Closedness of bounded convex sets of asymmetric normed linear spaces and the Hausdorff quasi-metric

Jesús Rodríguez-López Salvador Romaguera*

Abstract

If A is a (nonempty) bounded convex subset of an asymmetric normed linear space (X, q), we define the closedness of A as the set $\mathrm{cl}_q A \cap \mathrm{cl}_{q^{-1}} A$, and denote by $CB_0(X)$ the collection of the closednesses of all (nonempty) bounded convex subsets of (X, q). We show that $CB_0(X)$, endowed with the Hausdorff quasi-metric of q, can be structured as a quasi-metric cone. Then, and extending a classical embedding theorem of L. Hörmander, we prove that there is an isometric isomorphism from this quasi-metric cone into the product of two asymmetric normed linear spaces of bounded continuous real functions equipped with the asymmetric norm of uniform convergence.

1 Introduction and preliminaries

In the last decade several authors have successfully applied both asymmetric normed linear spaces and other related structures from topological algebra and nonsymmetric functional analysis as quasi-metric cones, algebraic $[0,\infty]$-modules and quasi-normed semilinear spaces to construct suitable mathematical models in theoretical

*The authors acknowledge the support of the Plan Nacional I+D+I and FEDER, under grant BFM2003-02302

Received by the editors February 2005.

Communicated by E. Colebunders.

2000 *Mathematics Subject Classification :* 54B20, 54C25, 54C35, 54E35, 46B10, 46B99.

Key words and phrases : The Hausdorff quasi-metric, asymmetric normed linear space, quasi-metric cone, bounded convex subset, closedness, isometric isomorphism.

computer science ([8], [17], [20], [21], [23], etc.) as well as in discussing some questions in approximation theory ([2], [5], [18], [19], [24]). Simultaneously, the interest in the study of hyperspaces and function spaces in quasi-uniform and quasi-metric spaces has increased considerably motivated in part for such applications (see [13], [15], [16], [22], etc). In this setting it then appears the unexplored but interesting problem of embedding (hyper)spaces of convex subsets of a given asymmetric normed linear space into appropriate function spaces endowed by the asymmetric norm of uniform convergence. In this paper we present a solution to that problem. Our main result extends to the asymmetric framework the famous embedding theorem of L. Hörmander [10] (see also Theorem 3.2.9 of [3]) which essentially establishes the existence of an algebraic and isometric embedding of the metric cone of the bounded convex and closed subsets of a normed linear space X, endowed with the Hausdorff metric, into the Banach space of bounded continuous real functions on the closed unit ball of the dual space of X equipped with the norm of uniform convergence. Here we prove that if X is an asymmetric normed linear space, the set of the closednesses of the bounded convex subsets of X endowed with the Hausdorff quasi-metric can be structured as a quasi-metric cone, and we construct an algebraic and isometric embedding from this quasi-metric cone into the product of two asymmetric normed linear spaces of bounded continuous real functions equipped with the asymmetric norm of uniform convergence.

In the following the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of real numbers, the set of nonnegative real numbers and the set of positive integer numbers, respectively.

According to the modern terminology by a quasi-metric on a (nonempty) set X we mean a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$, and (ii) $d(x,y) \leq d(x,z) + d(z,y)$.

If d can take the value ∞ then it is called a quasi-distance on X. Given a quasi-distance d on X, the function d^{-1}, defined on $X \times X$ by $d^{-1}(x,y) = d(y,x)$, is also a quasi-distance on X, called the conjugate of d, and the function d^s, defined on $X \times X$ by $d^s(x,y) = d(x,y) \vee d^{-1}(x,y)$, is a distance on X. If d is a quasi-metric, then d^{-1} and d^s are a quasi-metric and a metric on X, respectively.

A quasi-metric space is a pair (X, d) such that X is a (nonempty) set X and d is a quasi-metric on X.

Each quasi-distance d on X induces a T_0 topology τ_d on X which has as a base the family of d-balls $\{B_d(x,r) : x \in X, r > 0\}$ where $B_d(x,r) = \{y \in X \ : \ d(x,y) < r\}$.

If A is a subset of the quasi-metric space (X, d), the closure of A with respect to τ_d will be denoted by $\mathrm{cl}_d A$.

The reader might consult [7] and [13], for more information about quasi-metric spaces.

Let X be a linear space. We say that a function $q : X \to \mathbb{R}^+$ is an asymmetric norm on X ([8], [9]) if for all $x, y \in X$ and $r \in \mathbb{R}^+$: (i) $q(x) = q(-x) = 0$ if and only if $x = 0$; (ii) $q(rx) = rq(x)$, and (iii) $q(x+y) \leq q(x) + q(y)$.

Asymmetric norms are called quasi-norms in [1], [6], [18], etc.

An asymmetric normed linear space is a pair (X, q) such that X is a linear space and q is an asymmetric norm on X.

Given an asymmetric norm q on a linear space X, the function q^{-1} defined on X by $q^{-1}(x) = q(-x)$, for all $x \in X$, is also an asymmetric norm on X, called the conjugate of q, and the function q^s defined on X by $q^s(x) = \max\{q(x), q^{-1}(x)\}$ for all $x \in X$, is a norm on X. We say that (X, q) is a biBanach space if (X, q^s) is a Banach space ([9]).

The following is a simple but crucial instance of a biBanach space.

Example 1. Denote by u the function defined on $\mathbb{R}$ by $u(x) = x \vee 0$ for all $x \in \mathbb{R}$. Then u is an asymmetric norm on $\mathbb{R}$ such that u^s is the Euclidean norm on $\mathbb{R}$, i.e. $(\mathbb{R}, u^s)$ is the Euclidean normed space $(\mathbb{R}, |.|)$. Hence $(\mathbb{R}, u)$ is an asymmetric normed linear space (see, for instance, [6]).

It is well known that each asymmetric norm q on a linear space X induces a quasi-metric d_q on X given by $d_q(x, y) = q(y - x)$ for all $x, y \in X$. The d_q-ball $B_{d_q}(x, r)$, will be simply denoted by $B_q(x, r)$ and the set $\overline{B}_q(x, r) := \{y \in X : q(y - x) \leq r\}$ is said to be the closed ball of center x and radius r. Observe that $\overline{B}_q(x, r)$ is a $\tau_{d_{q^{-1}}}$-closed set. A subset A of (X, q) is called bounded if it is bounded in the normed linear space (X, q^s). If A is a subset of (X, q), the closure of A with respect to τ_{d_q} will be simply denoted by $\mathrm{cl}_q A$.

If A is a (nonempty) bounded convex subset of an asymmetric normed linear space (X, q), we define the closedness of A as the set $\mathrm{cl}_q A \cap \mathrm{cl}_{q^{-1}} A$, and denote by $CB_0(X)$ the collection of the closednesses of all (nonempty) bounded convex subsets of (X, q).

Following [11], a cone (a semilinear space in [18]) is a triple $(X, +, \cdot)$ such that $(X, +)$ is a commutative semigroup with neutral element $\mathbf{0}$ and $\cdot$ is a function from $\mathbb{R}^+ \times X$ into X which satisfies for all $r, s \in \mathbb{R}^+$ and $x, y \in X$: (i) $r \cdot (s \cdot x) = (rs) \cdot x$, (ii) $(r + s) \cdot x = r \cdot x + s \cdot x$, (iii) $r \cdot (x + y) = r \cdot x + r \cdot y$, (iv) $1 \cdot x = x$, and (v) $0 \cdot x = \mathbf{0}$ (see [12] for related structures).

By a quasi-metric cone we mean a quadruple $(X, +, \cdot, d)$ such that $(X, +, \cdot)$ is a cone and d is a quasi-metric on X such that $d(x+z, y+z) \leq d(x, y)$ and $d(rx, ry) \leq rd(x, y)$ for all $x, y, z \in X$ and $r \geq 0$.

2 On the structure of $CB_0(X)$ equipped with the Hausdorff quasi-metric

In the sequel we denote by $\mathcal{P}_0(X)$ the collection of all nonempty subsets of a given (nonempty) set X.

Let (X, d) be a quasi-metric space. Define

$$C_\cap(X) = \{\mathrm{cl}_d A \cap \mathrm{cl}_{d^{-1}} A : A \in \mathcal{P}_0(X)\}.$$

It is straightforward to show that if $A \in \mathcal{P}_0(X)$, then $A \in C_\cap(X)$ if and only if $A = \mathrm{cl}_d A \cap \mathrm{cl}_{d^{-1}} A$.

On the other hand, if (X, q) is an asymmetric normed linear space, we can easily describe the set $CB_0(X)$ in terms of $C_\cap(X)$.

Lemma 1. *Let (X, q) be an asymmetric normed linear space. Then*

$$CB_0(X) = \{A \in C_\cap(X) : A \textit{ is bounded and convex}\}.$$

Proof. Let $A \in C_\cap(X)$ such that A is bounded and convex. Since $A = \mathrm{cl}_q A \cap \mathrm{cl}_{q^{-1}} A$, we deduce that $A \in CB_0(X)$. Conversely, if $A \in CB_0(X)$, there is a bounded and convex nonempty subset B of X such that $A = \mathrm{cl}_q B \cap \mathrm{cl}_{q^{-1}} B$. Thus $A \in C_\cap(X)$. Moreover, boundedness of B clearly implies boundedness of A. Finally, given $a, a' \in A$ and $r \in [0,1]$, we deduce that $ra + (1-r)a' \in \mathrm{cl}_q B$ by convexity of B and the fact that $a, a' \in \mathrm{cl}_q B$. Similarly, we obtain that $ra + (1-r)a' \in \mathrm{cl}_{q^{-1}} B$. Therefore A is convex. ■

Example 2. Let (X, q) be the asymmetric normed linear space of Example 1. By using Lemma 1 it is easy to check that $CB_0(X)$ consists of all compact intervals of $(\mathbb{R}, |.|)$.

Note that for the space (X, q) of the above example, the set $CB_0(X)$ coincides with the set of bounded convex and closed (nonempty) subsets of (X, q^s). The next example shows that this is not the case, in general.

Example 3. Consider the classical Banach space $(\ell_1, \|.\|_1)$ of sequences of real numbers $\mathbf{x} := (x_n)_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} |x_n|$ converges. It is well known that the closed unit ball U is a bounded convex and closed subset of $(\ell_1, \|.\|_1)$. We split the norm $\|.\|_1$ as follows (see [6], [8]). For each $\mathbf{x} := (x_n)_{n\in\mathbb{N}} \in \ell_1$ let

$$q(\mathbf{x}) = \sum_{n=1}^{\infty} (x_n \vee 0).$$

Then q is an asymmetric norm on ℓ_1 such that $q^s(\mathbf{x}) \le \|\mathbf{x}\|_1 \le q(\mathbf{x}) + q^{-1}(\mathbf{x})$ for all $\mathbf{x} \in \ell_1$ ([8]). We show that $U \subsetneq \mathrm{cl}_q U \cap \mathrm{cl}_{q^{-1}} U$. Thus $U \notin CB_0(\ell_1)$ by Lemma 1. Indeed, choose $\mathbf{x} := (x_n)_n \in \ell_1$ such that $x_{2n-1} > 0$, $x_{2n} < 0$, $\|\mathbf{x}\|_1 > 1$, $q(\mathbf{x}) \le 1$ and $q^{-1}(\mathbf{x}) \le 1$. Then $q(\mathbf{y} - \mathbf{x}) = q(\mathbf{x} - \mathbf{z}) = 0$, where $\mathbf{y} := (y_n)_{n\in\mathbb{N}}$ and $\mathbf{z} := (z_n)_{n\in\mathbb{N}}$ satisfy $y_{2n-1} = 0$, $y_{2n} = x_{2n}$, $z_{2n-1} = x_{2n-1}$, and $z_{2n} = 0$ for all $n \in \mathbb{N}$. Since $\mathbf{y} \in U$ and $\mathbf{z} \in U$ it follows that $\mathbf{x} \in \mathrm{cl}_q U \cap \mathrm{cl}_{q^{-1}} U$. However $\mathbf{x} \notin U$.

Several parts of the proof of the next result follow exactly as in the case of normed linear spaces, hence some details will be omitted.

Proposition 1. *Let (X, q) be an asymmetric normed linear space. For each pair $A, B \in CB_0(X)$ and each $r \ge 0$ let*

$$A \oplus B = \mathrm{cl}_q(A + B) \cap \mathrm{cl}_{q^{-1}}(A + B) \quad \text{and} \quad r \cdot A = \{ra : a \in A\}.$$

Then $(CB_0(X), \oplus, \cdot)$ is a cone.

Proof. Let $A, B, C \in CB_0(X)$ and $r, s \ge 0$. Then $A, B \in C_\cap(X)$ and A and B are bounded and convex. It is immediate to show that $A \oplus B \in C_\cap(X)$, and that $A \oplus B$ is bounded and convex. So $A \oplus B \in CB_0(X)$ by Lemma 1. Clearly, we have that $A \oplus B = B \oplus A$. Furthermore

$$\begin{aligned} (A \oplus B) \oplus C &= \mathrm{cl}_q((A \oplus B) + C) \cap \mathrm{cl}_{q^{-1}}((A \oplus B) + C) \\ &= \mathrm{cl}_q((A + B) + C) \cap \mathrm{cl}_{q^{-1}}((A + B) + C) \\ &= \mathrm{cl}_q(A + (B + C)) \cap \mathrm{cl}_{q^{-1}}(A + (B + C)) \\ &= \mathrm{cl}_q(A + (B \oplus C)) \cap \mathrm{cl}_{q^{-1}}(A + (B \oplus C)) = A \oplus (B \oplus C). \end{aligned}$$

Note also that for $A \oplus \{\mathbf{0}\} = A$.

It is easily seen that $r \cdot A \in CB_0(X)$ and $r \cdot (s \cdot A) = (rs) \cdot A$. Moreover

$$\begin{aligned} r \cdot (A \oplus B) &= r \cdot (\mathrm{cl}_q(A+B) \cap \mathrm{cl}_{q^{-1}}(A+B)) \\ &= \mathrm{cl}_q(r \cdot (A+B)) \cap \mathrm{cl}_{q^{-1}}(r \cdot (A+B)) \\ &= \mathrm{cl}_q(r \cdot A + r \cdot B) \cap \mathrm{cl}_{q^{-1}}(r \cdot A + r \cdot B) = r \cdot A \oplus r \cdot B. \end{aligned}$$

In order to show that $(r+s) \cdot A = r \cdot A \oplus s \cdot A$, we first note that, by convexity of A, we have $(r+s) \cdot A = r \cdot A + s \cdot A$, so $(r+s) \cdot A \subseteq r \cdot A \oplus s \cdot A$. Now let $z \in r \cdot A \oplus s \cdot A$. Then there exist sequences $(a_n)_n, (a'_n)_n, (b_n)_n$ and $(b'_n)_n$ in A such that $q(ra_n + sa'_n - z) < 1/n$ and $q(z - rb_n - sb'_n) < 1/n$ for all $n \in \mathbb{N}$. Since $(ra_n + sa'_n)/(r+s) \in A$ (where we assume without loss of generality that $r+s > 0$) it follows that

$$q(\frac{ra_n + sa'_n}{r+s} - \frac{z}{r+s}) < \frac{r+s}{n},$$

for all $n \in \mathbb{N}$. Therefore $z/(r+s) \in \mathrm{cl}_q A$. Similarly, we deduce that $z/(r+s) \in \mathrm{cl}_{q^{-1}} A$. Hence $z/(r+s) \in \mathrm{cl}_q A \cap \mathrm{cl}_{q^{-1}} A = A$. We conclude that $(r+s) \cdot A = r \cdot A \oplus s \cdot A$.

Finally, it is obvious that $1 \cdot A = A$ and $0 \cdot A = \mathbf{0}$. Thus, we have proved that $(CB_0(X), \oplus, \cdot)$ is a cone. ∎

Remark 1. Note that the first part of the proof of Proposition 1 shows that if $(X,+)$ is a (commutative) semigroup, d is a quasi-metric on X and for each pair $A, B \in C_\cap(X)$, we define $A \oplus B = \mathrm{cl}_d(A+B) \cap \mathrm{cl}_{d^{-1}}(A+B)$, then $(C_\cap(X), \oplus)$ is a (commutative) semigroup.

Given a quasi-metric space (X,d), the construction of the Hausdorff quasi-distance on the set $C_\cap(X)$ may be found in [14] (see also [4], [15], [16], etc). We adapt this construction to our context as follows.

Let (X,q) be an asymmetric normed linear space. For each $A, B \in \mathcal{P}_0(X)$ define

$$H_q^+(A,B) = \sup_{b \in B} d_q(A,b), \quad H_q^-(A,B) = \sup_{a \in A} d_q(a,B),$$

and

$$H_q(A,B) = \max\{H_q^+(A,B), H_q^-(A,B)\}.$$

Then H_q is a quasi-distance on the set $C_\cap(X)$ (compare Lemma 2 of [14]) and it is a quasi-metric on the set of all bounded subsets of X that are in $C_\cap(X)$, hence in $CB_0(X)$. In this case we say that H_q is the Hausdorff quasi-metric of q on $CB_0(X)$.

Theorem 1. *Let (X,q) be an asymmetric normed linear space. Then $(CB_0(X), \oplus, \cdot, H_q)$ is a quasi-metric cone, where $\oplus$ and $\cdot$ are the operations defined in Proposition 1.*

Proof. Let $A, B, C \in CB_0(X)$. We shall prove that $H_q^+(A \oplus C, B \oplus C) \le H_q^+(A,B)$. Indeed, fix $\varepsilon > 0$. Choose any $z \in B \oplus C$. Then $z \in \mathrm{cl}_{q^{-1}}(B+C)$, so there exist $b \in B$ and $c \in C$ such that $d_q(b+c, z) < \varepsilon$. Now let $a \in A$ such that $d_q(a,b) < \varepsilon + d_q(A,b)$. Therefore

$$\begin{aligned} d_q(A \oplus C, z) &\le d_q(A+C, z) \le d_q(a+c, z) \le d_q(a+c, b+c) + d_q(b+c, z) \\ &< 2\varepsilon + d_q(A,b) \le 2\varepsilon + H_q^+(A,B). \end{aligned}$$

Hence $H_q^+(A \oplus C, B \oplus C) \leq H_q^+(A, B)$.

Similarly we prove that $H_q^-(A \oplus C, B \oplus C) \leq H_q^-(A, B)$, and thus

$$H_q(A \oplus C, B \oplus C) \leq H_q(A, B).$$

Finally, for $A, B \in CB_0(X)$ and $r \geq 0$ we immediately obtain

$$d_q(ra, r \cdot B) = rd_q(a, B) \quad \text{and} \quad d_q(r \cdot A, rb) = rd_q(A, b),$$

for all $a \in A$ and $b \in B$. This implies that

$$H_q(r \cdot A, r \cdot B) = rH_q(A, B).$$

We have proved that $(CB_0(X), \oplus, \cdot, H_q)$ is a quasi-metric cone. ■

3 Embedding the quasi-metric cone $(CB_0(X), \oplus, \cdot, H_q)$ into an asymmetric normed linear space

We start this section by giving some concepts and properties on the dual space of an asymmetric normed linear space which can be found in [9].

Given an asymmetric normed linear space (X, q) let

$$X^{s*} = \{f : (X, q^s) \to (\mathbb{R}, |.|) : f \text{ is linear and continuous}\},$$

and let

$$X^* = \{f : (X, q) \to (\mathbb{R}, u) : f \text{ is linear and continuous}\}.$$

It is well known that X^{s*} is a linear space. Note also that $f \in X^*$ if and only if it is a linear and upper semicontinuous real function on (X, q). Moreover, X^* is an algebraically closed subset of X^{s*}, and thus it is a cone.

Now, for each $f \in X^*$, put $q^*(f) = \sup\{f(x) : q(x) \leq 1\}$. Then q^* satisfies: (i') $q^*(f) = 0$ if and only if $f = \mathbf{0}$; and conditions (ii) and (iii) of the definition of an asymmetric normed linear space. Therefore (X^*, q^*) is a normed cone in the sense of [19] (a normed semilinear space in the sense of [17]), and it is called the dual space of (X, q).

Note that q^* induces a quasi-distance d_{q^*} on X^* given by $d_{q^*}(f, g) = q^*(g - f)$ if $g - f \in X^*$, and $d_{q^*}(f, g) = \infty$ otherwise.

Then, by continuity of a real function(al) on X^* we shall mean continuity with respect to the topology induced by d_{q^*} on X^*.

Lemma 2. *Let (X, q) be an asymmetric normed linear space and let $(X^{s*}, (q^s)^*)$ be the dual space of the normed linear space (X, q^s). Then $(q^s)^*(f) \leq q^*(f)$ for all $f \in X^*$.*

Proof. Let $f \in X^*$. Since for each $x \in X$, $|f(x)| = \max\{f(x), f(-x)\}$, and each $x \in X$ with $q^s(x) \leq 1$ satisfies $q(x) \leq 1$ and $q(-x) \leq 1$, we immediately deduce that

$$\sup\{|f(x)| : q^s(x) \leq 1\} \leq \sup\{f(x) : q(x) \leq 1\}.$$

Hence $(q^s)^*(f) \leq q^*(f)$. The proof is finished. ■

In the sequel and according to [9], we define $B_{X^*} := \{f \in X^* : q^*(f) \leq 1\}$, and $B_{X^{s*}}$ will denote the closed unit ball of $(X^{s*}, (q^s)^*)$, i.e. $B_{X^{s*}} = \{f \in X^{s*} : (q^s)^*(f) \leq 1\}$.

Similarly to the classical case, given an asymmetric normed linear space (X, q), for each (nonempty) bounded subset A of X, we define the support of A as the function $s(\cdot, A) : X^* \to \mathbb{R}$ given by

$$s(f, A) = \sup\{f(a) : a \in A\} \quad \text{for all } f \in X^*.$$

The following property of $s(\cdot, A)$ will be useful later on.

Proposition 2. *Let A be a (nonempty) bounded subset of an asymmetric normed linear space (X, q). Then $s(\cdot, A)$ is continuous from (X^*, q^*) into $(\mathbb{R}, |.|)$. Furthermore it is a bounded function on B_{X^*}.*

Proof. Let $d_{q^*}(f, f_n) \to 0$, where $f, f_n \in X^*$ for all $n \in \mathbb{N}$. Then $q^*(f_n - f) \to 0$, so, we can assume without loss of generality, that $f_n - f \in X^*$ for all $n \in \mathbb{N}$. On the other hand, since $f_n - f \in X^{s*}$ for all $n \in \mathbb{N}$, we deduce that

$$|(f_n - f)(x)| \leq (q^s)^*(f_n - f)q^s(x),$$

for all $x \in X$ and $n \in \mathbb{N}$. Now let $M > 0$ such that $q^s(a) \leq M$ for all $a \in A$. It then follows from Lemma 2 that

$$|(f_n - f)(a)| \leq Mq^*(f_n - f),$$

for all $a \in A$ and $n \in \mathbb{N}$. Choose an arbitrary $\varepsilon > 0$. Let $n_0 \in \mathbb{N}$ such that $q^*(f_n - f) < \varepsilon$ for all $n \geq n_0$. Therefore

$$|s(f_n, A) - s(f, A)| \leq \sup\{|(f_n - f)(a)| : a \in A\} \leq Mq^*(f_n - f) < M\varepsilon,$$

for all $n \geq n_0$. We conclude that $s(\cdot, A)$ is continuous from (X^*, q^*) into $(\mathbb{R}, |.|)$.

Finally, since $B_{X^*} \subseteq B_{X^{s*}}$ (Lemma 5 of [9]), it follows that for each $f \in B_{X^*}$ and each $x \in X$, $|f(x)| \leq q^s(x)$. So $|f(a)| \leq M$ for all $f \in B_{X^*}$ and $a \in A$. Therefore $|s(f, A)| \leq M$ for all $f \in B_{X^*}$. This concludes the proof. ■

The following asymmetric generalization of a classical theorem on separation of convex sets will be crucial later on (see Proposition 3 below).

Lemma 3. *Let A be a nonempty convex subset of an asymmetric normed linear space (X, q). If there is $\delta > 0$ such that $A \cap \overline{B}_q(\mathbf{0}, \delta) = \emptyset$, then there are an $f \in B_{X^*}$ and a constant $C > 0$ such that $f(x) \leq C$ for all $x \in \overline{B}_q(\mathbf{0}, \delta)$ and $C \leq f(x)$ for all $x \in A$.*

Proof. The closed ball $\overline{B}_q(\mathbf{0}, \delta)$ will simply be denoted by B. Fix $y_0 \in A$. Since A and B are convex sets, $B - A + y_0$ is a convex set. Moreover, it is absorbent because B is absorbent and $y_0 \in A$. Let p be the Minkowski functional for $B - A + y_0$. Thus $p(x) = \inf\{r > 0 : r^{-1}x \in B - A + y_0\}$ for all $x \in X$. In particular $p(y_0) \geq 1$.

Now consider the linear function g_0 defined on span$\{y_0\}$ by $g_0(ry_0) = rp(y_0)$ for all $r \in \mathbb{R}$. Then $g_0(ry_0) \leq p(ry_0)$ for all $r \in \mathbb{R}$. So, by Hahn-Banach's theorem, g_0

can be extended to a linear function g on X satisfying $g \le p$ on X. Clearly $g(y_0) \ge 1$ and $g(x) \le 1$ for all $x \in B - A + y_0$. By linearity of g we deduce that $g(x) \le c$ for all $x \in B$, where $c = 1 - g(y_0) + \inf_{x \in A} g(x)$. Clearly, $c \le g(x)$ for all $x \in A$.

Since $c \le 1$ we obtain $g(x) \le 1$ for all $x \in B$. From this relation it immediately follows that g is upper semicontinuous at $\mathbf{0}$, so it is continuous from (X, q) to $(\mathbb{R}, u)$. Hence $g \in X^*$. Note also that $c > 0$: Indeed, since $g(y_0) \le Mq(y_0)$ for some $M > 0$, we deduce that $q(y_0) > 0$, and thus the point $\delta y_0/q(y_0)$ is in B; consequently $0 < g(\delta y_0/q(y_0)) \le c$.

Finally, since $g(y_0) > 0$, $q^*(g) > 0$, and thus the function $f = g/q^*(g)$ satisfies $q^*(f) = 1$. Putting $C = c/q^*(g)$, we deduce that $f(x) \le C$ for all $x \in B$ and $C \le f(x)$ for all $x \in A$. This concludes the proof. ∎

Lemma 4 ([9]). *Let (X, q) be an asymmetric normed linear space. Then, for each $x \in X$, $q(x) = \sup\{f(x) : f \in B_{X^*}\}$.*

Proposition 3. *Let A and B be two nonempty bounded convex subsets of an asymmetric normed linear space (X, q). Then*

$$H_q^+(A, B) = \sup_{f \in B_{X^*}} (s(f, B) - s(f, A))$$

and

$$H_q^-(A, B) = \sup_{f \in B_{X^*}} (s(f, -A) - s(f, -B)).$$

Proof. Put $\lambda = \sup_{f \in B_{X^*}} (s(f, B) - s(f, A))$. Obviously $\lambda \ge 0$.

If $H_q^+(A, B) = 0$, then $H_q^+(A, B) \le \lambda$. If $H_q^+(A, B) > 0$, we choose an arbitrary $\delta > 0$ such that $H_q^+(A, B) > \delta$. Then there is $b_0 \in B$ such that $q(b_0 - a) > \delta$ for all $a \in A$. Since $b_0 - A$ is convex and $\overline{B}_q(\mathbf{0}, \delta) \cap (b_0 - A) = \emptyset$, it follows from Lemma 3 that there exist $f \in X^*$, with $q^*(f) = 1$, and $C > 0$ such that $f(x) \le C$ for all $x \in \overline{B}_q(\mathbf{0}, \delta)$, and $C \le f(b_0 - a)$ for all $a \in A$. Therefore

$$s(f, B) - s(f, A) \ge f(b_0) - \sup_{a \in A} f(a) \ge C \ge \sup_{x \in \overline{B}_q(\mathbf{0}, \delta)} f(x) = \delta.$$

Hence $\lambda \ge \delta$. We conclude that $\lambda \ge H_q^+(A, B)$.

Next we show that $\lambda \le H_q^+(A, B)$. Since this inequality is obvious for $\lambda = 0$, we will suppose $\lambda > 0$. Choose an arbitrary $\delta > 0$ such that $\delta < \lambda$. Then, there is $f \in B_{X^*}$ such that $s(f, B) - s(f, A) > \delta$, so $f(b_0) - s(f, A) > \delta$ for some $b_0 \in B$. Since, by Lemma 4, $q(b_0 - a) \ge f(b_0 - a)$ for all $a \in A$, it follows that

$$q(b_0 - a) \ge f(b_0) - f(a) \ge f(b_0) - s(f, A) > \delta,$$

for all $a \in A$. Consequently, $H_q^+(A, B) > \delta$, and, hence, $H_q^+(A, B) \ge \lambda$.

We have shown that $H_q^+(A, B) = \sup_{f \in B_{X^*}} (s(f, B) - s(f, A))$.

Now if we denote by $B_{X^*}^{-1}$ the unit ball of the dual of the asymmetric normed linear space (X, q^{-1}), then the first part of the proof shows that $H_{q^{-1}}^+(A, B) = \sup_{f \in B_{X^*}^{-1}} (s(f, B) - s(f, A))$, where the support function $s(\cdot, A)$ is now defined on the dual space of (X, q^{-1}). Since $f \in B_{X^*}^{-1}$ if and only if $-f \in B_{X^*}$, $s(f, -A) = \sup\{-f(a) : a \in A\}$ for $f \in B_{X^*}$, and $H_q^-(A,B) = H_{q^{-1}}^+(B,A)$, we deduce

$$H_q^-(A,B) = \sup_{f \in B_{X^*}^{-1}} (s(f,A) - s(f,B)) = \sup_{f \in B_{X^*}} (s(f,-A) - s(f,-B)).$$

The proof is complete. ∎

Similarly to [8], a map φ from a quasi-metric cone $(X, +, \cdot, d)$ into an asymmetric normed linear space (Y, q) is said to be an isometric isomorphism if φ is linear (i.e. $\varphi(a \cdot x + b \cdot y) = a\varphi(x) + b\varphi(y)$ whenever $x, y \in X$ and $a, b \in \mathbb{R}^+$), and it is an isometry (i.e. $d_q(\varphi(x), \varphi(y)) = d(x, y)$ for all $x, y \in X$).

Observe that if φ is an isometry then it is a one-to-one map.

Next we recall some concepts and results on the product of two asymmetric norms and the asymmetric norm of uniform convergence which will be useful in order to state our main result.

If q_1 and q_2 are asymmetric norms on a linear space X we define the product (or box) asymmetric norm $q_\times$ by $q_\times(x, y) = \max\{q_1(x), q_2(y)\}$.

As usual, if X is a nonempty set, we define the asymmetric norm of uniform convergence (or the supremum asymmetric norm) as the asymmetric norm $\|\cdot\|_\infty$ defined on the linear space $B\mathbb{R}^X$ of all bounded real functions on X by $\|f\|_\infty = \sup_{x \in X}(f(x) \vee 0)$ for all $f \in B\mathbb{R}^X$. Then, the conjugate asymmetric norm $\|.\|_\infty^{-1}$ of $\|.\|_\infty$ is defined by $\|f\|_\infty^{-1} = \sup_{x \in X}(-f(x) \vee 0)$. Moreover, since $\|f\|_\infty^s = \sup_{x \in X} |f(x)|$, then $(B\mathbb{R}^X, \|.\|_\infty^s)$ is a Banach space, so we have shown the following.

Proposition 4. *Let X be a nonempty set. Then $(B\mathbb{R}^X, \|.\|_\infty)$ is a biBanach space.*

If (X, q) is an asymmetric normed linear space, we shall denote by $C^*(B_{X^*})$ the linear space of bounded continuous real functions on (B_{X^*}, q^*). By Proposition 4 it easily follows that $(C^*(B_{X^*}), \|.\|_\infty)$ is a biBanach space.

The proof of the next lemma is analogous to the case of normed linear spaces (see, for instance, page 91 of [3]), so it is omitted.

Lemma 5. *Let (X, q) be an asymmetric normed linear space. Then, for each $A, B \in CB_0(X)$ and each $r \geq 0$ it follows*

$$s(\cdot, A \oplus B) = s(\cdot, A) + s(\cdot, B) \quad \text{and} \quad s(\cdot, r \cdot A) = r\ s(\cdot, A).$$

Theorem 2. *Let (X,q) be an asymmetric normed linear space. Then, the map*

$$A \mapsto (s(\cdot, A), s(\cdot, -A))$$

is an isometric isomorphism from the quasi-metric cone $(CB_0(X), \oplus, \cdot, H_q)$ into the biBanach space $(C^*(B_{X^*}) \times C^*(B_{X^*}), \|.\|_\times)$, where $\|.\|_\times$ is the product asymmetric norm given by $\|(F, G)\|_\times = \max\{\|F\|_\infty, \|G\|_\infty^{-1}\}$.

Proof. For each $A \in CB_0(X)$ put $\Psi(A) = (s(\cdot, A), s(\cdot, -A))$, where $s(\cdot, A)$ and $s(\cdot, -A)$ are restricted to B_{X^*}. By Proposition 2, for each $A \in CB_0(X)$, $s(\cdot, A)$ and $s(\cdot, -A)$ belong to $C^*(B_{X^*})$. Moreover, it follows from Lemma 5 that

$$\Psi(A \oplus B) = \Psi(A) + \Psi(B) \quad \text{and} \quad \Psi(r \cdot A) = r\Psi(A)$$

for all $A, B \in CB_0(X)$ and $r \geq 0$. Therefore Ψ is linear on the cone $(CB_0(X), \oplus, \cdot)$.
Finally, for each $A, B \in CB_0(X)$, we have, by Proposition 3, that

$$\begin{aligned} H_q(A,B) &= \max\left\{ \sup_{f \in B_{X^*}} (s(f,B) - s(f,A)), \sup_{f \in B_{X^*}} (s(f,-A) - s(f,-B)) \right\} \\ &= \max\left\{ \|s(\cdot,B) - s(\cdot,A)\|_\infty, \|s(\cdot,-A) - s(\cdot,-B)\|_\infty \right\} \\ &= \|(s(\cdot,B) - s(\cdot,A), s(\cdot,-B) - s(\cdot,-A))\|_\times = \|\Psi(B) - \Psi(A)\|_\times \\ &= d_{\|\cdot\|_\times}(\Psi(A), \Psi(B)). \end{aligned}$$

We conclude that Ψ is an isometric isomorphism from $(CB_0(X), \oplus, \cdot, H_q)$ into the biBanach space $(C^*(B_{X^*}) \times C^*(B_{X^*}), \|.\|_\times)$, where $\|(F,G)\|_\times = \max\{\|F\|_\infty, \|G\|_\infty^{-1}\}$. ■

Remark 2. Define $\varphi : (C^*(B_{X^*}), \|.\|_\infty^{-1}) \to (C^*(B_{X^*}^{-1}), \|.\|_\infty)$ by $\varphi(F) = -F$ for all $F \in C^*(B_{X^*})$. It is routine to check that φ is a bijective linear map between the linear spaces $C^*(B_{X^*})$ and $C^*(B_{X^*}^{-1})$ (recall that $B_{X^*}^{-1}$ is the unit closed ball of the dual space of (X, q^{-1}), as defined in the proof of Proposition 3). Furthermore, we have $\|F\|_\infty^{-1} = \|\varphi(F)\|_\infty$ for all $F \in C^*(B_{X^*})$. Then, it follows from Theorem 2 that the map $A \mapsto (s(\cdot,A), s(\cdot,-A))$ is an isometric isomorphism from the quasi-metric cone $(CB_0(X), \oplus, \cdot, H_q)$ into the product of the biBanach spaces $C^*(B_{X^*})$ and $C^*(B_{X^*}^{-1})$, when they are equipped with the asymmetric norm of uniform convergence.

Remark 3. Note that if (X,q) is a normed linear space, then the map $A \mapsto s(\cdot,A)$ is one-to-one; actually, in this case, Proposition 3 implies Corollary 3.2.8 of [3], that $H_q(A,B) = \sup_{f \in B_{X^*}} |s(f,A) - s(f,B)|$. Thus, we restate Hörmander's theorem that the map $A \mapsto s(\cdot,A)$ is an isometric isomorphism from the metric cone $(CB_0(X), \oplus, \cdot, H_q)$ into the Banach space of bounded continuous real functions on the unit ball of the dual of the normed linear space (X,q), equipped with the norm of uniform convergence.

We conclude the paper with an easy example which shows that in the asymmetric setting the map $A \mapsto s(\cdot,A)$ is not one-to-one and, hence, it is necessary to consider the map $A \mapsto (s(\cdot,A), s(\cdot,-A))$ to obtain the desired isometric isomorphism.

Example 4. Let $(\mathbb{R}, u)$ be the asymmetric normed linear space of Example 1. Let $A = \{0\}$ and $B = [-1, 0]$. Then $H_q^+(A,B) = H_q^+(B,A) = 0$. By Proposition 3, $s(f,A) = s(f,B)$ for all $f \in B_{X^*}$. Thus, the map $A \mapsto s(\cdot,A)$ is not one-to-one.

References

[1] C. Alegre, J. Ferrer and V. Gregori, On the Hahn-Banach theorem in certain linear quasi-uniform structures, Acta Math. Hungar. **82** (1999), 325-330.

[2] A. Alimov, On the structure of the complements of Chebyshev sets, Functional Anal. Appl. **35** (2001), 176-182.

[3] G. Beer, Topologies on Closed and Closed Convex Sets, vol. 268, Kluwer Acad. Publ. 1993.

[4] G. Berthiaume, On quasi-uniformities in hyperspaces, Proc. Amer. Math. Soc. **66** (1977), 335-343.

[5] E.P. Dolzhenko and E.A. Sevast'yanov, Sign-sensitive approximations, the space of sign-sensitive weights. The rigidity and the freedom of a system, Russian Acad. Sci. Dokl. Math. **48** (1994), 397-401.

[6] J. Ferrer, V. Gregori and C. Alegre, Quasi-uniform structures in linear lattices, Rocky Mountain J. Math. **23** (1993), 877-884.

[7] P. Fletcher and W.F. Lindgren, Quasi-Uniform Spaces, Marcel Dekker, New York, 1982.

[8] L.M.García-Raffi, S. Romaguera and E.A. Sánchez-Pérez, Sequence spaces and asymmetric norms in the theory of computational complexity, Math. Comput. Model. **36** (2002), 1-11.

[9] L.M. García-Raffi, S. Romaguera and E.A. Sánchez-Pérez, The dual space of an asymmetric normed linear space, Quaestiones Math. **26** (2003), 83-96.

[10] L. Hörmander, Sur le fonction d'appui des ensembles convexes dans un espace localement convexe, Arkiv för Mat. **3** (1955), 181-186.

[11] K. Keimel and W. Roth, Ordered Cones and Approximation, Lecture Notes Math., vol. 1517, Springer, Berlin, 1992.

[12] R. Kopperman, All topologies come from generalized metrics, Amer. Math. Monthly **95** (1988), 89-97.

[13] H.P.A. Künzi, Nonsymmetric distances and their associated topologies: About the origins of basic ideas in the area of asymmetric topology, in: C.E. Aull and R. Lowen (eds), Handbook of the History of General Topology, vol. 3, Kluwer, Dordrecht, 2001, pp. 853-968.

[14] H.P.A. Künzi and C. Ryser, The Bourbaki quasi-uniformity, Topology Proc. **20** (1995), 161-183.

[15] J. Rodríguez-López and S. Romaguera, Reconciling proximal convergence with uniform convergence in quasi-metric spaces, Houston J. Math. **27** (2001), 445-459.

[16] J. Rodríguez-López and S. Romaguera, Wijsman and hit-and-miss topologies of quasi-metric spaces, Set-Valued Analysis **11** (2003), 323-344.

[17] S. Romaguera, E.A. Sánchez-Pérez and O. Valero, Computing complexity distances between algorithms, Kybernetika **36** (2003), 369-382.

[18] S. Romaguera and M. Sanchis, Semi-Lipschitz functions and best approximation in quasi-metric spaces, J. Approx. Theory **103** (2000), 292-301.

[19] S. Romaguera and M. Sanchis, Properties of the normed cone of semi-Lipschitz functions, Acta Math. Hungar., **108** (2005), 55-70.

[20] S. Romaguera and M. Schellekens, Duality and quasi-normability of complexity spaces, Appl. Gen. Topology **3** (2002), 91-112.

[21] M. Schellekens, Complexity spaces revisited, in: Proc. 8th Prague topological symposium, Czech Republic 1996, Topology Atlas (1997). URL: http//at.yorku.ca/p/p/a/a/00.htm

[22] Ph. Sünderhauf, Discrete Approximation of Spaces. A Uniform Approach to Topologically Structured Data Types and their Function Spaces, Thesis, Darmstadt University, 1994.

[23] Ph. Sünderhauf, Tensor products and powerspaces in quantitative domain theory, Electronic Notes in Theoretical Computer Science **6,** 1997. URL: http://www.elsevier.nl/locate/entcs/volume6.html

[24] R. Tix, Continuous D-Cones: Convexity and Powerdomain Constructions, Thesis, Darmstadt University, 1999.

Escuela de Caminos, Departamento de Matemática Aplicada,
Instituto de Matemática Pura y Aplicada,
Universidad Politécnica de Valencia, 46022 Valencia, Spain.
E-mail: jrlopez@mat.upv.es, sromague@mat.upv.es