Birkhoff-Kellogg and Best Proximity Pair Results

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Abstract

The paper presents new Birkhoff-Kellogg type theorems for maps in the S-KKM class. Best proximity pair theorems are also established for the admissible class $\mathfrak{A}_c^\kappa$ and the PK class.

1 Introduction

The paper discusses maps in the S-KKM class and in the admissible class $\mathfrak{A}_c^\kappa$. We prove new Birkhoff-Kellogg type results on invariant direction for the class of S-KKM maps, which is a general class of maps including other important classes such as the composite class $\mathfrak{A}_c^\kappa$. We also obtain "invariant direction" results for countably condensing maps. We establish best proximity pair theorems for multimaps in the $\mathfrak{A}_c^\kappa$ and PK classes. The results given in this paper extend, generalize and complement various known results in the literature including those of [1, 7, 8, 10, 11, 13].

2 Preliminaries

Let X and Y be Hausdorff topological vector spaces. Recall a polytope P in X is any convex hull of a nonempty finite subset of X. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (the nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathfrak{A}$ of maps is defined by the following properties:

(i). $\mathfrak{A}$ contains the class $\mathcal{C}$ of single valued continuous functions;

(ii). each $F \in \mathfrak{A}_c$ is upper semicontinuous and compact valued; and

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(iii). for any polytope P, $F \in \mathfrak{A}_c(P,P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathfrak{A}$.

Definition 2.1. $F \in \mathfrak{A}_c^\kappa(X,Y)$ (i.e. F is $\mathfrak{A}_c^\kappa$–admissible) if for any compact subset K of X, there is a $G \in \mathfrak{A}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Definition 2.2. Let X be a convex subset of a Hausdorff topological vector space and Y a topological space. If $S, T : X \to 2^Y$ are two set-valued maps such that $T(co(A)) \subseteq S(A)$ for each finite subset A of X, then we say that S is a generalized KKM map w.r.t. T. The map $T : X \to 2^Y$ is said to have the KKM property if for any generalized KKM w.r.t. T map S, the family

$$\{\overline{S(x)} : x \in X\}$$

has the finite intersection property. We let

$$\mathrm{KKM}(X,Y) = \{T : X \to 2^Y : T \,\text{has the KKM property}\}.$$

Remark 2.1. If X is a convex space, then $\mathfrak{A}_c^\kappa(X,Y) \subset \mathrm{KKM}(X,Y)$ (see [6]).

Definition 2.3. Let X be a nonempty set, Y a nonempty convex subset of a Hausdorff topological vector space and Z a topological space. If $S : X \to 2^Y$, $T : Y \to 2^Z$, $F : X \to 2^Z$ are three set-valued maps such that $T(co(S(A))) \subseteq F(A)$ for each nonempty finite subset A of X, then F is called a generalized S-KKM map w.r.t. T. If the map $T : X \to 2^Z$ is such that for any generalized S-KKM w.r.t. T map F, the family

$$\{\overline{F(x)} : x \in X\}$$

has the finite intersection property, then T is said to have the S-KKM property. The class

$$\text{S-KKM}(X,Y,Z) = \{T : Y \to 2^Z : T \,\text{has the S-KKM property}\}.$$

Remark 2.2. Note that S-KKM(X,Y,Z) = KKM(X,Z) whenever $X = Y$ and S is the identity mapping $\mathbf{1}_X$. Moreover, KKM(Y,Z) is a proper subset of S-KKM(X,Y,Z) for any $S : X \to 2^Y$. S-KKM(X,Y,Z) also includes other important classes of multimaps (see [4, 5] for examples).

Remark 2.3. Let X be a convex space, Y a convex subset of a Hausdorff locally convex space, and Z a normal space. Suppose $s : Y \to Y$ is surjective, $F \in$ s-KKM(Y,Y,Z) is closed, and $f \in \mathcal{C}(X,Y)$. Then $F \circ f \in \mathbf{1}_X -$ KKM(X,X,Z) (see [5]).

Remark 2.4. Let X be a convex subset of a Hausdorff topological space, Y a convex space, and Z, W topological spaces and $S : X \to 2^Y$. If $F \in$ S-KKM(X,Y,Z) and $f \in \mathcal{C}(Z,W)$, then $f \circ F \in$ S-KKM(X,Y,W) (see [5]).

Let (E,d) be a pseudometric space. For any $C \subseteq E$, let $B(C,\epsilon) = \{x \in E : d(x,C) \le \epsilon\}$, here $\epsilon > 0$ The measure of noncompactness of the set $M \subseteq E$ is defined by $\alpha(M) = \inf Q(M)$, where

$$Q(M) = \{\epsilon > 0 : M \subseteq B(A,\epsilon) \text{ for some finite subset } A \text{ of } E\}.$$

Let C be a subset of a locally convex Hausdorff topological vector space E, and let $\mathcal{P}$ be a defining system of seminorms on E. Suppose $F : C \to 2^E$. Then F is called countably $\mathcal{P}$-concentrative mapping if $F(C)$ is bounded, and for $p \in \mathcal{P}$ and each countably bounded subset S of C, we have $\alpha_p(F(S)) \leq \alpha_p(S)$, and for $p \in \mathcal{P}$ for each countably bounded non-p-precompact subset S of C (i.e., S is not precompact in the pseudonormed space (E, p)) we have $\alpha_p(F(S)) < \alpha_p(S)$; here $\alpha_p(.)$ denotes the measure of noncompactness in the pseudonormed space (E, p).

Let Q be a subset of a Hausdorff topological space X. We let $\overline{Q}$ (respectively, ∂Q, $int(Q)$) denote the closure (respectively, boundary, interior) of Q.

Definition 2.4. Let Z and W be subsets of Hausdorff topological vector spaces E_1 and E_2 and F a set-valued map. We say that $F \in PK(Z, W)$ if W is convex, and there exists a map $S : Z \to W$ with

$$Z = \cup\{intS^{-1}(w) : w \in W\}\,,\; co(S(x)) \subset F(x)\,\text{for}\, x \in Z\,,$$

and $S(x) \neq \emptyset$ for each $x \in Z$; here $S^{-1}(w) = \{z : w \in S(z)\}$.

Remark 2.5. Suppose Z is paracompact, W is convex, and $F \in PK(Z, W)$. Then there exists a continuous (single valued) mapping $f : Z \to W$ such that $f(x) \in F(x)$ for each $x \in Z$ (see [9]).

A nonempty subset X of a Hausdorff topological vector space E is said to be admissible if for every compact subset K of X and every neighborhood V of 0, there exists a continuous map $h : K \to X$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E. X is said to be q-admissible if any nonempty compact, convex subset Ω of X is admissible.

The following results [2, 4] will be needed in the sequel.

Theorem 2.1. *Let Ω be an admissible convex subset of a Hausdorff topological vector space E and X a nonempty subset of Ω Suppose $s : X \to \Omega$ is surjective and $F \in s - KKM(X, \Omega, \Omega)$ is compact and closed. Then F has a fixed point in Ω.*

Theorem 2.2. *Let Ω be a q-admissible closed convex subset of a Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose $s : \Omega \to \Omega$ is surjective and $F \in s - KKM(\Omega, \Omega, \Omega)$ is closed with the following property holding:*

$$A \subseteq \Omega,\;\; A = \overline{co}\,(\{x_0\} \cup F(A)) \;\; \textit{implies} \;\; A \;\; \textit{is compact.} \tag{2.1}$$

Then F has a fixed point in Ω.

Theorem 2.3. *Let Ω be a closed convex bounded subset of a Fréchet vector space E ($\mathcal{P}$ is a defining family of seminorms) and $x_0 \in \Omega$. Suppose $s : \Omega \to \Omega$ is surjective and $F \in s - KKM(\Omega, \Omega, \Omega)$ is closed countably P-concentrative map. Then F has a fixed point in Ω.*

The following fixed point result is a particular case of a result established in [9].

Theorem 2.4. *Let Ω be a nonempty convex subset of a Hausdorff locally convex topological vector space and $F \in \mathfrak{A}_c^{\kappa}(\Omega, \Omega)$ a compact map. Then F has a fixed point.*

3 Birkhoff-Kellogg Type Results

We obtain a variety of the Birkhoff-Kellogg type results on invariant directions. Let E be a Hausdorff locally convex topological vector space, C a closed convex subset of E, and $U \subseteq C$ a convex, open subset of E with $0 \in U$. Since U is open in C, we have $int_C U = U$. Let $s : \overline{U} \to \overline{U}$ be surjective. We consider maps $F : \overline{U} \to K(C)$ which satisfies $F \in s - KKM(\overline{U}, \overline{U}, C)$; here $\overline{U}$ denotes the closure of U in C and $K(C)$ represents the family of nonempty closed subsets of C.

Throughout we will assume the map $F : \overline{U} \to K(C)$ satisfies one of the following conditions:
(H1). F is compact;
(H2). If $D \subseteq \overline{U}$ and $D \subseteq \overline{co}(\{0\} \cup F(co(\{0\} \cup D) \cap \overline{U}))$, then $\overline{D}$ is compact;
or
(H3). F is countably $\mathcal{P}$-concentrative and E is Fréchet (here $\mathcal{P}$ is a defining system of seminorms).

Fix $i \in \{1, 2, 3\}$. We say $F \in s - KKM^i(\overline{U}, \overline{U}, C)$ if $F \in s - KKM(\overline{U}, \overline{U}, C)$ satisfies (Hi).

Theorem 3.1. *Fix $i \in \{1, 2, 3\}$ and let E be a Hausdorff locally convex topological vector space, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E, and $0 \in U$. Suppose C is a normal space, $s : \overline{U} \to \overline{U}$ is surjective and $F \in s - KKM^i(\overline{U}, \overline{U}, C)$ is closed. Then either*
(i). F has a fixed point in $\overline{U}$;
or
(ii). there exists $x \in \partial U$ and $\lambda \in (0, 1)$ with $x \in \lambda F x$;
here ∂U denotes the boundary of U in C.

PROOF: Let μ be the Minkowski functional on $\overline{U}$ and let $r : E \to \overline{U}$ be defined by

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \text{ for } x \in E.$$

Let $G = Fr$. Then $G \in 1_C - KKM(C, C, C)$ by Remark 2.3. Furthermore G is closed. Next we show that G has a fixed point in C for $i \in \{1, 2, 3\}$.
Let $i = 1$. Since $F \in s - KKM(\overline{U}, \overline{U}, C)$ is compact and r is continuous, it follows that G is compact. Now Theorem 2.1 guarantees that there exists $y \in C$ such that $y \in G(y)$.
Let $i = 2$. Let $D \subseteq C$ and $D = \overline{co}(\{0\} \cup G(D))$. Then since $r(A) \subseteq co(\{0\} \cup A)$ for any subset A of E, we have

$$D \subseteq \overline{co}(\{0\} \cup F(co(\{0\} \cup D) \cap \overline{U})).$$

Since $F \in s - KKM^2(\overline{U}, \overline{U}, C)$, if follows that $\overline{D}$ is compact. Now Theorem 2.2 guarantees that there exists $y \in C$ such that $y \in G(y)$.
Now let $i = 3$. We show that G is countably $\mathcal{P}$-concentrative. To see this, let $p \in \mathcal{P}$ and Ω a countably bounded non-p-precompact subset of C. Then since

$$G(\Omega) \subseteq F(r(\Omega)) \subseteq F(co(\{0\} \cup \Omega) \cap \overline{U}),$$

we have

$$\alpha_p(G(\Omega)) \quad < \quad \alpha_p(\Omega).$$

Thus G is countably $\mathcal{P}$-concentrative. Now Theorem 2.3 guarantees that there exists $y \in C$ such that $y \in G(y)$.

Thus, in each case, we can find $y \in C$ with $y \in G(y) = Fr(y)$. Let $x = r(y)$. Then $x \in rF(x)$, i.e., $x = r(w)$ for some $w \in F(x)$. Now either $w \in \overline{U}$ or $w \notin \overline{U}$. If $w \in \overline{U} = U \cup \partial U$ (notice that $int_C U = U$ since U is open in E), then $r(w) = w$ and so $x = w \in F(x)$. If $w \notin \overline{U}$, then $x = r(w) = \frac{w}{\mu(w)}$ with $\mu(w) > 1$. Thus $x = \lambda w$ (i.e., $w \in \lambda F(w)$) with $0 < \lambda = \frac{1}{\mu(w)} < 1$. Notice that $x \in \partial U$ since $\mu(x) = \mu(\lambda w) = 1$ (note that $\partial U = \partial_E U$ since $int_C U = U$). Consequently, $x \in \lambda F(x)$ with $\lambda = \frac{1}{\mu(w)} \in (0,1)$ and $x \in \partial U$. □

Next we assume

$$\text{(3.1)} \qquad \begin{cases} \text{for any map } F \in s - KKM(\overline{U}, \overline{U}, C) \text{ and any} \\ \lambda \in \mathbf{R}, \text{ we have that } \lambda F \in s - KKM(\overline{U}, \overline{U}, C). \end{cases}$$

As an application of Theorem 3.1, we derive some Birkhoff-Kellogg type theorems.

Theorem 3.2. *Let E be a Hausdorff locally convex topological vector space, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E, and $0 \in U$. Suppose C is a normal space, $s : \overline{U} \to \overline{U}$ is surjective and $F \in s - KKM^1(\overline{U}, \overline{U}, C)$ is closed and assume* (3.1) *holds. In addition suppose the following condition holds*

$$\text{(3.2)} \qquad \textit{there exists } \mu \in \mathbf{R} \textit{ with } \mu F(\overline{U}) \cap \overline{U} = \emptyset.$$

Then there exists $\lambda \in (0,1)$ and $x \in \partial U$ with $(\lambda^{-1}\mu^{-1})x \in Fx$ (i.e. $F|_{\partial U}$ has an eigenvalue); here $\mu \neq 0$ is chosen as in (3.2).

PROOF: Choose $\mu \neq 0$ as in (3.2). By (3.1), we have $\mu F \in s - KKM(\overline{U}, \overline{U}, C)$. Also we have μF is closed and compact since $F \in s - KKM(\overline{U}, \overline{U}, C)$ is closed and compact. Now (3.2) guarantees that μF has no fixed points in $\overline{U}$. An application of Theorem 3.1 yields that there exists $\lambda \in (0,1)$ and $x \in \partial U$ with $x \in \lambda(\mu F)x$. Consequently, $(\lambda^{-1}\mu^{-1})x \in Fx$. This completes the proof. □

Theorem 3.3. *Fix $i \in \{2,3\}$ and let E be a Hausdorff locally convex topological vector space, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E, and $0 \in U$. Suppose C is a normal space, $s : \overline{U} \to \overline{U}$ is surjective and $F \in s - KKM^i(\overline{U}, \overline{U}, C)$ is closed. In addition suppose the following conditions are satisfied:*

$$\text{(3.3)} \qquad \begin{cases} \textit{for any map } F \in s - KKM(\overline{U}, \overline{U}, C) \textit{ and any } \lambda \in \mathbf{R} \\ \textit{with } |\lambda| \leq 1 \textit{ we have that } \lambda F \in s - KKM(\overline{U}, \overline{U}, C) \end{cases}$$

$$\text{(3.4)} \qquad \textit{there exists } \mu \in \mathbf{R} \textit{ with } |\mu| \leq 1 \textit{ and } \mu F(\overline{U}) \cap \overline{U} = \emptyset$$

and

$$\text{(3.5)} \qquad \begin{cases} \textit{if } i = 2, \textit{ assume either } \mu > 0 \textit{ in } (3.4) \\ \textit{or } -F(D) = F(D) \textit{ for any } D \subseteq \overline{U}. \end{cases}$$

Then there exists $\lambda \in (0,1)$ *and* $x \in \partial U$ *with* $(\lambda^{-1}\mu^{-1})x \in Fx$.

PROOF: Choose $\mu \neq 0$ as in (3.4) and notice that $\mu F \in s - KKM(\overline{U}, \overline{U}, C)$ from (3.3). We claim

$$\mu F \in s - KKM^i(\overline{U}, \overline{U}, C). \tag{3.6}$$

Let $i = 2$ and let $D \subseteq \overline{U}$ with $D \subseteq \overline{co}(\{0\} \cup \mu F(D))$. From (3.5), we have $\mu F(D) \subseteq co(\{0\} \cup F(D))$. As a result, we have

$$D \subseteq \overline{co}(\{0\} \cup co(\{0\} \cup F(D))) = \overline{co}(co(\{0\} \cup F(D))) = \overline{co}(\{0\} \cup F(D)).$$

Since $F \in s - KKM^2(\overline{U}, \overline{U}, C)$, $\overline{D}$ is compact. So (3.6) holds if $i = 2$. Now let $i = 3$, then (3.6) holds since $|\mu| \leq 1$. Now Theorem 3.1 guarantees that there exists $\lambda \in (0,1)$ and $x \in \partial U$ with $x \in \lambda(\mu F)x$. Hence $(\lambda^{-1}\mu^{-1})x \in Fx$. □

Remark 3.1. In Theorem 3.3, (3.5) can be replaced by the more general condition

$$\begin{cases} \text{if } i = 2, \text{ and if } D \subseteq \overline{U} \text{ with } D \subseteq \overline{co}(\{0\} \cup \mu F(D)), \\ \text{then } \overline{D} \text{ is compact; here } \mu \text{ is chosen as in (3.4)} \end{cases} \tag{3.7}$$

(with this assumption we do not require to assume $|\mu| \leq 1$ in (3.4) if $i = 2$). For example, if F is P-concentrative (here E is Fréchet), then clearly (3.7) is satisfied (if $|\mu| \leq 1$).

Theorem 3.4. *Let* E *be a normal locally convex topological vector space,* C *a closed convex subset of* E*,* $U \subseteq C$ *convex,* U *an open subset of* E*, and* $0 \in U$*. Suppose* $s : \overline{U} \to \overline{U}$ *is surjective and* $F \in s - KKM^1(\overline{U}, \overline{U}, C)$ *is closed. In addition suppose* (3.2) *holds Then there exists* $\lambda \in (0,1)$ *and* $x \in \partial U$ *with* $(\lambda^{-1}\mu^{-1})x \in Fx$.

PROOF: Choose $\mu \neq 0$ as in (3.2). Define $f(x) = \mu x$ for $x \in C$. Then $f \in \mathcal{C}(C, E)$. By Remark 2.4 we have $\mu F \in s - KKM(\overline{U}, \overline{U}, E)$. Furthermore μF is closed and compact. Now (3.2) guarantees that μF has no fixed points in $\overline{U}$. An application of Theorem 3.1 yields that there exists $\lambda \in (0,1)$ and $x \in \partial U$ with $x \in \lambda(\mu F)x$. As a result, we have $(\lambda^{-1}\mu^{-1})x \in Fx$. □

Essentially the same reasoning as above yields the following result.

Theorem 3.5. *Fix* $i \in \{2,3\}$ *and let* E *be a normal locally convex topological vector space,* C *a closed convex subset of* E*,* $U \subseteq C$ *convex,* U *an open subset of* E*, and* $0 \in U$*. Suppose* $s : \overline{U} \to \overline{U}$ *is surjective and* $F \in s - KKM^i(\overline{U}, \overline{U}, C)$ *is closed. In addition suppose* (3.4) *and* (3.5) *holds. Then there exists* $\lambda \in (0,1)$ *and* $x \in \partial U$ *with* $(\lambda^{-1}\mu^{-1})x \in Fx$.

In Theorem 3.2 (respectively Theorem 3.3) if $\mu > 0$ in (3.2) (respectively (3.4)), we say that $F|_{\partial U}$ has an invariant direction (i.e., has a positive eigenvalue). Some of the ideas here are borrowed from the literature (see [8] and the references therein).

Theorem 3.6. *Let* $E = (E, ||.||)$ *be an infinite dimensional normed linear space,* $C = E$*,* $U = B$*, Suppose* $s : \overline{B} \to \overline{B}$ *is surjective and* $F \in s - KKM^1(\overline{B}, \overline{B}, E)$ *is closed; here* $B = \{x \in E : ||x|| < 1\}$*. In addition suppose the following conditions are satisfied:*

$$0 \notin \overline{F(S)}; \tag{3.8}$$

here $S = \{x \in E : ||x|| = 1\}$. *Then* F *has an invariant direction.*

PROOF: It is known [3] that there exists a continuous retraction $r : \overline{B} \to S$. Let $G = Fr$. Then, as before, $G \in \mathbf{1}_{\overline{B}} - KKM(\overline{B}, \overline{B}, E)$. We claim that we can find a number $\mu > 0$ such that

$$\mu F(S) \cap \overline{B} = \emptyset. \tag{3.9}$$

If this is not true, then for each $n \in \{1, 2, 3, ...\}$, there exists $y_n \in F(S)$ and $w_n \in \overline{B}$ with $y_n = \frac{1}{n} w_n$. This implies that $0 \in \overline{F(S)}$. This contradicts (3.8).
Using (3.9), we have

$$\mu G(\overline{B}) \cap \overline{B} = \emptyset.$$

Now Theorem 3.4 (applied to G with $U = B$ and $C = E$) guarantees that there exists $\lambda \in (0, 1)$ and $x \in \partial B = S$ with $\lambda^{-1}\mu^{-1}x \in Gx = Frx = Fx$. Hence F has an invariant direction.

Remark 3.2. Let E be a normal locally convex topological vector space and U any open set with $0 \in U$. Theorem 3.6 remains valid if we replace B by U provided ∂U is a retract of $\overline{U}$. However, in this case, (3.8) is replaced by

$$\text{there exists } \mu > 0 \text{ with } \mu F(\partial U) \cap \overline{U} = \emptyset. \tag{3.10}$$

Remark 3.3. In Theorem 3.6, $F \in s - KKM^1(\overline{B}, \overline{B}, E)$ could be replaced by $F \in s - KKM^1(S, S, E)$.

It was shown [3] that if E is an infinite dimensional normed linear space, then there exists a Lipschitzian retraction $r : \overline{B} \to \overline{S}$ with Lipschitz constant $k_0(E)$, say; here B and S are as in Theorem 3.4. In fact, there exists a k_0 with $k_0(E) \leq k_0$ for any space E (as described above).

Let $r : \overline{B} \to S$ be a Lipschitzian retraction with Lipschitz constant $k_0(E)$.

Theorem 3.7. *Let* $E = (E, ||.||)$ *be an infinite dimensional normed linear space,* $C = E$, $U = B$, *Suppose* $s : \overline{B} \to \overline{B}$ *is surjective and* $F \in s - KKM(\overline{B}, \overline{B}, E)$ *is closed; here* $B = \{x \in E : ||x|| < 1\}$. *In addition suppose the following two conditions are satisfied:*

$$\begin{cases} F \text{ is countably } k\text{-set-contractive with } 0 \leq k < \frac{1}{k_0(E)}; \\ \text{here } k_0(E) \text{ is a Lipschitz constant as described above} \end{cases} \tag{3.11}$$

and

$$\text{there exists } \mu > 0 \text{ with } 0 < \mu \leq 1 \text{ and } \mu F(S) \cap \overline{B} = \emptyset. \tag{3.12}$$

Then F *has an invariant direction.*

PROOF: Let $G = Fr$, where r is a Lipschitzian retraction with Lipschitz constant $k_0(E)$. Then as before $G \in \mathbf{1}_{\overline{B}} - KKM(\overline{B}, \overline{B}, E)$. Clearly G is countably $kk_0(E)$-set contractive. Thus $G \in s - KKM^3(\overline{B}, \overline{B}, E)$. Now Theorem 3.4 (applied to G with $U = B$ and $C = E$) guarantees that there exists $\lambda \in (0,1)$ and $x \in \partial B = S$ with $\lambda^{-1}\mu^{-1}x \in Gx = Frx = Fx$. Hence F has an invariant direction.

Remark 3.4. In Theorem 3.7, $F \in s - KKM(\overline{B}, \overline{B}, E)$ could be replaced by $F \in s - KKM(S, S, E)$.

4 Best Proximity Pair Results

Let A and B be nonempty subsets of a normed space $E = (E, ||.||)$. Then A is called approximately compact if for each y in E and each $\{x_n\}$ in A with $||x_n - y|| \to d(y, A)$, there exists a subsequence of $\{x_n\}$ converging to an element of A. The set

$$P_A(x) = \{a \in A : ||a - x|| = d(x, A)\}$$

is the set of all best approximations in A to any element $x \in E$. It is known [12] that if A is an approximately compact convex subset of E, then $P_A(x)$ is a nonempty compact convex subset of A and the multivalued mapping $P_A : E \to 2^A$ is upper semicontinuous on E.

A mapping f from a topological space X to another topological space Y is called proper if $f^{-1}(K)$ is compact in X whenever K is compact in Y.

If B is convex, a mapping $f : B \to E$ is said to be quasi-affine if for every real number $r \geq 0$ and $x \in E$, the set $\{b \in B : ||f(b) - x|| \leq r\}$ is convex.

We recall the following notations (see [11, 13]).

$$d(A, B) = \inf\{||a - b|| : a \in A, b \in B\}$$

$$Prox(A, B) = \{(a, b) \in A \times B : ||a - b|| = d(A, B)\}$$

$$A_0 = \{a \in A : ||a - b|| = d(A, B) \text{ for some } b \in B\}$$

$$B_0 = \{b \in B : ||a - b|| = d(A, B) \text{ for some } a \in A\}.$$

Various sufficient conditions for the non-emptiness of the set $Prox(A, B)$ were explored by many authors, see, for example, [12, 14].

Remark 4.1. Note $P_A(B_0) \subset A_0$. Indeed, let $y \in P_A(B_0)$. Then $y \in P_A(b)$ for some $b \in B_0$. This implies that $||y-b|| = d(b, A)$. Since $b \in B_0$, we have $||a-b|| = d(A, B)$ for some $a \in A$ and so $||y - b|| = d(b, A) \leq ||a - b|| = d(A, B)$. On the other hand, $d(A, B) \leq ||y - b||$ for all $y \in A$ and $b \in B$. Consequently, $||y - b|| = d(A, B)$ and so $y \in A_0$.

Theorem 4.1. *Let $E = (E, ||.||)$ be a normed space. Let A be a nonempty, approximately compact convex subset of E and B a nonempty closed convex subset of E such that $Prox(A, B)$ is nonempty and A_0 is compact. Let C be a nonempty subset of E containing a nonempty convex set C_0. Assume that*
(a) $F \in PK(A, C)$ such that $F(A_0) \subset C_0$
(b) $G \in \mathfrak{A}_c^{\kappa}(C, B)$ such that $G(C_0) \subset B_0$
(c)$f : A \to A$ is a continuous, proper, quasi-affine, surjective single-valued map

such that $f^{-1}(A_0) \subset A_0$.
Then there exists $x_0 \in A_0$ *such that*

$$d(fx_0, GFx_0) = d(A, B).$$

PROOF: By Remark 2.5, there exists a continuous function $g : A \to C$ such that $g(x) \in F(x)$ for each $x \in A$. Let $T = HGg$, where $H = f^{-1}P_A : B_0 \to 2^{A_0}$. Then $T(A_0) \subset A_0$ since $G(C_0) \subset B_0$, $P_A(B_0) \subset A_0$ and $f^{-1}(A_0) \subset A_0$. Also $T : A_0 \to 2^{A_0}$ is a compact multifunction because A_0 is compact.

Now we show that H is a convex, compact-valued upper semicontinuous function. Let D be a closed subset of A_0 and $\{x_n\} \subset H^{-1}(D)$ such that $x_n \to x$ as $n \to \infty$. Then for each n, choose $y_n \in H(x_n) \cap D$ such that $||f(y_n) - x_n|| = d(x_n, A)$. Since D is compact, we may assume that $y_n \to y$ for some $y \in D$. Using the triangle inequality, we have

$$\begin{aligned} ||f(y) - x_n|| &\leq ||f(y) - f(y_n)|| + ||f(y_n) - x_n|| + |x_n - x|| \\ &= ||f(y) - f(y_n)|| + d(x_n, A) + ||x_n - x||, \end{aligned}$$

which on letting $n \to \infty$ yields

$$||f(y) - x|| \leq d(x, A)$$

since f is continuous and $d(x_n, A) \to d(x, A)$. On the other hand, $d(x, A) \leq ||f(y) - x||$. Thus

$$||f(y) - x|| = d(x, A)$$

and so $f(y) \in P_A(x)$. As a result $y \in H(x) \cap D$. Hence $x \in H^{-1}(D)$ and so H is upper semi-continuous.

Let $x_1, x_2 \in H(x)$ and $\lambda \in [0, 1]$. Then

$$||f(x_1) - x|| = d(x, A) = ||f(x_2) - x||.$$

Since f is quasi-affine, the set $\{a \in A : ||f(a) - x|| \leq d(x, A)\}$ is convex. Set $y = \lambda x_1 + (1 - \lambda)x_2$. Then

$$||f(y) - x|| = d(x, A)$$

and so $H(x)$ is convex.

Since f is a proper map and $P_A(x)$ is compact, $H(x)$ is compact. Hence $H \in \mathfrak{A}_c^\kappa(B_0, A_0)$. Since $\mathfrak{A}_c^\kappa$ is closed under compositions, $T \in \mathfrak{A}_c^\kappa(A_0, A_0)$. Next we show that A_0 is convex. Let $a_1, a_2 \in A_0$ and $\lambda \in [0, 1]$. Then $||a_1 - b_1|| = d(A, B)$ and $||a_2 - b_2|| = d(A, B)$ for some $b_1, b_2 \in B$. Since A and B are convex, $\lambda a_1 + (1 - \lambda)a_2 \in A$ and $\lambda b_1 + (1 - \lambda)b_2 \in B$. Now $||\lambda a_1 + (1 - \lambda)a_2 - [\lambda b_1 + (1 - \lambda)b_2]|| \leq \lambda||a_1 - b_1|| + (1 - \lambda)||a_2 - b_2|| = d(A, B)$. Consequently,

$$||\lambda a_1 + (1 - \lambda)a_2 - [\lambda b_1 + (1 - \lambda)b_2]|| = d(A, B)$$

and so $\lambda a_1 + (1 - \lambda)a_2 \in A_0$. Thus A_0 is convex. Now Theorem 2.4 guarantees that there exists $x_0 \in A_0$ such that $x_0 \in T(x_0)$. Set $y_0 = g(x_0)$. Then $y_0 \in F(x_0)$ and $x_0 \in H(z_0)$ for some $z_0 \in G(y_0)$. This implies that $f(x_0) \in P_A(z_0)$ and so

$$d(f(x_0), GF(x_0)) \leq ||f(x_0) - z_0|| = d(z_0, A).$$

Since $y_0 \in C_0$ and $G(C_0) \subset B_0$, we have $z_0 \in G(y_0) \subset G(C_0) \subset B_0$ and so there exists $a \in A$ with $||a - z_0|| = d(A, B)$. Therefore

$$d(f(x_0), GF(x_0)) \leq d(z_0, A) \leq ||a - z_0|| = d(A, B).$$

On the other hand, $d(A, B) \leq d(f(x_0), GF(x_0))$. Hence

$$d(f(x_0), GF(x_0)) = d(A, B).$$

Corollary 4.2. *Let $E = (E, ||.||)$ be a normed space. Let A be a nonempty, approximately compact convex subset of E and B a nonempty closed convex subset of E such that $Prox(A, B)$ is nonempty and A_0 is compact. Assume that*
(a) $F \in PK(A, B)$ such that $F(A_0) \subset B_0$
(b)$f : A \to A$ is a continuous, proper, quasi-affine, surjective single-valued map such that $f^{-1}(A_0) \subset A_0$.
Then there exists $x_0 \in A_0$ such that

$$d(fx_0, Fx_0) = d(A, B).$$

Corollary 4.3. *Let $E = (E, ||.||)$ be a normed space. Let A be a nonempty, compact convex subset of E and C a nonempty convex subset of E. Assume that*
(a) $F \in PK(A, C)$
(b) $G \in \mathfrak{A}_c^\kappa(C, A)$.
If the multifunction GF is closed-valued, then it has a fixed point and hence G and F have coincidence (i.e., there exist $x_0 \in A$ and $y_0 \in C$ such that $y_0 \in F(x_0)$ and $x_0 \in G(y_0)$).

PROOF: Theorem 4.1 (note $B = A$) guarantees that there exists $x_0 \in A_0 = A$ with $d(x_0, GFx_0) = 0$. Now since GF has closed values we get the result.

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