \in (0,1)$, $0 < \alpha \leq 2n-2$ there exists a natural number $K = K(C)$ such that if T, D are compact, circular, disjoint subsets of $\partial\mathbb{B}^n$, such that $\nu^\alpha(T) < \infty$ then there exists $m_0 \in \mathbb{N}$ such that homogeneous polynomials for (C,T) fulfills properties:*

1. *$|p_m(z)| \leq 2$ for $z \in \partial\mathbb{B}^n$ and $m \in \mathbb{N}$.*
2. $$\int_{\partial\mathbb{B}^n} |p_m|^2 \, d\sigma \leq \frac{6C^{2n}\left(\nu^\alpha_{\rho\sigma}(T) + \delta\right)}{m^{n-\frac{2+\alpha}{2}}}$$
   for $m \geq m_0$.
3. *$|p_m(z)| \leq 2^{-\sqrt{Km}}$ for all $z \in D$, $m \geq m_0$.*
4. *For $\alpha \geq 0$, $Km \geq m_0$ and $z \in T$ we have $\sum_{j=Km}^{K(m+1)-1} j^\alpha |p_j(z)|^2 \geq \frac{(Km)^\alpha}{4}$.*

Proof. Let $\beta = n - \frac{2+\alpha}{2}$. There exist $M, \varepsilon_0 > 0$ such that $M - \delta \leq \nu^\alpha_{\rho\sigma}(T)$ and $\sigma(K_\rho(T,\varepsilon)) \leq M\varepsilon^{2\beta}$ for $\varepsilon \in (0, 2\varepsilon_0)$. Denote $S := \partial\mathbb{B}^n \setminus K(T, \varepsilon_0)$. We may assume that ε_0 is so small that $D \subset S$.

Let $A = \{\xi_1, ..., \xi_s\}$ be $\frac{C}{\sqrt{N}}$-separated subset of T. Let

$$A_j(z) = \left\{\xi \in A : \frac{jC}{2\sqrt{N}} \leq \rho(z,\xi) < \frac{(j+1)C}{2\sqrt{N}}\right\}.$$

There exists $C_0 > 0$ such that

$$\exp\left(-\left(\frac{jC}{2}\right)^2\right)(j+2)^{2n-2} \leq \frac{1}{(j+2)^{2n}2^{j+n}}$$

for $C > C_0$ and $j \geq 1$.

Let N be so large that $\frac{C}{\sqrt{N}} \leq \varepsilon_0$ and $\rho(z,w) > \frac{1}{N^{0.1}}$ for $\xi \in A$, $w \in S$.

Due to Lemma 3.2 we can estimate

$$\begin{aligned} |p_m(z)| &\leq \sum_{\xi \in A} |\langle z, \xi_j\rangle|^m \leq \sum_{\xi \in A}\left(1 - \frac{1}{N^{0.2}}\right)^N \leq (2N)^{n-1}\left(1 - N^{0.2}\right)^{N^{0.2}N^{0.8}} \\ &\leq 2^{-N^{0.8}} \leq 2^{-\sqrt{Km}} \leq \frac{\delta}{2m^\beta} \end{aligned}$$

for $z \in S$, N high enough and $N \leq m \leq 2N$. We have proved the property (3). Moreover we may estimate

$$\int_S |p_m(z)|^2 \leq \int_S \frac{\delta}{2m^\beta} \leq \frac{\delta}{2m^\beta} \leq \frac{MC^{2n}}{m^\beta}.$$

Let us denote

$$\begin{aligned} B_0 &:= K_\rho\left(T, \frac{C}{2\sqrt{N}}\right) \\ B_{k+1} &:= K_\rho\left(T, \frac{(k+2)C}{2\sqrt{N}}\right) \setminus B_k. \end{aligned}$$

If $z \in B_{k+1}$ then $\rho(z,w) \geq \frac{(k+1)C}{2\sqrt{N}}$ for $w \in T$. In particular $A_j(z) = \emptyset$ for $j \leq k$. There exists $N_1 \in \mathbb{N}$ such that $K(T, \varepsilon_0) \subset \bigcup_{k=0}^{N_1} B_k \subset K(T, 2\varepsilon_0)$. We may estimate

$$\begin{aligned} |p_m(z)| &\leq \sum_{\xi \in A} |\langle z, \xi\rangle|^m \leq \sum_{j=0}^{\infty} \sum_{\xi \in A_j(z)} |\langle z, \xi\rangle|^m \\ &\leq \sum_{j=0}^{\infty} \sum_{\xi \in A_j(z)} \left(1 - \frac{j^2C^2}{4N}\right)^N \leq \sum_{j=0}^{\infty} \sum_{\xi \in A_j(z)} \exp\left(-\frac{j^2C^2}{4}\right) \\ &\leq \sum_{j=0}^{\infty} \#A_j(z) \exp\left(-\frac{j^2C^2}{4}\right) \\ &\leq 1 + \sum_{j=1}^{\infty} 2^{n-1}(j+2)^{2n-2} \exp\left(-\frac{j^2C^2}{4}\right) \leq 1 + \sum_{j=1}^{\infty} 2^{-j-1} \leq 2 \end{aligned}$$

for $z \in \partial\mathbb{B}^n$ and now we have the property (1). Moreover

$$|p_m(z)| \leq \sum_{j=k}^{\infty} 2^{n-1}(j+2)^{2n-2} \exp\left(-\frac{j^2C^2}{4}\right) \leq \sum_{j=k}^{\infty} (j+2)^{-2n} 2^{-j-1} \leq \frac{1}{(k+2)^{2n} 2^k}$$

for $z \in B_k$ and $k \geq 1$. Observe that

$$\sigma(B_k) \leq M \left(\frac{(k+1)C}{2\sqrt{N}}\right)^{2\beta} \leq M \frac{(k+1)^{2n} C^{2n}}{2^{2\beta} N^\beta} \leq M \frac{(k+1)^{2n} C^{2n}}{2^\beta m^\beta}$$

for $k \geq 0$, $N \leq m \leq 2N$. We can estimate

$$\begin{aligned} \int_{K_\rho(T,\varepsilon_0)} |p_m|^2 \, d\sigma &\leq \sum_{k=0}^{N_1} \int_{B_k} |p_m|^2 \, d\sigma \leq 4\sigma(B_0) + \sum_{k=1}^{N_1} \sigma(B_k)(k+2)^{-2n} 2^{-2k} \\ &\leq \frac{4MC^{2n}}{m^\beta} + \sum_{k=1}^{\infty} \frac{MC^{2n}}{m^\beta 2^{2k}} \leq \frac{5MC^{2n}}{m^\beta}. \end{aligned}$$

In particular we may prove the property (2):

$$\begin{aligned} \int_{\partial\mathbb{B}^n} |p_m|^2 \, d\sigma &\leq \int_{K_\rho(T,\varepsilon_0)} |p_m|^2 \, \sigma + \int_S |p_m|^2 \, \sigma \leq \frac{6MC^{2n}}{m^\beta} \\ &\leq \frac{6C^{2n} \left(\nu^\alpha(T) + \delta\right)}{m^\beta}. \end{aligned}$$

Now we prove the property (4).

Let $K = K(\alpha, \beta)$ be from Lemma 3.3 for $\alpha = 0.25$ and $\beta = C$. For $N = Km$ fix a maximal $1/(4\sqrt{N})$-separated subset $B \subset T$. Using Lemma 3.3 we can divide B into at least K disjoint $C/\sqrt{N}$-separated subsets $B_0, B_1, ..., B_{K-1}$. We define

$$p_{Km+j}(z) := \sum_{\xi \in B_j} \langle z, \xi \rangle^{Km+j}$$

for $j = 0, 1, ..., K-1$. There exists $C_0 > 0$ such that

$$\exp\left(-\left(\frac{kC}{2}\right)^2\right) k^{2n} 2^{3n} \leq \frac{1}{2^{k+3}}$$

for $C > C_0$ and $k \geq 1$.

Let

$$A_{i,j}(z) = \left\{\xi \in B_i : \frac{jC}{2\sqrt{N}} \leq \rho(z, \xi) < \frac{(j+1)C}{2\sqrt{N}}\right\}.$$

Due to Lemma 2.2 $\#A_{i,0} = 0$ and $\#A_{i,j} \leq 2^{n-1}(j+2)^{2n-2}$.

Due to Proposition 3.4 we have $\left(1 - \frac{1}{x+1}\right)^x > e^{-1} > \left(1 - \frac{1}{x+1}\right)^{x+1}$ for $x \geq 1$. Let $\xi \in B_j$. Let k_N be a maximal possible natural number such that $\frac{k_N^2 C^2}{4N} \leq \frac{1}{2}$. If $z \in K_\rho\left(\xi, \frac{1}{4\sqrt{N}}\right)$ then we may estimate:

$$\begin{aligned}
|p_{Km+j}(z)| &\geq |\langle z, \xi \rangle|^{Km+j} - \sum_{\eta \in B_j \setminus \{\xi\}} |\langle z, \eta \rangle|^{Km+j} \\
&\geq \left(1 - \frac{1}{16N}\right)^{Km+j} - \sum_{k=1}^{k_N} \left(1 - \frac{k^2 C^2}{4N}\right)^N 2^n (k+2)^{2n} - 2^{-N} N^n \\
&\geq \left(1 - \frac{1}{16N}\right)^{2N} - \sum_{k=1}^{\infty} \exp\left(-\left(\frac{kC}{2}\right)^2\right) k^{2n} 2^{3n} - 2^{-N} N^n \\
&\geq \exp\left(\frac{-2N}{16N-1}\right) - 2^{-N} N^n - \sum_{k=1}^{\infty} 2^{-k-3} \geq \frac{1}{2}
\end{aligned}$$

for $m_0 \leq N \leq m \leq 2N$ and m_0 high enough.

Since $B = \bigcup_{l=0}^{K-1} B_l$ is a maximal $1/(4\sqrt{N})$-separated subset of T we conclude that

$$\bigcup_{j=0}^{K-1} \bigcup_{\xi \in B_j} K_\rho\left(\xi; \frac{1}{4\sqrt{N}}\right) = \bigcup_{\xi \in B} K_\rho\left(\xi; \frac{1}{4\sqrt{N}}\right) \supset T$$

and from this follows that

$$\sum_{j=Km}^{K(m+1)-1} j^\alpha |p_j(z)|^2 \geq \frac{(Km)^\alpha}{4} \text{ for all } z \in T,\, m > m_0.$$

■

Now we are ready to prove our first, main result.

Theorem 3.6. *Let $0 < \alpha \leq 2n-2$. Let T be a compact, circular subset of $\partial\mathbb{B}^n$ such that $\nu^\alpha_{\rho\sigma}(T) = 0$. There exists $f \in \mathbb{O}(\mathbb{B}^n) \cap L^2(\mathbb{B}^n)$ such that $T = E^\beta(f)$ and $E^{\beta+\varepsilon}(f) = \emptyset$ for $\beta = n - \frac{2+\alpha}{2}$, $\varepsilon > 0$*

Proof. Let D_j be a sequence of compact, circular subsets of $\partial\mathbb{B}^n$ such that $D_j \cap T = \emptyset$, $D_j \subset D_{j+1}$ and $T = \bigcup_{j\in\mathbb{N}} D_j$. Due to Theorem 3.5 there exist numbers $C, M > 0$, a sequence of natural number $\{m_j\}_{j\in\mathbb{N}}$ and s sequence of polynomials $\{p_m\}_{m\in\mathbb{N}}$ such that

1. $m_j \geq 2^j$ and $K(m_j+1) \leq Km_{j+1}$
2. p_m is a homogeneous polynomial of degree m.
3. $\sum_{k\in I(i)} |p_k(z)|^2 \geq \frac{1}{4}$ for $z \in T$ and
$$I(i) := \{m \in \mathbb{N} : Km_i \leq m \leq K(m_i+1) - 1\}.$$
4. $|p_m(z)| \leq 2$ for $z \in \partial\mathbb{B}^n$ and $m \in \mathbb{N}$.
5. $\int_{\partial\mathbb{B}^n} |p_m|^2 \, d\sigma \leq MC^{2n}2^{-j}m^{-\beta}$ for $m \in I(j)$.
6. $|p_m(z)| \leq 2^{-j}$ for all $z \in D_j$, $m \in I(j)$

Let
$$f := \sum_{j=1}^{\infty} \sum_{k\in I(j)} \sqrt{k^{1+\beta}} p_k.$$

There exists a constant c_1 such that

$$\begin{aligned} c_1 \int_{\mathbb{B}^n} |f|^2 \, d\mathfrak{L}^{2n} &\leq \sum_{j=1}^{\infty} \sum_{k\in I(j)} k^{1+\beta} \int_{\partial\mathbb{B}^n} \frac{1}{k+1} \int_0^1 |p_k(tw)|^2 \, dt d\sigma(w) \\ &\leq \sum_{j=1}^{\infty} \sum_{k\in I(j)} \frac{k^{1+\beta}}{2k+1} \int_{\partial\mathbb{B}^n} |p_k|^2 \, d\sigma \\ &\leq \sum_{j=1}^{\infty} \sum_{k\in I(j)} \frac{k^\beta MC^{2n}2^{-j}}{2k^\beta} = \sum_{j=1}^{\infty} KMC^{2n}2^{-j-1} < \infty. \end{aligned}$$

There exist constants $c_2, c_3 > 0$ such that

$$\frac{c_2}{(k+1)^{\beta+1}.} \leq \int_0^1 t^{2k+1}(1-t^2)^\beta = 2\frac{(k+1)!(k+1)^\beta}{(k+1)(k+1)^\beta \prod_{j=1}^{k+1}(\beta+j)} \leq \frac{c_3}{(k+1)^{\beta+1}.}$$

Therefore we can estimate

$$\begin{aligned} \int_{\mathbb{D}z} |f|^2 \chi_\beta d\mathfrak{L}^2 &\geq \pi \sum_{j=1}^{\infty} \sum_{k\in I(j)} k^{1+\beta} \int_0^1 |p_k(tz)|^2 \, t(1-t^2)^\beta dt \\ &\geq \pi c_2 \sum_{j=1}^{\infty} \sum_{k\in I(j)} \frac{k^{1+\beta}}{(k+1)^{\beta+1}} |p_k(z)|^2 = \infty \end{aligned}$$

for $z \in T$. Moreover if $z \in \partial\mathbb{B}^n \setminus T$ then there exists a constant $c_4 = c_4(z) < \infty$ and j_0 such that $z \in D_j$ for $j \geq j_0$ and:

$$\begin{aligned}\int_{\mathbb{D}z} |f|^2 \chi_\beta d\mathfrak{L}^2 &\leq \pi c_2 \sum_{j=1}^{\infty} \sum_{k\in I(j)} \frac{k^{1+\beta}}{(k+1)^{\beta+1}} |p_k(z)|^2 \\ &\leq c_4(z) + \pi c_2 \sum_{j=j_0}^{\infty} \sum_{k=\in I(j)} 2^{-j} < \infty.\end{aligned}$$

We have proved that $T = E^\beta(f)$. Now let $\varepsilon > 0$. Then

$$\begin{aligned}\int_{\mathbb{D}z} |f|^2 \chi_{\beta+\varepsilon} d\mathfrak{L}^2 &\leq \pi c_2 \sum_{j=1}^{\infty} \sum_{k\in I(j)} \frac{k^{1+\beta}}{(k+1)^{\beta+\varepsilon+1}} |p_k(z)|^2 \\ &\leq \pi c_2 \sum_{j=1}^{\infty} \sum_{k=\in I(j)} \frac{4}{(K2^j)^\varepsilon} < \infty.\end{aligned}$$

for all $z \in \partial\mathbb{B}^n$. From this follows that $E^{\beta+\varepsilon}(f) = \emptyset$ for $\varepsilon > 0$. ■

Lemma 3.7. *Let U be an open, circular set and K be a compact, circular set such that $\nu^\alpha_{\rho\sigma}(K) < \infty$, $U, K \subset \partial\mathbb{B}^n$. Then there exists a sequence $\{T_i\}_{i\in\mathbb{N}}$ of compact, circular sets such that*

1. *$U \cap K = \bigcup_{i\in\mathbb{N}} T_i$.*
2. *If $T_i \cap T_j \neq \emptyset$ then $|i-j| < 2$.*
3. *$\sum_{i=1}^{\infty} \nu^s_{\rho\sigma}(T_i) = 0$ for $s > \alpha$.*

Proof. Let

$$\begin{aligned}T_{-1} &:= \left\{ z \in K \cap U : \inf_{w\in\partial U} \rho(z,w) \geq 1 \right\} \\ T_i &:= \left\{ z \in K \cap U : 2^{-i-1} \leq \inf_{w\in\partial U} \rho(z,w) \leq 2^{-i} \right\}.\end{aligned}$$

Observe that $U \cap K = \bigcup_{i\in\mathbb{N}} T_i$ and $T_i \cap T_j = \emptyset$ when $|i-j| \geq 2$. Moreover $\nu^\alpha_{\rho\sigma}(T_i) \leq \nu^\alpha_{\rho\sigma}(K)$ and therefore $\nu^s(T_i) = 0$ for $s > \alpha$. ■

Theorem 3.8. *Let $0 < \alpha < 2n-2$ and $\beta = n - \frac{2+\alpha}{2}$. Let E be a circular set of type G_δ such that $E \subset \partial\mathbb{B}^n$ and $\Theta^s_{\rho\sigma}(E) = 0$ for $s > \alpha$. There exists $f \in \mathbb{O}(\mathbb{B}^n) \cap L^2(\mathbb{B}^n)$ such that $E^\beta(f) = \emptyset$ and $E = E^s(f)$ for $0 \leq s < \beta$.*

Proof. Let $\alpha_i = \alpha + \frac{1}{i+2}(2n-2-\alpha)$ and $\beta_i = n - \frac{2+\alpha_i}{2}$. There exists a sequence $\{U_i\}_{i\in\mathbb{N}}$ of open, circular subsets of $\partial\mathbb{B}^n$ such that $E = \bigcap_{i=1}^{\infty} U_i$ and $U_{i+1} \subset U_i$. There exists a sequence $\{S_i\}_{i\in\mathbb{N}}$ of compact, circular subsets of $\partial\mathbb{B}^n$ such that $E \subset \bigcup_{j\in\mathbb{N}} S_{\lfloor i,j\rfloor}$ and $\sum_{j\in\mathbb{N}} \nu^{\alpha_{\lfloor i,j\rfloor -1}}_{\rho\sigma}(S_{\lfloor i,j\rfloor}) \leq 2^{-i}$. We denote $\lfloor i,j,k\rfloor := \lfloor \lfloor i,j\rfloor, k\rfloor$. Due to Lemma 3.7 there exists a sequence $\{T_i\}_{i\in\mathbb{N}}$ of compact, circular subsets of $\partial\mathbb{B}^n$ such that

1. $T_{\lfloor i,j,k\rfloor} \subset U_{\lfloor i,j\rfloor}$.

2. $S_{\lfloor i,j \rfloor} \cap U_{\lfloor i,j \rfloor} = \bigcup_{k\in\mathbb{N}} T_{\lfloor i,j,k \rfloor}$.
3. $T_{\lfloor i,j,k \rfloor} \cap T_{\lfloor i,j,l \rfloor} = \emptyset$ when $|l-k| \geq 2$.
4. $\nu_{\rho\sigma}^{\alpha_{\lfloor i,j,k \rfloor}}(T_{\lfloor i,j,k \rfloor}) = 0$.

Let $T_{-1} = \emptyset$ and

$$D_{\lfloor i,j,k \rfloor} = \overline{\partial\mathbb{B}^n \setminus \left(T_{\lfloor i,j,k-1 \rfloor} \cup T_{\lfloor i,j,k \rfloor} \cup T_{\lfloor i,j,k+1 \rfloor}\right)}$$

for $i,j,k \in \mathbb{N}$. Observe that $D_{\lfloor i,j,k \rfloor} \cap T_{\lfloor i,j,k \rfloor} = \emptyset$. Therefore due to Theorem 3.5 there exists a number $C > 0$, a sequence of natural number $\{m_j\}_{j\in\mathbb{N}}$ and a sequence of polynomials $\{p_m\}_{m\in\mathbb{N}}$ such that

1. $m_j^{\beta-\beta_j} \geq 2^j$ and $K(m_j+1) \leq Km_{j+1}$
2. p_m is a homogeneous polynomial of degree m.
3. $\sum_{m\in I(i,j,k)} |p_m(z)|^2 \geq \frac{1}{4}$ for $z \in T_{\lfloor i,j,k \rfloor}$ and
$$I(i,j,k) := \left\{ l \in \mathbb{N} : Km_{\lfloor i,j,k \rfloor} \leq l \leq K(m_{\lfloor i,j,k \rfloor}+1) - 1 \right\}.$$
4. $|p_m(z)| \leq 2$ for $z \in \partial\mathbb{B}^n$ and $m \in \mathbb{N}$.
5. $\int_{\partial\mathbb{B}^n} |p_m|^2 \, d\sigma \leq 6C^{2n} 2^{-\lfloor i,j,k \rfloor} m^{-\beta_{\lfloor i,j,k \rfloor}}$ for $m \in I(i,j,k)$.
6. $|p_m(z)| \leq 2^{-\sqrt{m}}$ for all $z \in D_{\lfloor i,j,k \rfloor}$, $m \in I(i,j,k)$

Let

$$f := \sum_{i,j\in\mathbb{N}} \sum_{m\in I(i,j,k)} \sqrt{m^{1+\beta_{\lfloor i,j,k \rfloor}}} p_m.$$

We denote

$$\phi(f,z,s) := \int_{\mathbb{D}z} |f|^2 \chi_s d\mathfrak{L}^2.$$

There exists a constant $c_1 > 0$ such that

$$\begin{aligned} c_1 \int_{\mathbb{B}^n} |f|^2 \, d\mathfrak{L}^{2n} &\leq \sum_{i,j,k\in\mathbb{N}} \sum_{m\in I(i,j,k)} m^{1+\beta_{\lfloor i,j,k \rfloor}} \int_{\partial\mathbb{B}^n} \frac{1}{m+1} \int_0^1 |p_m(tw)|^2 \, dt d\sigma(w) \\ &\leq \sum_{i,j,k\in\mathbb{N}} \sum_{m\in I(i,j,k)} \frac{m^{1+\beta_{\lfloor i,j,k \rfloor}}}{2m+1} \int_{\partial\mathbb{B}^n} |p_m|^2 \, d\sigma \\ &\leq \sum_{i,j.k\in\mathbb{N}} \sum_{m\in I(i,j,k)} \frac{m^{\beta_{\lfloor i,j,k \rfloor}} 6C^{2n} 2^{-\lfloor i,j,k \rfloor}}{2m^{\beta_{\lfloor i,j,k \rfloor}}} \leq 3C^{2n} \sum_{i\in\mathbb{N}} 2^{-i} < \infty. \end{aligned}$$

Let $0 \leq s < \beta$. We can use the similar arguments as in [3, Lemma 2.1,2.3] to conclude that there exist constants $c_2, c_3 > 0$ such that

$$\frac{c_2}{\pi(k+1)^{r+1}.} \leq \int_0^1 t^{2k+1}(1-t^2)^r = 2\frac{(k+1)!(k+1)^r}{(k+1)(k+1)^r \prod_{j=1}^{k+1}(r+j)} \leq \frac{c_3}{\pi(k+1)^{r+1}}$$

for $0 \le r < \beta$. Moreover

$$\sum_{\substack{i,j,k\in\mathbb{N}\\ z\in T_{\lfloor i,j,k\rfloor}}} 1 \ge \sum_{\substack{i,j\in\mathbb{N}\\ \beta_{\lfloor i,j,k\rfloor}>s\\ z\in S_{\lfloor i,j\rfloor}\cap U_{\lfloor i,j\rfloor}}} 1 = \infty.$$

We may estimate

$$\begin{aligned}\phi(f,z,s) &\ge \pi \sum_{i,j,k\in\mathbb{N}} \sum_{m\in I(i,j,k)} m^{1+\beta_{\lfloor i,j,k\rfloor}} \int_0^1 |p_m(tz)|^2\, t(1-t^2)^s dt\\ &\ge c_2 \sum_{\substack{i,j,k\in\mathbb{N}\\ \beta_{\lfloor i,j,k\rfloor}>s\\ z\in T_{\lfloor i,j,k\rfloor}}} \sum_{m\in I(i,j,k)} \frac{m^{1+\beta_{\lfloor i,j,k\rfloor}}}{(m+1)^{1+s}} |p_m(z)|^2 = \infty\end{aligned}$$

for $z \in E$. Let now $z \in \partial\mathbb{B}^n \setminus E$. There exists a minimal $\eta(z) \in \mathbb{N}$ such that $z \in \partial\mathbb{B}^n \setminus U_{\lfloor i,j\rfloor}$ for $\lfloor i,j\rfloor \ge \eta(z)$. Observe that $z \in U_{\lfloor i,j\rfloor}$ for $\lfloor i,j\rfloor < \eta$. In particular there exists $k_{i,j}$ such that $z \in T_{\lfloor i,j,k_{i,j}\rfloor}$ for $\lfloor i,j\rfloor < \eta(z)$. Let

$$J(\eta(z)) := \{(i,j,l) : \lfloor i,j\rfloor < \eta(z), |l-k_{i,j}| \le 1\}.$$

Observe that $\#J(\eta(z)) \le 3\eta(z)$. If $(i,j,k) \notin J(\eta(z))$ then $z \in D_{\lfloor i,j,k\rfloor}$. Therefore we may estimate:

$$\begin{aligned}c_3^{-1}\phi(f,z,s) &\le \sum_{i,j,k\in\mathbb{N}} \sum_{m\in I(i,j,k)} \frac{m^{\beta_{\lfloor i,j,k\rfloor}+1}}{(m+1)^{s+1}} |p_m(z)|^2\\ &\le \sum_{\substack{(i,j,k)\in J(\eta(z))\\ m\in I(i,j,k)}} m^\beta |p_m(z)|^2 + \sum_{\substack{(i,j,k)\notin J(\eta(z))\\ m\in I(i,j,k)}} m^\beta |p_m(z)|^2\\ &\le \sum_{\substack{(i,j,k)\in J(\eta(z))\\ m\in I(i,j,k)}} 4m^\beta + \sum_{\substack{(i,j,k)\notin J(\eta(z))\\ m\in I(i,j,k)}} Km^\beta 2^{-2\sqrt{m}} < \infty.\end{aligned}$$

We have proved that $E = E^s(f)$. Moreover

$$\begin{aligned}\phi(f,z,\beta) &\le c_3 \sum_{\substack{i,j,k\in\mathbb{N}\\ m\in I(i,j,k)}} m^{\beta_{\lfloor i,j,k\rfloor}-\beta} |p_m(z)|^2\\ &\le 4c_3 \sum_{\substack{i,j,k\in\mathbb{N}\\ m\in I(i,j,k)}} \frac{1}{\left(Km_{\lfloor i,j,k\rfloor}\right)^{\beta-\beta_{\lfloor i,j,k\rfloor}}} \le 4Kc_3 \sum_{i,j,k\in\mathbb{N}} 2^{-\lfloor i,j,k\rfloor} < \infty.\end{aligned}$$

for all $z \in \partial\mathbb{B}^n$. We conclude that $E^\beta(f) = \emptyset$. ∎

Lemma 3.9. *Let U be an open, circular subset of $\partial\mathbb{B}^n$. Let M be a compact, circular subset of $\partial\mathbb{B}^n$ and η a probability measure on M, such that M is (α,ρ,η)-regular. There exists a constant $c>0$ such that if K is a compact, circular set such that $\nu^\alpha_{\rho\sigma}(K)<\infty$, $K\subset M$ then there exists a sequence $\{T_i\}_{i\in\mathbb{N}}$ of compact, circular sets such that*

1. *$U\cap K=\bigcup_{i\in\mathbb{N}}T_i$.*
2. *If $T_i\cap T_j\neq\emptyset$ then $|i-j|<2$.*
3. *$\sum_{i=1}^\infty\nu^\alpha_{\rho\sigma}(T_i)\le c\nu^\alpha_{\rho\sigma}(K)$.*

Proof. Observe that $\partial\mathbb{B}^n$ is $(2n-2,\rho,\sigma)$-regular. Due to Lemma 2.2 there exist constants $c_1,c_2>0$ such that $c_1^{-1}\nu^\alpha_{\rho\sigma}(K)\le\nu^\alpha_{\rho\eta}(K)\le c_2\nu^\alpha_{\rho\sigma}(K)$ for a closed, circular K subset of M. We denote

$$\begin{aligned}T_0 &:= \left\{z\in K\cap U:\ \inf_{w\in\partial U}\rho(z,w)\ge 1\right\}\\ T_{i+1} &:= \left\{z\in K\cap U:\ 2^{-i-1}\le\inf_{w\in\partial U}\rho(z,w)\le 2^{-i}\right\}.\end{aligned}$$

Observe that $U\cap K=\bigcup_{i\in\mathbb{N}}T_i$ and $\rho(T_i,T_j)>0$ when $|i-j|\ge 2$. We may estimate

$$\sum_{i=0}^\infty\nu^\alpha_{\rho\eta}(T_{2i})+\sum_{i=0}^\infty\nu^\alpha_{\rho\eta}(T_{2i+1})=\nu^\alpha_{\rho\eta}\left(\bigcup_{i=0}^\infty T_{2i}\right)+\nu^\alpha_{\rho\eta}\left(\bigcup_{i=0}^\infty T_{2i+1}\right)\le 2\nu^\alpha_{\rho\eta}\left(\bigcup_{i=0}^\infty T_i\right).$$

In particular

$$\begin{aligned}\sum_{i=1}^\infty\nu^\alpha_{\rho\sigma}(T_i) &\le c_1\sum_{i=1}^\infty\nu^\alpha_{\rho\eta}(T_i)\le 2c_1\nu^\alpha_{\rho\eta}\left(\bigcup_{i=0}^\infty T_i\right)\\ &\le 2c_1\nu^\alpha_{\rho\eta}\left(\overline{\bigcup_{i=0}^\infty T_i}\right)\le 2c_1c_2\nu^\alpha_{\rho\sigma}\left(\overline{\bigcup_{i=0}^\infty T_i}\right)\le 2c_1c_2\nu^\alpha_{\rho\sigma}(K).\end{aligned}$$

■

Theorem 3.10. *Let $0<\alpha\le 2n-2$ and $\beta=n-\frac{2+\alpha}{2}$. Let E be a circular set of type G_δ such that $E\subset\partial\mathbb{B}^n$ and $\Theta^\alpha_{\rho\sigma}(E)=0$. Assume that there exists M - a compact, circular subset of $\partial\mathbb{B}^n$ and η a probability measure on M, such that M is (α,ρ,η)-regular and $E\subset M$. There exists $f\in\mathbb{O}(\mathbb{B}^n)\cap L^2(\mathbb{B}^n)$ such that $E^\beta(f)=E$ and $E^s(f)=\emptyset$ for $s>\beta$.*

Proof. Let $c>0$ be a constant from Lemma 3.9. There exists a sequence $\{U_i\}_{i\in\mathbb{N}}$ of open, circular subsets of $\partial\mathbb{B}^n$ such that $E=\bigcap_{i=1}^\infty U_i$ and $U_{i+1}\subset U_i$. There exists a sequence $\{S_i\}_{i\in\mathbb{N}}$ of compact, circular subsets of $\partial\mathbb{B}^n$ such that $E\subset\bigcup_{j\in\mathbb{N}}S_{\lfloor i,j\rfloor}$ and $\sum_{j\in\mathbb{N}}\nu^\alpha_{\rho\sigma}(S_{\lfloor i,j\rfloor})\le 2^{-i}$. We denote $\lfloor i,j,k\rfloor:=\lfloor\lfloor i,j\rfloor,k\rfloor$. Due to Lemma 3.9 there exists a sequence $\{T_i\}_{i\in\mathbb{N}}$ of compact, circular subsets of $\partial\mathbb{B}^n$ such that

1. $T_{\lfloor i,j,k\rfloor}\subset U_{\lfloor i,j\rfloor}$.
2. $S_{\lfloor i,j\rfloor}\cap U_{\lfloor i,j\rfloor}=\bigcup_{k\in\mathbb{N}}T_{\lfloor i,j,k\rfloor}$.

3. $T_{\lfloor i,j,k\rfloor} \cap T_{\lfloor i,j,l\rfloor} = \emptyset$ when $|l-k| \geq 2$.

4. $\sum_{k\in\mathbb{N}} \nu^{\alpha}_{\rho\sigma}(T_{\lfloor i,j,k\rfloor}) \leq c\nu^{\alpha}_{\rho\sigma}(S_{\lfloor i,j\rfloor})$.

Let $T_{-1} = \emptyset$ and

$$D_{\lfloor i,j,k\rfloor} = \overline{\partial\mathbb{B}^n \setminus \left(T_{\lfloor i,j,k-1\rfloor} \cup T_{\lfloor i,j,k\rfloor} \cup T_{\lfloor i,j,k+1\rfloor}\right)}$$

for $i, j, k \in \mathbb{N}$. Observe that $D_{\lfloor i,j,k\rfloor} \cap T_{\lfloor i,j,k\rfloor} = \emptyset$. Therefore due to Theorem 3.5 there exists a number $C > 0$, a sequence of natural number $\{m_j\}_{j\in\mathbb{N}}$ and a sequence of polynomials $\{p_m\}_{m\in\mathbb{N}}$ such that

1. $m_j \geq 2^j$ and $K(m_j+1) \leq Km_{j+1}$

2. p_m is a homogeneous polynomial of degree m.

3. $\sum_{m\in I(i,j,k)} |p_m(z)|^2 \geq \frac{1}{4}$ for $z \in T_{\lfloor i,j,k\rfloor}$ and

$$I(i,j,k) := \left\{l \in \mathbb{N} : Km_{\lfloor i,j,k\rfloor} \leq l \leq K(m_{\lfloor i,j,k\rfloor}+1) - 1\right\}.$$

4. $|p_m(z)| \leq 2$ for $z \in \partial\mathbb{B}^n$ and $m \in \mathbb{N}$.

5. $\int_{\partial\mathbb{B}^n} |p_m|^2 \, d\sigma \leq 6C^{2n}\left(\nu^{\alpha}_{\rho\sigma}(T_{\lfloor i,j,k\rfloor}) + 2^{-\lfloor i,j,k\rfloor}\right) m^{-\beta}$ for $m \in I(i,j,k)$.

6. $|p_m(z)| \leq 2^{-\sqrt{m}}$ for all $z \in D_{\lfloor i,j,k\rfloor}$, $m \in I(i,j,k)$

Let

$$f := \sum_{i,j\in\mathbb{N}} \sum_{m\in I(i,j,k)} \sqrt{m^{1+\beta}} p_m.$$

We denote

$$\phi(f,z,s) := \int_{\mathbb{D}z} |f|^2 \chi_s d\mathfrak{L}^2.$$

There exists a constant $c_1 > 0$ such that

$$\begin{aligned} c_1 \int_{\mathbb{B}^n} |f|^2 \, d\mathfrak{L}^{2n} &\leq \sum_{i,j,k\in\mathbb{N}} \sum_{m\in I(i,j,k)} m^{1+\beta} \int_{\partial\mathbb{B}^n} \frac{1}{m+1} \int_0^1 |p_m(tw)|^2 \, dt d\sigma(w) \\ &\leq \sum_{i,j,k\in\mathbb{N}} \sum_{m\in I(i,j,k)} \frac{m^{1+\beta}}{2m+1} \int_{\partial\mathbb{B}^n} |p_m|^2 \, d\sigma \\ &\leq \sum_{i,j.k\in\mathbb{N}} \sum_{m\in I(i,j,k)} \frac{m^{\beta} 6C^{2n}\left(\nu^{\alpha}_{\rho\sigma}(T_{\lfloor i,j,k\rfloor}) + 2^{-\lfloor i,j,k\rfloor}\right)}{2m^{\beta}} \\ &\leq 3C^{2n}(1+c)\sum_{i\in\mathbb{N}} 2^{-i} < \infty. \end{aligned}$$

Due to [3, Lemma 2.1,2.3] there exist constants $c_2, c_3 > 0$ such that

$$\frac{c_2}{\pi(k+1)^{r+1}.} \leq \int_0^1 t^{2k+1}(1-t^2)^{\beta} = 2\frac{(k+1)!(k+1)^{\beta}}{(k+1)(k+1)^r \prod_{j=1}^{k+1}(\beta+j)} \leq \frac{c_3}{\pi(k+1)^{\beta+1}}.$$

Moreover

$$\sum_{\substack{i,j,k\in\mathbb{N}\\ z\in T_{\lfloor i,j,k\rfloor}}} 1 \geq \sum_{\substack{i,j\in\mathbb{N}\\ z\in S_{\lfloor i,j\rfloor}\cap U_{\lfloor i,j\rfloor}}} 1 = \infty.$$

We may estimate

$$\begin{aligned}
\phi(f,z,\beta) &\geq \pi \sum_{i,j,k\in\mathbb{N}} \sum_{m\in I(i,j,k)} m^{1+\beta}\int_0^1 |p_m(tz)|^2\, t(1-t^2)^s dt \\
&\geq c_2 \sum_{\substack{i,j,k\in\mathbb{N}\\ z\in T_{\lfloor i,j,k\rfloor}}} \sum_{m\in I(i,j,k)} \frac{m^{1+\beta}}{(m+1)^{1+\beta}} |p_m(z)|^2 = \infty
\end{aligned}$$

for $z\in E$. Let now $z\in\partial\mathbb{B}^n\setminus E$ and $0\leq s$. There exists a minimal $\eta(z)\in\mathbb{N}$ such that $z\in\partial\mathbb{B}^n\setminus U_{\lfloor i,j\rfloor}$ for $\lfloor i,j\rfloor\geq\eta(z)$. Observe that $z\in U_{\lfloor i,j\rfloor}$ for $\lfloor i,j\rfloor<\eta$. In particular there exists $k_{i,j}$ such that $z\in T_{\lfloor i,j,k_{i,j}\rfloor}$ for $\lfloor i,j\rfloor<\eta(z)$. Let

$$J(\eta(z)) := \left\{(i,j,l) : \lfloor i,j\rfloor<\eta(z), |l-k_{i,j}|\leq 1\right\}.$$

Observe that $\#J(\eta(z))\leq 3\eta(z)$. If $(i,j,k)\notin J(\eta(z))$ then $z\in D_{\lfloor i,j,k\rfloor}$. Therefore we may estimate:

$$\begin{aligned}
c_3^{-1}\phi(f,z,s) &\leq \sum_{i,j,k\in\mathbb{N}} \sum_{m\in I(i,j,k)} \frac{m^{\beta+1}}{(m+1)^{s+1}} |p_m(z)|^2 \\
&\leq \sum_{\substack{(i,j,k)\in J(\eta(z))\\ m\in I(i,j,k)}} m^\beta |p_m(z)|^2 + \sum_{\substack{(i,j,k)\notin J(\eta(z))\\ m\in I(i,j,k)}} m^\beta |p_m(z)|^2 \\
&\leq \sum_{\substack{(i,j,k)\in J(\eta(z))\\ m\in I(i,j,k)}} 4m^\beta + \sum_{\substack{(i,j,k)\notin J(\eta(z))\\ m\in I(i,j,k)}} K m^\beta 2^{-2\sqrt{m}} < \infty.
\end{aligned}$$

We have proved that $E=E^s(f)$ for $0\leq s\leq\beta$. Moreover

$$\begin{aligned}
\phi(f,z,\beta+\varepsilon) &\leq c_3 \sum_{\substack{i,j,k\in\mathbb{N}\\ m\in I(i,j,k)}} m^{-\varepsilon} |p_m(z)|^2 \\
&\leq 4c_3 \sum_{\substack{i,j,k\in\mathbb{N}\\ m\in I(i,j,k)}} \frac{1}{\left(Km_{\lfloor i,j,k\rfloor}\right)^\varepsilon} \leq 4Kc_3 \sum_{i,j,k\in\mathbb{N}} 2^{-\varepsilon\lfloor i,j,k\rfloor} < \infty.
\end{aligned}$$

for all $z\in\partial\mathbb{B}^n$. We conclude that $E^{\beta+\varepsilon}(f)=\emptyset$ for $\varepsilon>0$. ∎

4 Examples

We consider a pseudometric $\rho(z,w) = \sqrt{1-|\langle z,w\rangle|}$ and a natural measure σ on $\partial\mathbb{B}^n$. Let us define $\phi : \mathbb{C}^{n-1}\times\mathbb{R} \ni (z,\theta) \to \phi(z,\theta) \in \Omega = \partial\mathbb{B}^n \setminus \mathbb{C}^{n-1}\times\{0\} \subset \mathbb{C}^n$:

$$\phi(z,\theta) = \exp(2\pi i\theta)\left(\frac{z_1}{\sqrt{1+\|z\|^2}},...,\frac{z_{n-1}}{\sqrt{1+\|z\|^2}},\frac{1}{\sqrt{1+\|z\|^2}}\right).$$

Let $M \subset \mathbb{C}^{n-1}$ be such that $\Im\langle z,w\rangle = 0$ for $z,w \in M$. Let $z,w \in M$ be such that $\|\phi(z)-\phi(w)\| < 2$. Observe that $\|\phi(z)-\phi(w)\|^2 = 2-2\Re\langle\phi(z),\phi(w)\rangle < 2$. In particular

$$2\rho(\phi(z),\phi(w))^2 = 2-2\,|\langle\phi(z),\phi(w)\rangle| = \|\phi(z)-\phi(w)\|^2 \tag{4.1}$$

for $z,w \in M$ such that $\|\phi(z)-\phi(w)\| < 2$.

We prove the following fact:

Lemma 4.1. *Let us consider the maximum norm $\|\circ\|$ on $\mathbb{R}^m$. We have the property: $\nu^\alpha_{\mathfrak{L}^m}(T) = \nu^\alpha_{\mathfrak{L}^{m+1}}(T\times[0,1])$.*

Proof. Let $\varepsilon > 0$. Observe that $K(T,\varepsilon)\times[0,1] \subset K(T\times[0,1],\varepsilon) \subset K(T,\varepsilon)\times[-\varepsilon,1+\varepsilon]$. We may estimate

$$\begin{aligned}\frac{\mathfrak{L}^m(K(T,\varepsilon))}{\varepsilon^{m-\alpha}} &= \frac{\mathfrak{L}^{m+1}(K(T,\varepsilon)\times[0,1])}{\varepsilon^{m-\alpha}} \le \frac{\mathfrak{L}^{m+1}(K(T\times[0,1],\varepsilon))}{\varepsilon^{m+1-(\alpha+1)}}\\ &\le \frac{\mathfrak{L}^{m+1}(K(T,\varepsilon)\times[-\varepsilon,1-\varepsilon])}{\varepsilon^{m+1-(\alpha+1)}} = \frac{\mathfrak{L}^m(K(T,\varepsilon))}{\varepsilon^{m-\alpha}}(1+2\varepsilon).\end{aligned}$$

This proves the required property. ■

Example 4.2. Let $0 < \alpha < 2n-2$ and $\beta = n-1-\frac{\alpha}{2}$. Let E be a set of type G_δ such that $E \subset M$ and $Q^s(E) = 0$ for $s > \alpha$. There exists $f \in \mathbb{O}(\mathbb{B}^n)\cap L^2(\mathbb{B}^n)$ such that $E^\beta(f) = \emptyset$ and $\phi(E\times[0,1]) = E^s(f)$ for $0 \le s < \beta$.

Proof. Due to Lemma 4.1 and Lemma 2.2 we conclude that $Q^s(E\times[0,1]) = 0$ for $s > \alpha+1$.

Let K be a compact subset of M. There exist constants $r_1 = r_1(K), r_2 = r_2(K) > 0$ such that

$$r_1\|\xi_1-\xi_2\| \le \|\phi(\xi_1)-\phi(\xi_2)\| \le r_2\|\xi_1-\xi_2\|.$$

In particular due to Lemma 2.6 we have $Q^s(\phi(E\times[0,1])) = 0$ for $s > \alpha+1$. Due to (4.1) we conclude that $Q^s_\rho(\phi(E\times[0,1])) = 0$ for $s > \alpha+1$. In particular due to Lemma 2.2 we have $\Theta^s_{\rho\sigma}(\phi(E\times[0,1])) = 0$ for $s > \alpha+1$. Now due to Theorem 3.8 there exists a function f with the required properties. ■

Example 4.3. There exists E - a compact, uncountable, circular set of type G_δ in $\partial\mathbb{B}^n$, a function $f \in \mathbb{O}(\mathbb{B}^n)\cap L^2(\mathbb{B}^n)$ such that $E^{n-1}(f) = \emptyset$ and $E = E^s(f)$ for $0 \le s < n-1$.

Proof. Due to Lemma 2.5 there exists a compact, uncountable set K such that $K \subset [0,1]$ and $Q^s(K) = 0$ for $s > 0$. Now it is enough to use the Example 4.2 for $\alpha = 0$. ■

Example 4.4. There exists E - a set of type G_δ and a holomorphic function $f \in \mathbb{O}(\mathbb{B}^n) \cap L^2(\mathbb{B}^n)$ such that $E^{n-1}(f) = \emptyset$ and $E = E^s(f)$ for $0 \le s < n-1$. Moreover $\Theta^\alpha_{\rho\sigma}(E) = \infty$ for $0 \le \alpha < 2n-2$.

Proof. We denote $\chi_s(z) = \left(1 - \|z\|^2\right)^s$. Let $e_1 = (1, 0, ..., 0)$ and

$$g(z_1, ..., z_n) = \sum_{m=2}^{\infty} \frac{2^{mn}}{m} z_1^{2^{2m}}.$$

First we show that $g \in \mathbb{O}(\mathbb{B}^n) \cap L^2(\mathbb{B}^n)$ and $e_1 \in E^s(g)$ for $0 \le s < n-1$.

Using [3, Theorem 2.2] we may estimate

$$\int_{\mathbb{B}^n} |g|^2 \, d\mathfrak{L}^{2n} \;=\; \sum_{m=2}^{\infty} \frac{2^{2mn}\pi^n (2^{2m})!}{m^2 (2^{2m}+n)!} \le \sum_{m=2}^{\infty} \frac{1}{m^2} < \infty.$$

Let $0 < \varepsilon < n-1$ and $s = n-1-\varepsilon$. Due to [3, Theorem 2.2, Lemma 2.3] there exists $c > 0$ such that

$$\begin{aligned} \int_{\mathbb{D}e_1} |g|^2 \chi_s d\mathfrak{L}^2 &= \sum_{m=2}^{\infty} \frac{2^{2mn}\pi (2^{2m})!}{m^2 \left(s + 2^{2m} + 1\right) \prod_{i=1}^{2^{2m}} (s+i)!} \\ &\ge c \sum_{m=2}^{\infty} \frac{2^{2mn} (2^{2m})!}{m^2 \left(n + 2^{2m}\right) 2^{2ms} \left(2^{2m}\right)!} \\ &\ge c \sum_{m=2}^{\infty} \frac{2^{2mn}}{m^2 n 2^{2m(s+1)}} = cn^{-1} \sum_{m=2}^{\infty} 2^{2m\varepsilon} m^{-2} = \infty. \end{aligned}$$

There exists a sequence $T = \{\xi_i\}_{i\in\mathbb{N}}$ dense in $\partial\mathbb{B}^n$ and such that $\xi_{\lfloor i,j \rfloor} = \xi_{\lfloor i,1 \rfloor}$ for $i, j \in \mathbb{N}$. Let now

$$f_k(z) := \sum_{m=k+1}^{\infty} \frac{2^{mn}}{m} \langle z, \xi_k \rangle^{2^{2m}+2^{2k}}$$

and $A_k := \left\{2^{2m} + 2^{2k}\right\}_{m=k+1}^{\infty}$. Observe that $\int_{\mathbb{B}^n} |f_k|^2 \, d\mathfrak{L}^{2n} \le \int_{\mathbb{B}^n} |g|^2 \, d\mathfrak{L}^{2n}$ and $\xi_k \in E^s(f_k)$ for $0 \le s < n-1$. Moreover $A_i \cap A_j = \emptyset$ for $i \ne j$. Let

$$f = \sum_{k\in\mathbb{N}} 2^{-k} f_k.$$

We can estimate

$$\int_{\mathbb{B}^n} |f|^2 \, d\mathfrak{L}^{2n} = \sum_{k\in\mathbb{N}} 2^{-2k} \int_{\mathbb{B}^n} |f_k|^2 \, d\mathfrak{L}^{2n} \le \int_{\mathbb{B}^n} |g|^2 \, d\mathfrak{L}^{2n}.$$

In particular due to [3, Theorem 2.7] we conclude that $E^{n-1}(f) = \emptyset$.

We may estimate $\int_{\mathbb{D}\xi_k} |f|^2 \chi_s d\mathfrak{L}^2 = \sum_{m\in\mathbb{N}} \int_{\mathbb{D}\xi_k} |f_m|^2 \chi_s d\mathfrak{L}^2 \ge \int_{\mathbb{D}\xi_k} |f_k|^2 \chi_s d\mathfrak{L}^2 = \infty$ for $0 \le s < n-1$. In particular $T \subset E^s(f)$ for $0 \le s < n-1$.

Let $0 < \alpha < 2n-2$. It is known that $E^s(f)$ is a circular set of type G_δ in $\partial\mathbb{B}^n$. Let $\delta > 0$ and $\{K_i\}_{i\in\mathbb{N}}$ be a sequence of compact, circular sets such that $T \subset E^s(f) \subset \bigcup_{i\in\mathbb{N}} K_i \subset \partial\mathbb{B}^n$ and $d_\rho(K_i) \le 2\delta$. Due to Bair's Theorem we conclude that there exists K_{i_0} with a non empty interior in $\partial\mathbb{B}^n$. In particular due to $0 < H^{2n-2}_\rho(\partial\mathbb{B}^n) < \infty$ we have $H^\alpha_\rho(K_{i_0}) = \infty$ and $\sum_{i\in\mathbb{N}} \tau^\alpha_\rho(K_i) \ge \tau^\alpha_\rho(K_{i_0}) \ge H^\alpha_\rho(K_{i_0}) = \infty$. Therefore $Q^\alpha_\rho(E^s(f)) = \infty$ and $\Theta^\alpha_{\rho\sigma}(E^s(f)) = \infty$ for $0 \le \alpha < 2n-2$. ∎

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