On classifying involutive locally m-convex algebras, via cones

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Abstract

We show that any hermitian *-*l.m.c.a.*, the set of positive elements of which is a locally bounded cone, is necessarily a Q-algebra (the converse is not true). We also obtain that the algebra of complex numbers is the unique locally C^*-algebra without zero-divisors.

1 Introduction

Some characterizations, in hermitian algebras can be obtained by examining the set A_+ of their positive elements. For example, in [4], it is shown that a unital, hermitian and complete Q-*-*l.m.c.a.* whose positive elements is a normal cone is necessarily a C^*-algebra. In this perspective we consider, in the third section, the notion of a locally bounded cone. We show that a unital and hermitian *-*l.m.c.a.* A, with A_+ a locally bounded cone is necessarily a Q-algebra (Theorem 3.6). As a consequence we obtain that a locally C^*-algebra is a C^*-algebra if, and only if, the cone A_+ satisfies this property. The converse of Theorem 3.6 is not valid in general (Remark 3.10). In the fourth section we first show that there is no non-trivial hermitian algebra in which the set A_+ defines a total order. As a consequence we obtain (Theorem 4.4) that the algebra of complex numbers is the unique locally C^*-algebra containing no algebraic zero-divisors.

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2 Preliminaries

Let (E, τ) be a Hausdorff locally (convex space) and K a cone in E such that $K \cap (-K) = \{0\}$. The cone K defines an order in E by $x \leq y$ if, and only if, $y - x \in K$. This order will be said total if, for every $x, y \in K$, one has $x \leq y$ or $y \leq x$. Now let us fix some of cone notions. A cone is said to be normal if there exists a family $(q_i)_i$ of seminorms defining the topology τ of E and such that $q_i(x) \leq q_i(y)$, for every i, whenever $0 \leq x \leq y$. It will be said nuclear (supernormal) if, for every i, there is $\mu_i > 0$ such that $q_i(x + y) \geq \mu_i q_i(x) + q_i(y)$ for every x, y in K. Now K is called locally bounded if there is a zero-neighborhood U such that $U \cap K$ is bounded. Finally K is said to be well-based if there exists a convex and bounded set S such that $0 \notin \overline{S}$ and $K = \bigcup_{\lambda \geq 0} \lambda S$.

A locally convex algebra (*l.c.a.*) is an algebra which is a locally convex space such that the ring multiplication is separately continuous. A *l.m.c.a.* is a locally convex algebra the topology of which is defined by family $(|.|_\lambda)_\lambda$of sub-multiplicative seminorms i.e., $|xy|_\lambda \leq |x|_\lambda |y|_\lambda$, for every x, y in A and every λ. Now, a unital algebra is said to be a Q-algebra if the set of its invertible elements is open in A. We denote by *-*l.c.a.* (resp. *-*l.m.c.a.*) a *l.c.a.* (resp. *l.m.c.a.*) endowed with a continuous involution $x \longmapsto x^*$; it will be said hermitian (resp. symmetric) if the spectrum of any self-adjoint element is real (resp. $e + xx^*$ is invertible in A for every x). The set of self-adjoint elements and that one of positive elements will be denoted by $H(A)$ and A_+, respectively, that is, we put, $H(A) = \{x \in A : x = x^*\}$ and $A_+ = \{x \in A : x = x^* \text{ and } Sp(x) \subset \mathbb{R}_+\}$. A locally C^*-algebra is a complete *l.m.c.a.* $(A, (|.|_\lambda)_\lambda)$ endowed with an involution $x \longmapsto x^*$ such that for every λ, $|xx^*|_\lambda = |x|_\lambda^2$, for every $x \in A$. Concerning involutive *l.m.c.a.*'s the reader is referred to [10]. In the sequel $Sp(x)$ and $\rho(x)$ will stand respectively, for the spectrum and the spectral radius in A, with the assumption that $Sp(x) \neq \varnothing$, for every x in A.

3 Locally bounded cone and Q-algebra structure

In a hermitian and normed algebra A the set A_+ is locally bounded. We show that the converse is true for some classes of hermitian algebras.

Proposition 3.1. Let $\left(A, (|.|_\lambda)_\lambda\right)$ be a unital hermitian *l.m.c.a.*. If A_+ is a locally bounded cone which is stable by product (: $A_+ A_+ \subset A_+$), then $\left(A, (|.|_\lambda)_\lambda\right)$ can be endowed with a finer algebra norm.

Proof. A_+ being locally bounded, there exists a zero-neighborhood U which is convex, balanced, idempotent and such that $U \cap A_+$ is bounded. Now consider

$$B = \left\{ \sum_{i=1}^{n} \lambda_i a_i, \text{ with } a_i \in U \cap A_+ \text{ and } \lambda_i \in \mathbb{R}\text{:} \sum_{i=1}^{n} |\lambda_i| \leq 1 \right\}$$

Since A_+ is stable by product, the set B is an idempotent and bounded disk. It is moreover absorbent in the real vectorial space $H(A)$. Indeed, for $h \in H(A)$, there are $p, q \in A_+$ such that $h = p - q$. Since U is absorbent, there exist $\alpha_1, \alpha_2 > 0$ such that $p \in \alpha_1 U$ and $q \in \alpha_2 U$. Let $\alpha = \max(\alpha_1, \alpha_2)$. Then $p, q \in \alpha U$. Hence

$p, q \in \alpha (U \cap A_+)$; it follows that, there exist $u, v \in U \cap A_+$ such that $p = \alpha u$ and $q = \alpha v$ and therefore $h = p - q = 2\alpha \left(\frac{u}{2} - \frac{v}{2}\right)$. Let now $J = \overline{iB + iB}$. Then J is an absorbing, bounded and idempotent barrel in A. Its gauge function, p_J, defines in $\left(A, (|.|_\lambda)_\lambda\right)$ a finer algebra norm. ■

In the more general case of hermitian *l.c.a.* with A_+ a locally bounded cone, not necessarily stable by product, the set J remains an absorbing bounded barrel. Whence the following result.

Proposition 3.2. Let $\left(A, (|.|_\lambda)_\lambda\right)$ be a unital hermitian *l.c.a.*. If A_+ is a locally bounded cone, then $\left(A, (|.|_\lambda)_\lambda\right)$ can be endowed with a finer vectorial space norm.

Corollary 3.3. Let $\left(A, (|.|_\lambda)_\lambda\right)$ be a unital hermitian and barrelled (take e.g. Fréchet) *l.c.a.*. If A_+ is a locally bounded cone, then $\left(A, (|.|_\lambda)_\lambda\right)$ is a normed algebra.

Proof. A being barrelled, the set J is a zero-neighborhood in the algebra $\left(A, (|.|_\lambda)_\lambda\right)$. It follows that $\left(A, (|.|_\lambda)_\lambda\right)$ is A-normed. We conclude by ([10], Corollary 5.3, p. 38). ■

The Q-property is one of the most important notions in the theory of topological algebras. Indeed several properties of Banach algebras are due to this property. In the following we give conditions on the set, of positive elements of a hermitian *-*l.m.c.a.* to be a Q-algebra. We first provide, in our context, a direct and short proof of a result, of G. Isac ([8], Theorem 6).

Proposition 3.4. Let $\left(A, (|.|_\lambda)_\lambda\right)$ be a *l.c.a.* and K a cone in A. The following assertions are equivalent.

1. K is locally bounded.

2. $(\exists \lambda_0 \in \Lambda)\, (\forall \lambda \in \Lambda)\, (\exists c_\lambda > 0) : |x|_\lambda \leq c_\lambda\, |x|_{\lambda_0}$, for every $x \in K$.

Proof. 1. $\Longrightarrow$ 2. K being locally bounded, there exists a zero-neighborhood U such that $U \cap K$ is bounded. Since the family $(|.|_\lambda)_\lambda$ is directed, we suppose that $U = B_{\lambda_0}(0, \alpha) = \{x \in A : |x|_{\lambda_0} < \alpha\}$ with $\lambda_0 \in \Lambda$ and α>0. So that, for every $\lambda \in \Lambda$, there is M_λ>0 such that $|x|_\lambda < M_\lambda$ for every $x \in B_{\lambda_0}(0, \alpha) \cap K$. Let $x \in K$ with $|x|_{\lambda_0} \neq 0$. One has $\frac{\alpha x}{2|x|_{\lambda_0}} \in B_{\lambda_0}(0, \alpha) \cap K$. Hence $|x|_\lambda < 2\alpha^{-1} M_\lambda\, |x|_{\lambda_0}$. On the other hand, if $x \in K$ and $|x|_{\lambda_0} = 0$, then $tx \in B_{\lambda_0}(0, \alpha) \cap K$, for every $t \in \mathbb{R}_+$. Whence, for every λ and every $t \in \mathbb{R}_+$, $|tx|_\lambda < M_\lambda$; this implies $|x|_\lambda = 0$, for every λ. 2. $\Longrightarrow$ 1. It is easily seen that $B_{\lambda_0}(0, \alpha) \cap K$ is bounded. ■

Now here is a property of the spectral radius, when A_+ is locally bounded.

Proposition 3.5. Let $\left(A, (|.|_\lambda)_\lambda\right)$ be a complete hermitian and unital *l.m.c.a.*. If A_+ is a locally bounded cone, then $\rho/H(A)$ is finite.

Proof. According to Proposition 3.4, there is $\lambda_0 \in \Lambda$ and $c_\lambda > 0$ such that, for every λ, $|x|_\lambda \leq c_\lambda |x|_{\lambda_0}$, for every $x \in A_+$. Since $x^n \in A_+$, for every $x \in A_+$, one has

$$\rho(x) = \sup_\lambda \left[\lim_n (|x^n|_\lambda)^{\frac{1}{n}}\right] \leq \lim_n \left[c_\lambda^{\frac{1}{n}} \left(|x^n|_{\lambda_0}\right)^{\frac{1}{n}}\right] \leq |x|_{\lambda_0}. \tag{1}$$

Now, for every $h \in H(A)$, we have $\rho(h) = [\rho(h^2)]^{\frac{1}{2}} < |h|_{\lambda_0} < +\infty$. ■

If we suppose, additionally, the involution continuous, we obtain the Q-algebra structure.

Theorem 3.6. Every unital complete hermitian *-*l.m.c.a.*, having the cone of positive elements locally bounded, is a Q-algebra.

Proof. The involution being continuous, we suppose that $|x^*|_\lambda = |x|_\lambda$ for every $x \in A$ and every λ. Since A is hermitian, one has $\rho(y) \leq [\rho(yy^*)]^{\frac{1}{2}}$, for every $y \in A$ ([2], Theorem 4.4). Whence, by (1)

$$\rho(y) \leq \left[|yy^*|_{\lambda_0}\right]^{\frac{1}{2}} \leq |y|_{\lambda_0},$$

for every $y \in A$. We conclude by ([14], Theorem 4.1, "Tsertos inequality"). ■

A compilation of the notions of locally bounded and normal cones yield now

Corollary 3.7. A unital complete hermitian *-*l.m.c.a.*, whose positive cone is normal and locally bounded, is a C^*-algebra.

Proof. By Theorem 3.6 and Corollary 2.2 of [4]. ■

Theorem 3.8. Every locally C^*-algebra, having a locally bounded positive cone, is a C^*-algebra.

Corollary 3.9. Let $\left(A, (|.|_\lambda)_\lambda\right)$ be a complete hermitian and unital *-*l.m.c.a.*. If A_+ is a well-based cone, then A is a finite dimensional C^*-algebra.

Proof. By Proposition 3.3.1 of [12], the cone A_+ is both locally bounded and nuclear. We conclude with ([5], Theorem 3.1). ■

Remark 3.10.

1. In Proposition 3.1 and Proposition 3.5 the converse is not true as it is shown by the algebra $l^{\infty}(\mathbb{R})$ endowed with the usual operations and the C^*-seminorms, p_n, $n \in \mathbb{N}$, given by $p_n[(x_p)_p] = \sup\{|x_p|, 1 \le p \le n\}$. Indeed, $l^{\infty}(\mathbb{R})$ can be endowed with a finer algebra norm and $\rho/H(l^{\infty}(\mathbb{R}))$ is finite but $[l^{\infty}(\mathbb{R})]_+$ is not locally bounded by Theorem 3.8.

2. The converse of Theorem 3.6 is also not true. Indeed, consider the algebra $C^{\infty}[0,1]$ endowed with the usual operations and the family $(q_n)_n$ of seminorms given by $q_n(f) = \frac{1}{2^{n-1}} \sup_{0 \le t \le 1} \left| f^{(n)}(t) \right|$. It is a Fréchet hermitian *-*l.m.c.a.* which is a Q-algebra, but the cone of its positive elements is not locally bounded. Otherwise, By Corollary 3.3, $C^{\infty}[0,1]$ should be a normed algebra, which is not true ([6], Example 2.1).

4 Locally C^*-algebra containing no zero-divisors

In ([11], Proposition 11.12), it is shown that every complete Q-*l.m.c.a.* which does not contain topological zero-divisors is isomorphic to the algebra of complex numbers. In the following we consider just algebraic zero-divisors and we obtain the same result in some particular cases. We start by examining involutive algebra endowed with a certain total order,

Proposition 4.1. Let A be an involutive and unital algebra. If $(A_+, \le)$ is totally ordered, then $A_+ = \mathbb{R}_+$.

Proof. Let us first show that $\rho(x) < +\infty$, for every $x \in A_+$. Suppose that there is $x \in A_+$ such that $\rho(x) = +\infty$. Since $(A_+, \le)$ is totally ordered, for every $n \in \mathbb{N}$, one has $n \le x$ or $x \le n$. But since $\rho(x) = +\infty$, we have necessarily $n \le x$, for every $n \in \mathbb{N}$; this implies that $Sp(x) \subset [n, +\infty[$, for every n. It follow that $Sp(x) = \varnothing$; which is ruled out by hypothesis (cf. preliminaries). Now let $x \in A_+$ such that $0 \in Sp(x)$. For every $\alpha > 0$, one has $x \le \alpha$, for otherwise one gets $\alpha < 0$. Hence $Sp(x) = \{0\}$; it follows that $x \in A_+ \cap (-A_+) = \{0\}$. Suppose now that $x \in A_+$ and $0 \notin Sp(x)$. Put $\mu = \inf\{\beta : \beta \in Sp(x)\}$. Then $0 \in Sp(x-\mu)$, otherwise $x - \mu$ should be invertible and then $\rho((x-\mu)^{-1}) = \frac{1}{\inf(Sp(x-\mu))} = +\infty$; which can not be. Hence $x - \mu = 0$ and so $x = \mu \in \mathbb{R}_+$. ∎

Here is an application of the previous.

Corollary 4.2. Let A be an involutive and unital algebra. If $(A_+, \le)$ is totally ordered, then

1. $\{x \in H(A) : Sp(x) \subset \mathbb{R}\} = \mathbb{R}$.

2. If A is hermitian, then $A = \mathbb{C}$.

To continue, we need the following.

Proposition 4.3. Let $(A, (|.|_\lambda)_\lambda)$ be a unital locally C^*-algebra. For every $h \in H(A)$, there are $p, q \in A_+$ such that $h = p - q$ with $pq = qp = 0$.

Proof. Let $h \in H(A)$ and C_h the commutative and maximal sub-algebra containing h. By ([7], Proposition 4.4), there is $u \in A_+ \cap C_h$ such that $u^2 = h^2$. Let χ be a character of C_h. One has

$$[\chi(h)]^2 = \chi\left(h^2\right) = \chi\left(u^2\right) = [\chi(u)]^2 .$$

Whence

$$Sp_A(u \pm h) = Sp_{C_h}(u \pm h) = \{\chi(u) \pm \chi(h) : \chi \in M(A)\}.$$

Now, since $u \in A_+$, one has $Sp_A(u \pm h) \subset \mathbb{R}_+$ and therefore $u \pm h \in A_+$. Finally $h = \frac{1}{2}(u + h) - \frac{1}{2}(u - h)$. ■

Theorem 4.4. A unital locally C^*-algebra containing no zero-divisor is isomorphic to $\mathbb{C}$.

Proof. According to Corollary 4.2, it suffices to prove that $(A_+, \leq)$ is totally ordered. For $a, b \in A_+$, one has $a - b \in H(A)$. By Proposition 4.3, there are $p, q \in A_+$ such that $a - b = p - q$ with $pq = qp = 0$. But since there is no zero-divisor in A, one gets $p = 0$ or $q = 0$; that is $a \leq b$ or $b \leq a$. ■

Remark 4.5. The result of Theorem 4.4 remains true for any hermitian algebra satisfying,

1. A_+ defines an order in A.
2. For every $h \in H(A)$, there are $p, q \in A_+$ such that $h = p - q$ with $pq = qp = 0$.

These two conditions being satisfied by any unital and commutative GB^*-algebra ([1], Theorem 3.14), we obtain, in the commutative case, an extension of Theorem 4.4.

Theorem 4.6. A unital and commutative GB^*-algebra A containing no zero-divisors is isomorphic to $\mathbb{C} + Rad(A)$.

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