Regularity of solutions for a fourth order parabolic equation

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Abstract

In this paper, we study the regularity of solutions for a fourth order parabolic equation. Based on the Schauder type estimates and Campanato spaces, we prove the global existence of classical solutions.

1 Introduction

This paper concerns the study of a fourth order parabolic equation

$$\frac{\partial u}{\partial t} + \operatorname{div}[m(u)k\nabla\Delta u - |\nabla u|^{p-2}\nabla u] = 0, \quad x \in \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ and k is the positive coefficients. On the basis of physical consideration, the equation is supplemented by the zero mass flux boundary condition, the natural boundary condition

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = \frac{\partial \Delta u}{\partial n}\Big|_{\partial\Omega} = 0,$$

and initial value condition

$$u(x,0) = u_0(x), \quad x \in \Omega.$$

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The equation arises in epitaxial growth of nanoscale thin films [1, 2, 3], where $u(x,t)$ denotes the height from the surface of the film in epitaxial growth. The term $\operatorname{div}(m(u)\nabla\Delta u)$ denotes the capillarity-driven surface diffusion and the term $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the upward hopping of atoms. If the nonlinear relation $|\nabla u|^{p-2}\nabla u$ is replaced by a term of the form $f(u)\nabla u$, we obtain the well known Cahn-Hilliard equation[4, 6, 8, 9].

B. B. King, O. Stein and M. Winkler[3] studied the following equation

$$\frac{\partial u}{\partial t} + \Delta^2 u - \operatorname{div}(f(\nabla u)) = g(x,t),$$

where reasonable choice of $f(z)$ is $f(z) = |z|^{p-2}z - z$. They proved the existence, uniqueness and regularity of solutions in an appropriate function space for initial-boundary value problem.

In this paper, we study the problem for one-dimensional case. i.e.

$$\frac{\partial u}{\partial t} + D[m(u)kD^3u - |Du|^{p-2}Du] = 0, \quad (x,t) \in Q_t, p > 2, \tag{1.1}$$

with boundary condition

$$Du\Big|_{x=0,1} = D^3u\Big|_{x=0,1} = 0, \tag{1.2}$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in (0,1). \tag{1.3}$$

where $Q_t = (0,1) \times (0,t)$, $D = \dfrac{\partial}{\partial x}$.

Our main purpose is to establish the global existence of classical solutions under much general assumptions. The main difficulties for treating the problem (1.1)–(1.3) are caused by the nonlinearity of the principal part and the lack of maximum principle. Due to the nonlinearity of the principal part, there are more difficulties in establishing the global existence of classical solutions. Our method for investigating the regularity of solutions is based on uniform Schauder type estimates for local in time solutions, which are relatively less used for such kind of parabolic equations of fourth order. Our approach lies in the combination of the energy techniques with some methods based on the framework of Campanato spaces.

Now, we state the main results in this paper.

Theorem 1.1. *Assume that*

$(H1)$ $$m(s) \in C^{2+\alpha}(\mathbb{R}), \quad 0 < m(s),$$

$(H2)$ $$u_0 \in C^{4+\alpha}, \quad D^iu_0(0) = D^iu_0(1) = 0 \ (i = 1,3),$$

where C_1, C_2, q are positive constants. Then the problem (1.1)-(1.3) admits a unique classical solution $u \in C^{4+\alpha,1+\alpha/4}(\bar{Q}_T)$.

This paper is organized as follows. We first present a key step for the priori estimates on the Hölder norm of solutions in Section 2, and then give the proof of our main theorem subsequently in Section 3.

2 Hölder Estimates

As an important step, in this section, we give the Hölder norm estimate on the local in time solutions. From the classical approach, it is not difficult to conclude that the problem admits a unique classical solution local in time. So, it is sufficient to make a priori estimates.

Proposition 2.1. *Assume that (H1)–(H2) holds, and u is a smooth solution of the problem. Then there exists a constant C depending only on the known quantities, such that for any $(x_1, t_1), (x_2, t_2) \in Q_T$ and some $0 < \alpha < 1$,*

$$|u(x_1, t_1) - u(x_2, t_2)| \leq C(|t_1 - t_2|^{\alpha/4} + |x_1 - x_2|^{\alpha}).$$

Proof. Multiplying both sides of the equation by D^2u and then integrating resulting relation with respect to x over $(0, 1)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 (Du)^2 dx + k\int_0^1 m(u)(D^3u)^2 dx \\ = &-\int_0^1 D[|Du|^{p-2}Du]D^2u dx, \end{aligned}$$

that is

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (Du)^2 dx + k\int_0^1 m(u)(D^3u)^2 dx = -(p-1)\int_0^1 |Du|^{p-2}(D^2u)^2 dx,$$

hence

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (Du)^2 dx + k\int_0^1 m(u)(D^3u)^2 dx \leq 0$$

we obtain

$$\sup_{0<t<T}\int_0^1 (Du)^2 dx \leq C. \tag{2.1}$$

$$\int\int_{Q_T} m(u)(D^3u)^2 dx \leq C. \tag{2.2}$$

The integration of (1.1) over the interval $(0, 1)$ yields $\displaystyle\int_0^1 \frac{\partial u}{\partial t} dx = 0$, hence we obtain

$$\int_0^1 u(x, t)dx = \int_0^1 u_0(x)dx.$$

Applying the mean value theorem, we see that for some $x_t^* \in (0, 1)$

$$u(x_t^*, t) = \int_0^1 u_0(x)dx = M.$$

Therefore

$$\begin{aligned} |u(x, t)| &\leq |u(x, t) - u(x_t^*, t)| + |u(x_t^*, t)| \\ &\leq \left|\int_{x_t^*}^x Du(t, y)dy\right| + M. \end{aligned}$$

Taking this into account and using (2.2), it follows that

$$\sup_{Q_T} |u(x,t)| \le C. \tag{2.3}$$

By (2.1), we have

$$|u(x_1,t) - u(x_2,t)| \le C|x_1 - x_2|^\alpha, \quad 0 < \alpha < 1. \tag{2.4}$$

Integrating the equation (1.1) with respect to x over $(y, y + (\Delta t)^{1/4}) \times (t_1, t_2)$, where $0 < t_1 < t_2 < T$, $\Delta t = t_2 - t_1$, we see that

$$\begin{aligned}
&\int_y^{y+(\Delta t)^{1/4}} [u(z,t_2) - u(z,t_1)]dz \\
=& -\int_{t_1}^{t_2} [(m(u(y',s))kD^3u(y',s) - |Du|^{p-2}Du(u(y',s))) \\
&- (m(u(y,s))kD^3u(y,s) - |Du|^{p-2}Du(y,s))]ds.
\end{aligned} \tag{2.5}$$

Set

$$\begin{aligned}
N(s,y) =&(m(u(y',s))kD^3u(y',s) - |Du|^{p-2}Du(y',s)) \\
&- (m(u(y,s))kD^3u(y,s) - |Du|^{p-2}Du(y,s)),
\end{aligned}$$

where $y' = y + (\Delta t)^{1/4}$.

Then (2.5) is converted into

$$\begin{aligned}
&(\Delta t)^{1/4} \int_0^1 [u(y + \theta(\Delta t)^{1/4}, t_2) - u(y + \theta(\Delta t)^{1/4}, t_1))]d\theta \\
=& -\int_{t_1}^{t_2} N(s,y)ds.
\end{aligned}$$

Integrating the above equality with respect to y over $(x, x + (\Delta t)^{1/4})$, we get

$$(\Delta t)^{1/2}(u(x^*,t_2) - u(x^*,t_1)) = -\int_{t_1}^{t_2}\int_x^{x+(\Delta t)^{1/4}} N(s,y)dyds.$$

Here, we have used the mean value theorem, where $x^* = y^* + \theta^*(\Delta t)^{1/4}$, $y^* \in (x, x + (\Delta t)^{1/4})$, $\theta \in (0,1)$. Hence by Hölder inequality and (2.2),(2.3),(2.4), we get

$$\Big|u(x^*,t_2) - u(x^*,t_1)\Big| \le C(\Delta t)^{\alpha/4}, \quad 0 < \alpha < 1. \tag{2.6}$$

The proof is complete. ∎

3 Proof of the result

In this section, we prove the theorem that there exists a classical solution of the problem (1.1)–(1.3), under our assumptions on m and u_0.

Now, we consider the following linear problem

$$\frac{\partial u}{\partial t} + D^2(a(x,t)D^2u) = D^2 f, \tag{3.1}$$

$$u\Big|_{x=0,1} = D^2u\Big|_{x=0,1} = 0, \tag{3.2}$$

$$u(x,0) = 0 \tag{3.3}$$

Here we do not restrict the smoothness of the given functions $a(x,t)$ and $f(x,t)$, but simply assume that they are sufficiently smooth. Our main purpose is to find the relation between the Hölder norm of the solution u and $a(x,t)$, $f(x,t)$.

The crucial step is to establish the estimates on the Hölder norm of u. Let $(t_0, x_0) \in (0,T) \times (0,1)$ be fixed and define

$$\varphi(\rho) = \int\int_{S_\rho} \Big(|u - u_\rho|^2 + \rho^4 |D^2u|^2 \Big)\, dtdx, \quad (\rho > 0)$$

where

$$S_\rho = (t_0 - \rho^4, t_0 + \rho^4) \times B_\rho(x_0), \quad u_\rho = \frac{1}{|S_\rho|}\int\int_{S_\rho} u\, dtdx$$

and $B_\rho(x_0) = (x_0 - \rho, x_0 + \rho)$.

Let u be the solution of the problem (3.1),(3.2),(3.3). We split u on S_R into $u = u_1 + u_2$, where u_1 is the solution of the problem

$$\frac{\partial u_1}{\partial t} + a(t_0, x_0) D^4 u_1 = 0, \quad (t,x) \in S_R \tag{3.4}$$

$$u_1 = u, \quad D^2u_1 = D^2u, \quad (t,x) \in (t_0 - R^4, t_0 + R^4) \times \partial B_R(x_0) \tag{3.5}$$

$$u_1 = u, \quad t = t_0 - R^4, \;\; x \in B_R(x_0), \tag{3.6}$$

and u_2 solves the problem

$$\frac{\partial u_2}{\partial t} + a(t_0,x_0)D^4u_2 = D^2\Big[(a(t_0,x_0) - a(t,x))D^2u\Big] + D^2 f, \;\; (t,x) \in S_R, \tag{3.7}$$

$$u_2 = 0, \quad D^2u_2 = 0, \quad (t,x) \in (t_0 - R^4, t_0 + R^4) \times \partial B_R(x_0), \tag{3.8}$$

$$u_2 = 0, \quad t = t_0 - R^4, \;\; x \in B_R(x_0). \tag{3.9}$$

By classical linear theory, the above decomposition is uniquely determined by u.

We need several lemmas on u_1 and u_2.

Lemma 3.1. *Assume that*

$$|a(t,x) - a(t_0,x_0)| \le a_\sigma \Big(|t - t_0|^{\sigma/4} + |x - x_0|^\sigma \Big), t \in (t_0 - R^4, t_0 + R^4), \quad x \in B_R(x_0).$$

Then

$$\sup_{(t_0-R^4,t_0+R^4)}\int_{B_R(x_0)} u_2^2(t,x)\,dx+\int\int_{S_R}(D^2u_2)^2\,dtdx$$
$$\le CR^{2\sigma}\int\int_{S_R}(D^2u)^2\,dtdx+C\sup_{S_R}|f|^2R^5.$$

Proof. Multiply the equation (3.7) by u_2 and integrate the resulting relation over $(t_0-R^4,t)\times B_R(x_0)$. Integrating by parts, we have

$$\frac{1}{2}\int_{B_R}u_2^2(x,t)dx+a(x_0,t_0)\int_{t_0-R^4}^t ds\int_{B_R}(D^2u_2)^2dx$$
$$=\int_{t_0-R^4}^t ds\int_{B_R}[a(t_0,x_0)-a(t,x)]D^2uD^2u_2dx+\int_{t_0-R^4}^t ds\int_{B_R}fD^2u_2dx.$$

Noticing that

$$\left|\int_{t_0-R^4}^t ds\int_{B_R}[a(t_0,x_0)-a(t,x)]D^2uD^2u_2dx\right|$$
$$\le\varepsilon\int\int_{S_R}(D^2u_2)^2dsdx+C_\varepsilon a_\sigma^2R^{2\sigma}\int\int_{S_R}(D^2u)^2dsdx,$$

and

$$\left|\int_{t_0-R^4}^t ds\int_{B_R}fD^2u_2dx\right|\le\varepsilon\int\int_{S_R}(D^2u_2)^2dsdx+C_\varepsilon R^5\sup|f|^2,$$

hence we obtain the estimate and the proof is complete. ∎

Lemma 3.2. *For any* (t_1,x_1), $(t_2,x_2)\in S_\rho$,

$$\frac{|u_1(t_1,x_1)-u_1(t_2,x_2)|^2}{|t_1-t_2|^{1/4}+|x_1-x_2|}$$
$$\le C\sup_{(t_0-\rho^4,t_0+\rho^4)}\int_{B_\rho(x_0)}(Du_1(t,x))^2\,dx+C\int\int_{S_\rho}(D^3u_1)^2\,dtdx.$$

Proof. From the Sobolev embedding theorem, we have for any $(x_1,t),(x_2,t)\in S_\rho$,

$$\frac{|u_1(t,x_1)-u_1(t,x_2)|^2}{|x_1-x_2|}\le C\sup_{(t_0-\rho^4,t_0+\rho^4)}\int_{B_\rho(x_0)}(Du_1(t,x))^2\,dx. \tag{3.10}$$

Integrating the equation (3.4) with respect to x over $(y,y+(\Delta t)^{1/4})\times(t_1,t_2)$, where $0<t_1<t_2<T$, $\Delta t=t_2-t_1$, we see that

$$\int_y^{y+(\Delta t)^{1/4}}[u_1(z,t_2)-u_1(z,t_1)]dz+a(x_0,t_0)\int_{t_1}^{t_2}[D^3u_1(y',s)-D^3u_1(y,s)]ds=0,$$

where $y'=y+(\Delta t)^{1/4}$.

That is

$$(\Delta t)^{1/4}\int_0^1 u_1(y+\theta(\Delta t)^{1/4},t_2)-u_1(y+\theta(\Delta t)^{1/4},t_1))d\theta$$
$$+a(x_0,t_0)\int_{t_1}^{t_2}[D^3u_1(y+(\Delta t)^{1/4},s)-D^3u_1(y,s)]ds=0.$$

Integrating the above equality with respect to y over $(x,x+(\Delta t)^{1/4})$, we get

$$\begin{aligned}&(\Delta t)^{1/2}(u_1(x^*,t_2)-u_1(x^*,t_1))\\ =&a(x_0,t_0)\int_{t_1}^{t_2}\int_x^{x+(\Delta t)^{1/4}}[D^3u_1(y+(\Delta t)^{1/4},s)-D^3u_1(y,s)]ds.\end{aligned}$$

Hence,

$$\Big|u_1(x^*,t_2)-u_1(x^*,t_1)|\le C|t_1-t_2|^{1/4}\int\int_{S_\rho}(D^3u_1)^2\,dtdx$$

where $x^*=y^*+\theta^*(\Delta t)^{1/4}$, $y^*\in(x,x+(\Delta t)^{1/4})$, $\theta\in(0,1)$. This and (3.10) yields the desired conclusion and the proof is complete. ∎

Lemma 3.3. *(Caccioppoli type inequality)*

$$\begin{aligned}&\sup_{(t_0-(R/2)^4,t_0+(R/2)^4)}\int_{B_{R/2}(x_0)}|u_1(t,x)-(u_1)_R|^2\,dx+\int\int_{S_{R/2}}|D^2u_1|^2\,dtdx\\ \le&\frac{C}{R^4}\int\int_{S_R}|u_1(t,x)-(u_1)_R|^2\,dtdx\\ &\sup_{(t_0-(R/2)^4,t_0+(R/2)^4)}\int_{B_{R/2}(x_0)}|Du_1|^2\,dx+\int\int_{S_{R/2}}|D^3u_1|^2\,dtdx\\ \le&\frac{C}{R^4}\int\int_{S_R}|Du_1|^2\,dtdx\le\frac{C}{R^6}\int\int_{S_{2R}}|u_1(t,x)-(u_1)_R|^2\,dtdx.\end{aligned}$$

Lemma 3.4. *Assume that*

$$\begin{aligned}|a(t,x)-a(t_0,x_0)|\le&a_\sigma\Big(|t-t_0|^{\sigma/4}+|x-x_0|^\sigma\Big),\\ &t\in(t_0-R^4,t_0+R^4),\quad x\in B_R(x_0).\end{aligned}$$

Then for any $\rho\in(0,R)$,

$$\begin{aligned}&\frac{1}{\rho^6}\int\int_{S_\rho}(|u_1-(u_1)_\rho|^2+\rho^4|D^2u_1|^2)\,dtdx\\ \le&\frac{C}{R^6}\int\int_{S_R}(|u_1-(u_1)_R|^2+R^4|D^3u_1|^2)\,dtdx.\end{aligned}$$

Lemma 3.5. *For $\lambda \in (5,6)$,*

$$\varphi(\rho) \le C_\lambda \left(\varphi(R_0) + \sup_{S_{R_0}} |f|^2\right) \rho^\lambda, \qquad \rho \le R_0 = min\Big(dist(x_0, \partial\Omega), t_0^{1/4}\Big),$$

where C_λ depends on λ, R_0 and the known quantities.

The proof of lemma 3.3–3.5 is quite similar to the corresponding part in [5], and we omit the details.

Similar to the discussion about the Campanato spaces in [7], we first conclude from Lemma 3.5 that

$$\frac{|u(t_1,x_1) - u(t_2,x_2)|}{|t_1 - t_2|^{(\lambda-5)/4} + |x_1 - x_2|^{(\lambda-5)}} \le C\left(1 + \sup_{S_{R_0}} |f|\right) \tag{3.11}$$

Proof of Theorem 1.1. The key estimate is the Hölder estimate for Du, which can be obtained by the above result. In face, let $w = Du - Du_0$ satisfies

$$\frac{\partial w}{\partial t} + D^2(m(u)D^2 w) = D^2 f,$$
$$w\Big|_{x=0,1} = D^2 w\Big|_{x=0,1} = 0,$$
$$w(x,0) = 0,$$

where

$$f = -km(u)D^3 u_0 + |Du|^{p-2} Du.$$

Hence by (2.3), (3.11) and using the interpolation inequality, we thus obtain

$$|Du(x_1,t_1) - Du(x_2,t_2)| \le C(|x_1 - x_2|^{\alpha/2} + |t_1 - t_2|^{\alpha/8}).$$

The conclusion follows immediately from the classical theory, since we can transform the equation (1.1) into the form

$$\frac{\partial u}{\partial t} + a_1(t,x)D^4 u + b_1(t,x)D^3 u + a_2(t,x)D^2 u = 0,$$

where the Hölder norms on

$$a_1(t,x) = km(u(t,x)), \quad b_1(t,x) = km'(u(t,x))Du(t,x)$$
$$a_2(t,x) = -(p-1)|Du|^{p-2},$$

have been estimated in the above discussion. The proof is complete. ∎

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