Sandwich-type theorems for a class of integral operators

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Abstract

Let $H(\mathrm{U})$ be the space of all analytic functions in the unit disk U. For a given function $h \in \mathcal{A}$ we define the integral operator $\mathrm{I}_{h;\beta} : \mathcal{K} \to H(\mathrm{U})$, with $\mathcal{K} \subset H(\mathrm{U})$, by

$$\mathrm{I}_{h;\beta}[f](z) = \left[\beta \int_0^z f^{\beta}(t) h^{-1}(t) h'(t) \,\mathrm{d}\, t\right]^{1/\beta},$$

where $\beta \in \mathbb{C}$ and all powers are the principal ones.

We will determine sufficient conditions on g_1, g_2 and β such that

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_1(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_2(z)$$

implies

$$\mathrm{I}_{h;\beta}[g_1](z) \prec \mathrm{I}_{h;\beta}[f](z) \prec \mathrm{I}_{h;\beta}[g_2](z),$$

where the symbol "$\prec$" stands for subordination. We will call such a kind of result a *sandwich-type theorem.*

In addition, $\mathrm{I}_{h;\beta}[g_1]$ will be the *largest* function and $\mathrm{I}_{h;\beta}[g_2]$ the *smallest* function so that the left-hand side, respectively the right-hand side of the above implication hold, for all f functions satisfying the differential subordination, respectively the differential superordination of the assumption.

We will give some particular cases of the main result obtained for appropriate choices of the h, that also generalize classic results of the theory of differential subordination and superordination.

The concept of differential superordination was introduced by S. S. Miller and P. T. Mocanu in [5] like a dual problem of differential subordination [4].

Received by the editors January 2005.
Communicated by F. Brackx.
1991 *Mathematics Subject Classification :* Primary 30C80; Secondary 30C25, 30C45.
Key words and phrases : Differential subordination, univalent function, starlike function, integral operator.

1 Introduction

Let $H(\mathrm{U})$ be the class of analytic functions in the unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$. We denote by A the class of analytic functions in U and usually normalized, i.e. $A = \{f \in H(\mathrm{U}) : f(0) = 1, f'(0) = 1\}$ and let

$$\mathcal{A} = \{h \in A : h(z)h'(z) \neq 0, 0 < |z| < 1\}.$$

For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in H(\mathrm{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}.$$

For a function $h \in \mathcal{A}$ we define the integral operator $\mathrm{I}_{h;\beta} : \mathcal{K}_{h;\beta} \to H(\mathrm{U})$ by

$$\mathrm{I}_{h;\beta}[f](z) = \left[\beta \int_0^z f^\beta(t) h^{-1}(t) h'(t)\,\mathrm{d}\,t\right]^{1/\beta}, \tag{1.1}$$

where $\mathcal{K}_{h;\beta} \subset H(\mathrm{U})$ will be determined in Lemma 3.1 and Lemma 3.2, such that this integral operator is well defined (all powers in the above formula are the principal ones).

For $f, g \in H(\mathrm{U})$ we say that the function f is *subordinate* to g, or g is *superordinate* to f, if there exists a function $w \in H(\mathrm{U})$, with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathrm{U}$, such that $f(z) = g(w(z))$ for all $z \in \mathrm{U}$. In such a case we write $f(z) \prec g(z)$. If g is univalent in U, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathrm{U}) \subseteq g(\mathrm{U})$.

In [2] the author determined conditions on the h and g functions and on the parameter β, such that

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z) \Rightarrow \mathrm{I}_{h;\beta}[f](z) \prec \mathrm{I}_{h;\beta}[g](z). \tag{1.2}$$

In the present paper we will improve the above result, then we will study the reverse problem to determine simple sufficient conditions on h, g and β, such that

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \Rightarrow \mathrm{I}_{h;\beta}[g](z) \prec \mathrm{I}_{h;\beta}[f](z),$$

and we will prove that under our assumptions this result is sharp.

Combining these results we will obtain a so called *sandwich-type theorem*, and we will give some interesting particular results obtained for convenient choices of the h function.

2 Preliminaries

In order to prove our main results, we will need the following definitions and lemmas presented in this section.

Let $c \in \mathbb{C}$ with $\operatorname{Re} c > 0$, and let $N = N(c) = \dfrac{|c|\sqrt{1 + 2\operatorname{Re} c} + \operatorname{Im} c}{\operatorname{Re} c}$. If k is the univalent function $k(z) = \dfrac{2Nz}{1 - z^2}$, then we define the *open door function* R_c by

$$R_c(z) = k\left(\frac{z + b}{1 + \bar{b}z}\right), \ z \in \mathrm{U}, \tag{2.1}$$

where $b = k^{-1}(c)$.

Remark that R_c is univalent in U, $R_c(0) = c$ and $R_c(\mathrm{U}) = k(\mathrm{U})$ is the complex plane slit along the half-lines $\operatorname{Re} w = 0$, $\operatorname{Im} w \geq N$ and $\operatorname{Re} w = 0$, $\operatorname{Im} w \leq -N$, i.e.

$$R_c(\mathrm{U}) = k(\mathrm{U}) = \mathbb{C} \setminus \{w \in \mathbb{C} : \operatorname{Re} w = 0, |\operatorname{Im} w| \geq N\}.$$

Lemma 2.1. *[1, Lemma 3.1.] Let* $\beta, \gamma \in \mathbb{C}$ *with* $\beta \neq 0$, $\operatorname{Re}(\beta + \gamma) > 0$ *and let* $h \in A$ *with* $h(z)h'(z)/z \neq 0$, $z \in \mathrm{U}$. *If* $f \in A$ *and*

$$\beta \frac{zf'(z)}{f(z)} + (\gamma - 1)\frac{zh'(z)}{h(z)} + 1 + \frac{zh''(z)}{h'(z)} \prec R_{\beta+\gamma}(z)$$

then,

$$F \in A, \quad \frac{F(z)}{z} \neq 0,\ z \in \mathrm{U} \quad \textit{and} \quad \operatorname{Re}\left[\beta \frac{zF'(z)}{F(z)} + \gamma \frac{zh'(z)}{h(z)}\right] > 0,\ z \in \mathrm{U}$$

where

$$F(z) = \left[\frac{\beta + \gamma}{h^{\gamma}(z)} \int_0^z f^{\beta}(t) h^{\gamma - 1}(t) h'(t)\, \mathrm{d}\, t\right]^{1/\beta},$$

and all powers are the principal ones.

We denote by $\mathcal{Q}$ the set of functions q that are analytic and injective on $\overline{\mathrm{U}} \setminus E(q)$, where

$$E(q) = \{\zeta \in \partial \mathrm{U} : \lim_{z \to \zeta} q(z) = \infty\},$$

and such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathrm{U} \setminus E(q)$. The subclass of $\mathcal{Q}$ for which $q(0) = a$ is denoted by $\mathcal{Q}(a)$.

Like in [3] or [4], let $\Omega \subset \mathbb{C}$, $q \in \mathcal{Q}$ and n be a positive integer. Then, the *class of admissible functions* (in the sense of subordination) $\Psi_n[\Omega, q]$ is the class of those functions $\psi : \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}$ that satisfy the *admissibility* condition

$$\psi(r, s, t; z) \notin \Omega,$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $\operatorname{Re} \dfrac{t}{s} + 1 \geq m \operatorname{Re}\left[\dfrac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right]$, $z \in \mathrm{U}$, $\zeta \in \partial\mathrm{U} \setminus E(q)$ and $m \geq n$.

We write $\Psi[\Omega, q] \equiv \Psi_1[\Omega, q]$. For the special case when $\Omega \neq \mathbb{C}$ is a simply connected domain and h is a conformal mapping of U onto Ω, we denote this class by $\Psi_n[h, q]$.

Remark 2.1. If $\psi : \mathbb{C}^2 \times \mathrm{U} \to \mathbb{C}$, then the above defined admissibility condition reduces to

$$\psi(q(\zeta), m\zeta q'(\zeta); z) \notin \Omega,$$

when $z \in \mathrm{U}$, $\zeta \in \partial\mathrm{U} \setminus E(q)$ and $m \geq n$.

The next lemma is a key result in the theory of *sharp* differential subordinations.

Lemma 2.2. *[3], [4] Let h be univalent in* U *and $\psi : \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\psi(q(z), zq'(z), z^2q''(z); z) = h(z)$$

has a solution q, with $q(0) = a$, and one of the following conditions is satisfied:

(*i*) $q \in \mathcal{Q}$ *and* $\psi \in \Psi[h, q]$

(*ii*) q *is univalent in* U *and* $\psi \in \Psi[h, q_\rho]$, *for some* $\rho \in (0, 1)$, *where* $q_\rho(z) = q(\rho z)$, *or*

(*iii*) q *is univalent in* U *and there exists* $\rho_0 \in (0, 1)$ *such that* $\psi \in \Psi[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$, *where* $h_\rho(z) = h(\rho z)$ *and* $q_\rho(z) = q(\rho z)$.

If $p(z) = a + a_1z + \ldots \in H(\mathrm{U})$ and $\psi(p(z), zp'(z), z^2p''(z); z) \in H(\mathrm{U})$, then

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \quad \textit{implies} \quad p(z) \prec q(z)$$

and q is the best dominant.

Lemma 2.3. *[3] Let $q \in \mathcal{Q}$, with $q(0) = a$, and let $p(z) = a + a_nz^n + \ldots$ be analytic in* U *with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 \in \mathrm{U}$ and $\zeta_0 \in \partial\mathrm{U} \setminus E(q)$, and an $m \geq n \geq 1$ for which $p(|z| < |z_0|) \subset q(\mathrm{U})$, and*

(*i*) $p(z_0) = q(\zeta_0)$,

(*ii*) $z_0p'(z_0) = m\zeta_0q'(\zeta_0)$,

(*iii*) $\mathrm{Re}\,\dfrac{z_0p''(z_0)}{p'(z_0)} + 1 \geq m\,\mathrm{Re}\left[\dfrac{\zeta_0q''(\zeta_0)}{q'(\zeta_0)} + 1\right]$.

The function $f \in H(\mathrm{U})$, with $f(0) = 0$, is called to be *starlike in* U, or simply *starlike*, if f is univalent in U and $f(\mathrm{U})$ is a starlike domain with respect to the origin. It is well-known that a function $f \in H(\mathrm{U})$, with $f(0) = 0$, is starlike if and only if $f'(0) \neq 0$ and $\mathrm{Re}\,\dfrac{zf'(z)}{f(z)} > 0$, $z \in \mathrm{U}$.

The function $f \in H(\mathrm{U})$ is called to be *convex in* U, or simply *convex*, if f is univalent in U and $f(\mathrm{U})$ is a convex domain. A function $f \in H(\mathrm{U})$ is convex if and only if $f'(0) \neq 0$ and $\mathrm{Re}\,\dfrac{zf''(z)}{f'(z)} + 1 > 0$, $z \in \mathrm{U}$.

For $\alpha \in \mathbb{R}$, a function $f \in H(\mathrm{U})$ with $f(0) = 0$ and $f'(0) \neq 0$ is called to be an α-*convex* (not necessarily normalized) *function* [7], if

$$\mathrm{Re}\left[(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] > 0,\ z \in \mathrm{U},$$

and we denote this class by $\mathcal{M}_\alpha$. Note that all α-convex functions are univalent and starlike [6], i.e.

$$\mathcal{M}_\alpha \subset \mathcal{M}_0. \tag{2.2}$$

Lemma 2.4. *[4, Lemma 1.2c.] Let $n \geq 0$ be an integer and let $\gamma \in \mathbb{C}$, with* $\operatorname{Re}\gamma > -n$. *If $f(z) = \sum_{m \geq n} a_m z^m$ is analytic in* U *and F is defined by*

$$F(z) = \frac{1}{z^\gamma} \int_0^z f(t) t^{\gamma-1} \, \mathrm{d}\, t,$$

then $F(z) = \sum_{m \geq n} \frac{a_m z^m}{m+\gamma}$ is analytic in U.

As in [5], let $\Omega \subset \mathbb{C}$ and $q \in \mathcal{H}[a,n]$, where n is a positive integer. Then, the *class of admissible functions* (in the sense of superordination) $\Phi_n[\Omega, q]$ is the class of those functions $\varphi : \mathbb{C}^3 \times \overline{\mathrm{U}} \to \mathbb{C}$ that satisfy the *admissibility* condition

$$\varphi(r, s, t; \zeta) \in \Omega,$$

whenever $r = q(z)$, $s = \dfrac{zq'(z)}{m}$, $\operatorname{Re} \dfrac{t}{s} + 1 \leq \dfrac{1}{m} \operatorname{Re} \left[\dfrac{zq''(z)}{q'(z)} + 1 \right]$, $\zeta \in \partial\mathrm{U}$, $z \in \mathrm{U}$ and $m \geq n$.

We write $\Phi[\Omega, q] \equiv \Phi_1[\Omega, q]$. For the special case when $\Omega \neq \mathbb{C}$ is a simply connected domain and h is a conformal mapping of U onto Ω, we denote this class by $\Phi_n[h, q]$.

Remark 2.2. If $\varphi : \mathbb{C}^2 \times \overline{\mathrm{U}} \to \mathbb{C}$, then the above defined admissibility condition reduces to

$$\varphi(q(z), zq'(z)/m; \zeta) \in \Omega,$$

when $\zeta \in \partial\mathrm{U}$, $z \in \mathrm{U}$ and $m \geq n$.

This last lemma gives us a very important result in the theory of *sharp* differential superordinations.

Lemma 2.5. *[5, Theorem 5.] Let $h \in H(\mathrm{U})$, $q \in \mathcal{H}[a,n]$ and let $\varphi \in \Phi_n[h, q]$, i.e. $\varphi : \mathbb{C}^2 \times \overline{\mathrm{U}} \to \mathbb{C}$ and satisfies the condition*

$$\varphi(q(z), tzq'(z); \zeta) \in h(\mathrm{U}),$$

for $z \in \mathrm{U}$, $\zeta \in \partial\mathrm{U}$ and $0 < t \leq 1/n \leq 1$. If $p \in \mathcal{Q}(a)$ and $\varphi(p(z), zp'(z); z)$ is univalent in U, *then*

$$h(z) \prec \varphi(p(z), zp'(z); z) \Rightarrow q(z) \prec p(z).$$

Furthermore, if $\varphi(q(z), zq'(z); z) = h(z)$ has a univalent solution $q \in \mathcal{Q}(a)$, then q is the best subordinant.

3 Main results

For a given $\beta \in \mathbb{C}$ with $\operatorname{Re}\beta > 0$, first we need to find sufficient conditions for the h function in order to determine the correspondent subset $\mathcal{K}_{h;\beta} \subset H(\mathrm{U})$, such that the integral operator $\mathrm{I}_{h;\beta}$ defined by (1.1) is well defined on $\mathcal{K}_{h;\beta}$.

Lemma 3.1. *Let $\beta \in \mathbb{C}$ with $\operatorname{Re}\beta > 0$, let $h \in \mathcal{A}$ and denote by*

$$J(\gamma, h)(z) = (\gamma - 1)\frac{zh'(z)}{h(z)} + 1 + \frac{zh''(z)}{h'(z)}.$$

If R_β represents the open door *function defined by (2.1) and if*

$$\widetilde{\mathcal{K}}_{h;\beta} = \left\{ f \in A : \beta\frac{zf'(z)}{f(z)} + J(0,h)(z) \prec R_\beta(z) \right\}, \text{ for } \beta \neq 1,$$
$$\widetilde{\mathcal{K}}_{h;1} = \{ f \in H(\mathrm{U}) : f(0) = 0 \}, \text{ for } \beta = 1,$$

then the integral operator $\mathrm{I}_{h;\beta}$ is well-defined on $\widetilde{\mathcal{K}}_{h;\beta}$.

Proof. The case $\beta \neq 1$ represents Lemma 2.1 for $\gamma = 0$. If $\beta = 1$, denoting $t = wz$ we have

$$\mathrm{I}_{h;1}[f](z) = \frac{z}{h(z)} \int_0^1 \left[\frac{wz}{h(wz)} h'(wz) \right] f(wz) w^{-1} \,\mathrm{d}\, w,$$

and according to Lemma 2.4 for the special case $\gamma = 0$ and $n = 1$, we obtain our result. ■

Lemma 3.2. *Let $\beta \in \mathbb{C}$ with $\operatorname{Re}\beta > 0$, and let $h \in \mathcal{A}$. If*

$$\mathcal{K}_{h;\beta} = \widetilde{\mathcal{K}}_{h;\beta}, \text{ for } \beta \neq 1,$$
$$\mathcal{K}_{h;1} = \left\{ f \in \widetilde{\mathcal{K}}_{h;1} : f'(0) \neq 0 \right\}, \text{ for } \beta = 1,$$

then the integral operator $\mathrm{I}_{h;\beta}$ is well-defined on $\mathcal{K}_{h;\beta}$ and satisfies the following conditions:

$$F = \mathrm{I}_{h;\beta}[f] \in A, \quad \frac{F(z)}{z} \neq 0,\ z \in \mathrm{U}, \quad \operatorname{Re}\left[\beta\frac{zF'(z)}{F(z)}\right] > 0,\ z \in \mathrm{U}, \quad \text{for } \beta \neq 1,$$

and

$$F(z) = \mathrm{I}_{h;\beta}[f](z) = f'(0)z + \dots,\ z \in \mathrm{U}, \quad \text{for } \beta = 1.$$

Proof. The case $\beta \neq 1$ follows from Lemma 3.1 and Lemma 2.1 for $\gamma = 0$. For $\beta = 1$, since $f(z) = a_1 z + \dots$, $z \in \mathrm{U}$, a simple computation shows that

$$F(z) = \mathrm{I}_{h;1}[f](z) = a_1 z + \dots,\ z \in \mathrm{U}.$$

■

The next main result deals with the subordination of the form (1.2) and gives us an extension of Theorem 1 of [2].

Theorem 3.1. *Let $\beta > 0$ and let $h \in \mathcal{A}$. Let $f, g \in \mathcal{K}_{h;\beta}$ and suppose that*

$$\operatorname{Re} \frac{zg'(z)}{g(z)} > -\frac{1}{\beta} \operatorname{Re} J(0,h)(z), \; z \in \mathrm{U}. \tag{3.1}$$

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z) \Rightarrow \mathrm{I}_{h;\beta}[f](z) \prec \mathrm{I}_{h;\beta}[g](z),$$

and the function $\mathrm{I}_{h;\beta}[g]$ is the best dominant of the subordination.

Proof. Denoting by $F(z) = \mathrm{I}_{h;\beta}[f](z)$, $G(z) = \mathrm{I}_{h;\beta}[g](z)$, $\psi(z) = \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$ and $\varphi(z) = \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z)$, we need to prove that

$$\psi(z) \prec \varphi(z) \Rightarrow F(z) \prec G(z).$$

Since $f, g \in \mathcal{K}_{h;\beta}$ and $h \in \mathcal{A}$ then $\psi, \varphi \in H(\mathrm{U})$, and by Lemma 3.2 we have $F, G \in H(\mathrm{U})$ with $F(0) = G(0) = 0$, $F'(0) \neq 0$ and $G'(0) \neq 0$.

Differentiating the relations $\varphi(z) = \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z)$ and $G(z) = \mathrm{I}_{h;\beta}[g](z)$ we obtain

$$\frac{z\varphi'(z)}{\varphi(z)} = \frac{1}{\beta} J(0,h)(z) + \frac{zg'(z)}{g(z)} = \left(1 - \frac{1}{\beta}\right) \frac{zG'(z)}{G(z)} + \frac{1}{\beta}\left(1 + \frac{zG''(z)}{G'(z)}\right). \tag{3.2}$$

From the assumption (3.1), according to (2.2) and the second part of the above equality, we deduce that $G \in \mathcal{M}_{1/\beta} \subset \mathcal{M}_0$, hence G is a starlike (univalent) function in U.

Since $h \in \mathcal{A}$ and $g \in \mathcal{K}_{h;\beta}$, then $\varphi(0) = 0$, $\varphi'(0) \neq 0$. Hence, combining the inequality (3.1) of the assumption together with the first part of (3.2), we obtain that φ is a starlike (univalent) function U.

From $G(z) = \mathrm{I}_{h;\beta}[g](z)$, a simple differentiation shows that

$$g(z) = G(z) \left[\frac{1}{\chi(z)} \frac{zG'(z)}{G(z)}\right]^{1/\beta}, \quad \text{where} \quad \chi(z) = \frac{zh'(z)}{h(z)},$$

then

$$\varphi(z) = \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z) = G(z) \left[\frac{zG'(z)}{G(z)}\right]^{1/\beta}. \tag{3.3}$$

Similarly, we obtain

$$f(z) = F(z) \left[\frac{1}{\chi(z)} \frac{zF'(z)}{F(z)}\right]^{1/\beta}$$

and,

$$\psi(z) = \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) = F(z) \left[\frac{zF'(z)}{F(z)}\right]^{1/\beta}. \tag{3.4}$$

Now, by using Lemma 2.2, we will show that $F(z) \prec G(z)$. Without loss of generality, we can assume that φ and G are analytic and univalent in $\overline{\mathrm{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$. If not, then we could replace φ with $\varphi_\rho(z) = \varphi(\rho z)$ and G with $G_\rho(z) = G(\rho z)$, where $\rho \in (0, 1)$. These new functions will have the desired properties and we would prove our result using part (iii) of Lemma 2.2.

With our assumption, we will use part (i) of the Lemma 2.2. Denoting by

$$\phi(G(z), zG'(z)) = G(z) \left[\frac{zG'(z)}{G(z)}\right]^{1/\beta} = \varphi(z),$$

we only need to show that $\phi \in \Psi[\varphi, G]$, i.e. ϕ is an admissible function (in the sense of subordination).

If we suppose that $F(z) \not\prec G(z)$, then by Lemma 2.3 there exist points $z_0 \in \mathrm{U}$ and $\zeta_0 \in \partial\mathrm{U}$, and a number $m \geq 1$, such that

$$F(z_0) = G(\zeta_0)$$
$$z_0 F'(z_0) = m\zeta_0 G'(\zeta_0).$$

Using the equalities (3.3) and (3.4) together with the above two relations, we obtain

$$\psi(z_0) = F(z_0) \left[\frac{z_0 F'(z_0)}{F(z_0)}\right]^{1/\beta} = G(\zeta_0) \left[\frac{m\zeta_0 G'(\zeta_0)}{G(\zeta_0)}\right]^{1/\beta} = m^{1/\beta}\varphi(\zeta_0). \tag{3.5}$$

Since we already proved that φ is a starlike function in U, then $\varphi(\mathrm{U})$ is a starlike domain with respect to the origin, and from the fact that $\beta > 0$ the relation (3.5) gives us

$$\psi(z_0) = m^{1/\beta}\varphi(\zeta_0) \notin \varphi(\mathrm{U}).$$

According to the Remark 2.1, we have $\phi \in \Psi[\varphi, G]$ and, using Lemma 2.2, we obtain that $F(z) \prec G(z)$.

Furthermore, since the G function, with $G(0) = F(0)$, is a univalent solution of the differential equation $\phi(q(z), zq'(z)) = \varphi(z)$, then G is the best dominant of $\psi(z) \prec \varphi(z)$ differential subordination, that completes the proof of the Theorem. ■

The next theorem represents a dual result of Theorem 3.1, in the sense that the subordinations are replaced by superordinations.

Theorem 3.2. *Let $\beta > 0$ and let $h \in \mathcal{A}$. Let $g \in \mathcal{K}_{h;\beta}$ and suppose that*

$$\operatorname{Re}\frac{zg'(z)}{g(z)} > -\frac{1}{\beta}\operatorname{Re} J(0, h)(z),\ z \in \mathrm{U}. \tag{3.6}$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ such that $\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$ and $\mathrm{I}_{h;\beta}[f](z)$ are univalent functions in U*.*

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \Rightarrow \mathrm{I}_{h;\beta}[g](z) \prec \mathrm{I}_{h;\beta}[f](z),$$

and the function $\mathrm{I}_{h;\beta}[g]$ is the best subordinant of the superordination.

Proof. Using the same notation as in the previous proof, i.e. $G(z) = \mathrm{I}_{h;\beta}[g](z)$, $F(z) = \mathrm{I}_{h;\beta}[f](z)$, $\varphi(z) = \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z)$ and $\psi(z) = \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$, our conclusion becomes

$$\varphi(z) \prec \psi(z) \Rightarrow G(z) \prec F(z).$$

From $f, g \in \mathcal{K}_{h;\beta}$ and $h \in \mathcal{A}$ it follows that $\psi, \varphi \in H(\mathrm{U})$, and by Lemma 3.2 we have $F, G \in H(\mathrm{U})$ with $F(0) = G(0) = 0$, $F'(0) \neq 0$ and $G'(0) \neq 0$.

Using the assumption (3.6) and according to (2.2), from the second part of the equality (3.2) we deduce that $G \in \mathcal{M}_{1/\beta} \subset \mathcal{M}_0$, hence G is a starlike (univalent) function in U.

Since $h \in \mathcal{A}$ and $g \in \mathcal{K}_{h;\beta}$, then $\varphi(0) = 0$, $\varphi'(0) \neq 0$. From the inequality (3.6) of the assumption together with the first part of (3.2), we obtain that φ is a starlike (univalent) function in U, hence $\varphi(\mathrm{U})$ is a starlike domain with respect to the origin.

By using Lemma 2.5 we will show that $G(z) \prec F(z)$. Without loss of generality, we can assume that φ and G are analytic and univalent in $\overline{\mathrm{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$. If not, then we could replace φ with $\varphi_\rho(z) = \varphi(\rho z)$ and G with $G_\rho(z) = G(\rho z)$, where $\rho \in (0,1)$. These new functions will have the desired properties and by letting $\rho \to 1$ we will obtain our result.

Letting

$$\phi(G(z), zG'(z)) = G(z) \left[\frac{zG'(z)}{G(z)}\right]^{1/\beta} = \varphi(z),$$

we only need to show that $\phi \in \Phi[\varphi, G]$, i.e. ϕ is an admissible function (in the sense of superordination).

A simple calculus shows that

$$\phi(G(z), tzG'(z)) = G(z) \left[\frac{tzG'(z)}{G(z)}\right]^{1/\beta} = t^{1/\beta}\varphi(z). \tag{3.7}$$

Using the fact that φ is a starlike function, then $\varphi(\mathrm{U})$ is a starlike domain with respect to the origin, and from the assumption $\beta > 0$ the relation (3.7) gives us

$$\phi(G(z), tzG'(z)) = t^{1/\beta}\varphi(z) \in \varphi(\mathrm{U}),$$

whenever $0 < t \leq 1$. From the Remark 2.2 we get $\phi \in \Phi[\varphi, G]$, then applying Lemma 2.5 we obtain that $G(z) \prec F(z)$.

Furthermore, since the G function, with $G(0) = F(0)$, is a univalent solution of the differential equation $\phi(G(z), zG'(z)) = \varphi(z)$, then G is the best subordinant of $\varphi(z) \prec \psi(z)$ differential superordination, hence the proof of the Theorem is complete. ∎

If we combine these two results we obtain the following *sandwich-type theorem*.

Theorem 3.3. *Let $\beta > 0$ and let $h \in \mathcal{A}$. Let $g_1, g_2 \in \mathcal{K}_{h;\beta}$ and suppose that the next two conditions are satisfied*

$$\mathrm{Re}\,\frac{zg_k'(z)}{g_k(z)} > -\frac{1}{\beta}\,\mathrm{Re}\,J(0,h)(z),\ z \in \mathrm{U}, \quad \textit{for} \quad k = 1, 2. \tag{3.8}$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ *such that* $\left[\dfrac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$ *and* $\mathrm{I}_{h;\beta}[f](z)$ *are univalent functions in* U.

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_1(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_2(z)$$

implies

$$\mathrm{I}_{h;\beta}[g_1](z) \prec \mathrm{I}_{h;\beta}[f](z) \prec \mathrm{I}_{h;\beta}[g_2](z).$$

Moreover, the functions $\mathrm{I}_{h;\beta}[g_1]$ *and* $\mathrm{I}_{h;\beta}[g_2]$ *are respectively the best subordinant and the best dominant.*

Since in the assumption of the above Theorem we need to suppose that the functions $\left[\dfrac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$ and $\mathrm{I}_{h;\beta}[f](z)$ are univalent in U, the next similar result will give us, in addition, sufficient conditions that imply the univalence of these functions.

Corollary 3.1. *Let* $\beta > 0$ *and let* $h \in \mathcal{A}$*. Let* $g_1, g_2 \in \mathcal{K}_{h;\beta}$ *and suppose that the conditions (3.8) are satisfied.*

Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ *such that*

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > -\frac{1}{\beta}\,\mathrm{Re}\,J(0,h)(z),\ z \in \mathrm{U}. \tag{3.9}$$

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_1(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_2(z)$$

implies

$$\mathrm{I}_{h;\beta}[g_1](z) \prec \mathrm{I}_{h;\beta}[f](z) \prec \mathrm{I}_{h;\beta}[g_2](z).$$

Moreover, the functions $\mathrm{I}_{h;\beta}[g_1]$ *and* $\mathrm{I}_{h;\beta}[g_2]$ *are respectively the best subordinant and the best dominant.*

Proof. In order to use Theorem 3.3 to prove this Corollary, we only need to prove that the functions $\psi(z) = \left[\dfrac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$ and $F(z) = \mathrm{I}_{h;\beta}[f](z)$ are univalent in U. Next we will show that the assumption (3.9) implies the univalence of both of these functions.

A simple calculus shows that

$$\frac{z\psi'(z)}{\psi(z)} = \frac{1}{\beta}J(0,h)(z) + \frac{zf'(z)}{f(z)} = \left(1 - \frac{1}{\beta}\right)\frac{zF'(z)}{F(z)} + \frac{1}{\beta}\left(1 + \frac{zF''(z)}{F'(z)}\right). \tag{3.10}$$

From the assumption (3.9), according to (2.2) and (3.10), we deduce that $F \in \mathcal{M}_{1/\beta} \subset \mathcal{M}_0$, hence F is a starlike (univalent) function in U.

Since $h \in \mathcal{A}$ and $f \in \mathcal{K}_{h;\beta}$, we get $\psi(0) = 0$, $\psi'(0) \neq 0$. Then, combining the inequality (3.9) of the assumption together with (3.10), we obtain that ψ is a starlike (univalent) function in U. ∎

4 Particular cases

In this section we will discuss some particular cases of Theorem 3.3 obtained for appropriate choices of the h function.

4.1 The special case $h(z) = z\exp(\lambda z)$, $|\lambda| < 1$.

Then it is easy to show that $h \in \mathcal{A}$, and for $\beta > 0$ and $|\lambda| < 1$ we have

$$-\frac{1}{\beta}\operatorname{Re} J(0,h)(z) = -\frac{1}{\beta}\operatorname{Re}\frac{\lambda z}{1+\lambda z} < -\frac{1}{\beta}\inf\left\{\operatorname{Re}\frac{\lambda z}{1+\lambda z} : z \in \mathrm{U}\right\}$$

$$\operatorname{Re}\frac{\lambda z}{1+\lambda z} > \frac{|\lambda|}{|\lambda|-1},\ z \in \mathrm{U}.$$

It follows that

$$-\frac{1}{\beta}\operatorname{Re} J(0,h)(z) < \frac{1}{\beta}\frac{|\lambda|}{1-|\lambda|},\ z \in \mathrm{U},$$

and for this special case, from Theorem 3.3 we obtain the next example:

Example 4.1. *Let $\beta > 0$ and $g_1, g_2 \in \mathcal{K}_{z\exp(\lambda z);\beta}$, where $|\lambda| < 1$. Suppose that the next two conditions are satisfied*

$$\operatorname{Re}\frac{zg_k'(z)}{g_k(z)} > \frac{1}{\beta}\frac{|\lambda|}{1-|\lambda|},\ z \in \mathrm{U}, \quad \textit{for} \quad k = 1,2.$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{z\exp(\lambda z);\beta}$ such that $(1+\lambda z)^{1/\beta} f(z)$ and $\left[\beta\int_0^z f^\beta(t)\frac{1+\lambda t}{t}\,\mathrm{d}\,t\right]^{1/\beta}$ are univalent functions in U*.*

Then,

$$(1+\lambda z)^{1/\beta} g_1(z) \prec (1+\lambda z)^{1/\beta} f(z) \prec (1+\lambda z)^{1/\beta} g_2(z)$$

implies

$$\left[\beta\int_0^z g_1^\beta(t)\frac{1+\lambda t}{t}\,\mathrm{d}\,t\right]^{1/\beta} \prec \left[\beta\int_0^z f^\beta(t)\frac{1+\lambda t}{t}\,\mathrm{d}\,t\right]^{1/\beta} \prec \left[\beta\int_0^z g_2^\beta(t)\frac{1+\lambda t}{t}\,\mathrm{d}\,t\right]^{1/\beta}.$$

Moreover, the functions $\left[\beta\int_0^z g_1^\beta(t)\frac{1+\lambda t}{t}\,\mathrm{d}\,t\right]^{1/\beta}$ and $\left[\beta\int_0^z g_2^\beta(t)\frac{1+\lambda t}{t}\,\mathrm{d}\,t\right]^{1/\beta}$ are respectively the best subordinant and the best dominant.

Remarks 4.1. 1. According to Corollary 3.1, if $f \in \mathcal{Q} \cap \mathcal{K}_{z\exp(\lambda z);\beta}$ satisfies the condition

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{\beta}\frac{|\lambda|}{1-|\lambda|},\ z \in \mathrm{U},$$

then it is not necessary to assume that $(1+\lambda z)^{1/\beta} f(z)$ and $\left[\beta\int_0^z f^\beta(t)\frac{1+\lambda t}{t}\,\mathrm{d}\,t\right]^{1/\beta}$ are univalent functions in U.

2. For the special case $\beta = 1$ and $\lambda = 0$, the right-hand side of the Example 4.1 represents a generalization of a result due to Suffridge [8]. In addition, the left-hand side generalizes Theorem 9 from [5].

4.2 The special case $h(z) = \dfrac{z}{1+\lambda z}$, $|\lambda| \leq 1$.

For this case we have $h \in \mathcal{A}$ and $J(0,h)(z) = -\dfrac{\lambda z}{1+\lambda z}$. If we denote by

$$l(\zeta) = -\frac{\zeta}{1+\zeta},\ |\zeta| < |\lambda| \leq 1,$$

then,

$$\operatorname{Re}\frac{\zeta l''(\zeta)}{l'(\zeta)} + 1 = \operatorname{Re}\frac{1-\zeta}{1+\zeta} > 0,\ |\zeta| < |\lambda| \leq 1,$$

and $l'(0) \neq 0$, which shows that l is a convex (univalent) function in U. Since $l(\overline{\zeta}) = \overline{l(\zeta)}$, it follows that $l\left(|\zeta| \leq |\lambda|\right)$ is a convex domain symmetric with respect to the real axis, hence

$$\operatorname{Re} l(\zeta) > l(|\lambda|) = -\frac{|\lambda|}{1+|\lambda|},\ |\zeta| < |\lambda| \leq 1.$$

Hence, we deduce that

$$-\frac{1}{\beta}\operatorname{Re} J(0,h)(z) < \frac{1}{\beta}\frac{|\lambda|}{1+\lambda|},\ z \in \mathrm{U},$$

and from Theorem 3.3 we have:

Example 4.2. *Let $\beta > 0$ and $g_1, g_2 \in \mathcal{K}_{z/(1+\lambda z);\beta}$, where $|\lambda| \leq 1$. Suppose that the next two conditions are satisfied*

$$\operatorname{Re}\frac{z g_k'(z)}{g_k(z)} > \frac{1}{\beta}\frac{|\lambda|}{1+|\lambda|},\ z \in \mathrm{U}, \quad \textit{for} \quad k = 1, 2.$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{z/(1+\lambda z);\beta}$ such that $\dfrac{f(z)}{(1+\lambda z)^{1/\beta}}$ and $\left[\beta\displaystyle\int_0^z \frac{f^\beta(t)}{t(1+\lambda t)}\,\mathrm{d}\,t\right]^{1/\beta}$ are univalent functions in U.

Then,

$$\frac{g_1(z)}{(1+\lambda z)^{1/\beta}} \prec \frac{f(z)}{(1+\lambda z)^{1/\beta}} \prec \frac{g_2(z)}{(1+\lambda z)^{1/\beta}}$$

implies

$$\left[\beta\int_0^z \frac{g_1^\beta(t)}{t(1+\lambda t)}\,\mathrm{d}\,t\right]^{1/\beta} \prec \left[\beta\int_0^z \frac{f^\beta(t)}{t(1+\lambda t)}\,\mathrm{d}\,t\right]^{1/\beta} \prec \left[\beta\int_0^z \frac{g_2^\beta(t)}{t(1+\lambda t)}\,\mathrm{d}\,t\right]^{1/\beta}.$$

Moreover, the functions $\left[\beta\displaystyle\int_0^z \frac{g_1^\beta(t)}{t(1+\lambda t)}\,\mathrm{d}\,t\right]^{1/\beta}$ and $\left[\beta\displaystyle\int_0^z \frac{g_2^\beta(t)}{t(1+\lambda t)}\,\mathrm{d}\,t\right]^{1/\beta}$ are respectively the best subordinant and the best dominant.

Remarks 4.2. 1. From the Corollary 3.1 we deduce that, if $f \in \mathcal{Q} \cap \mathcal{K}_{z/(1+\lambda z);\beta}$ satisfies the condition

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{1}{\beta} \frac{|\lambda|}{1+|\lambda|}, \ z \in \mathrm{U},$$

then it is not necessary to assume that $\dfrac{f(z)}{(1+\lambda z)^{1/\beta}}$ and $\left[\beta \displaystyle\int_0^z \frac{f^\beta(t)}{t(1+\lambda t)} \,\mathrm{d}\, t\right]^{1/\beta}$ are univalent functions in U.

2. For the special case $\beta = 1$ and $\lambda = 0$, the right-hand side of this Example generalizes a result due to Suffridge [8], and the left-hand side generalizes Theorem 9 from [5].

4.3 The special case $h(z) = z \exp \int_0^z \frac{e^{\lambda t}-1}{t} \,\mathrm{d}\, t$, $\lambda \in \mathbb{C}$.

We may easily show that $h \in \mathcal{A}$ and $\operatorname{Re} J(0,h)(z) = \operatorname{Re}(\lambda z) > -|\lambda|, \ z \in \mathrm{U}$. Taking in Theorem 3.3 this special case we get:

Example 4.3. *Let* $\beta > 0$ *and* $g_1, g_2 \in \mathcal{K}_{h;\beta}$, *where* $h(z) = z \exp \int_0^z \frac{e^{\lambda t}-1}{t} \,\mathrm{d}\, t$ *and* $\lambda \in \mathbb{C}$. *Suppose that the next two conditions are satisfied*

$$\operatorname{Re} \frac{zg_k'(z)}{g_k(z)} > \frac{|\lambda|}{\beta}, \ z \in \mathrm{U}, \quad \textit{for} \quad k = 1, 2.$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ *such that* $f(z)\exp(\lambda z/\beta)$ *and* $\left[\beta \displaystyle\int_0^z f^\beta(t) \frac{\exp(\lambda t)}{t} \,\mathrm{d}\, t\right]^{1/\beta}$ *are univalent functions in* U.

Then,

$$g_1(z)\exp(\lambda z/\beta) \prec f(z)\exp(\lambda z/\beta) \prec g_2(z)\exp(\lambda z/\beta)$$

implies

$$\left[\beta \int_0^z g_1^\beta(t) \frac{\exp(\lambda t)}{t} \,\mathrm{d}\, t\right]^{1/\beta} \prec \left[\beta \int_0^z f^\beta(t) \frac{\exp(\lambda t)}{t} \,\mathrm{d}\, t\right]^{1/\beta} \prec \left[\beta \int_0^z g_2^\beta(t) \frac{\exp(\lambda t)}{t} \,\mathrm{d}\, t\right]^{1/\beta}.$$

Moreover, the functions $\left[\beta \displaystyle\int_0^z g_1^\beta(t) \frac{\exp(\lambda t)}{t} \,\mathrm{d}\, t\right]^{1/\beta}$ *and* $\left[\beta \displaystyle\int_0^z g_2^\beta(t) \frac{\exp(\lambda t)}{t} \,\mathrm{d}\, t\right]^{1/\beta}$ *are respectively the best subordinant and the best dominant.*

Remarks 4.3. 1. As in the previous remarks, from Corollary 3.1 we obtain that if $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$, where $h(z) = z \exp \int_0^z \frac{e^{\lambda t}-1}{t} \,\mathrm{d}\, t$, satisfies the condition

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{|\lambda|}{\beta}, \ z \in \mathrm{U},$$

then it is not necessary to assume that $f(z)\exp(\lambda z/\beta)$ and $\left[\beta \displaystyle\int_0^z f^\beta(t) \frac{\exp(\lambda t)}{t} \,\mathrm{d}\, t\right]^{1/\beta}$ are univalent functions in U.

2. For the special case $\beta = 1$ and $\lambda = 0$, the right-hand side of the Example 4.3 extends a result of Suffridge [8]. In addition, the left-hand side is an extension of Theorem 9 from [5].

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