A New Hilbert-Type Inequality

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Abstract

This paper deals with a new Hilbert- type inequality by introducing a parameter and the Beta function. As applications, the equivalent form, the reversions and some particular results are considered. All the theorems provide some new estimates on this type of inequalities.

1 Introduction

If $a_n, b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then the famous Hilbert's inequality is given by

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left\{\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2\right\}^{1/2}, \tag{1}$$

where the constant factor π is the best possible (see [1]). Inequality (1) was generalized by Hardy-Riesz [2] in 1925 with (p,q)- parameter as:

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}\left\{\sum_{n=1}^{\infty} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} b_n^q\right\}^{\frac{1}{q}}, \tag{2}$$

where the constant factor $\pi/\sin(\pi/p)$ is the best possible. When p=q=2, (2) reduces to (1). Inequality (2) is named of Hardy-Hilbert's inequality, which is important in

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analysis and its applications (see [3]). In 1997- 1998, by estimating the weight coefficient, Yang and Gao [4,5] gave a strengthened version of (2) as:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{\frac{1}{p}}}\right] a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty}\left[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{\frac{1}{q}}}\right] b_n^q\right\}^{\frac{1}{q}}, \quad (3)$$

where γ is Euler constant, and $1-\gamma = 0.42278433^{+}$. In recent years, by introducing a parameter λ and the Beta function, Yang [6] gave a generalization of (2) as:

If $2-\min\{p,q\} < \lambda \leq 2, 0 < \sum_{n=1}^{\infty} n^{1-\lambda}a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{1-\lambda}b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < k_{\lambda}(p)\left\{\sum_{n=1}^{\infty} n^{1-\lambda}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}, \quad (4)$$

where the constant factor $k_{\lambda}(p)(= B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}))$ is the best possible(B(u,v) is the Beta function). And the equivalent form was built as (see [7]):

$$\sum_{n=1}^{\infty} n^{(p-1)(\lambda-1)}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m+n)^{\lambda}}\right]^p < [k_{\lambda}(p)]^p \sum_{n=1}^{\infty} n^{1-\lambda}a_n^p, \quad (5)$$

where the constant factor $[k_{\lambda}(p)]^p(= [B(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q})]^p)$ is the best possible. When $\lambda = 1$, inequality (4) reduces to(2), and (5) reduces to the equivalent form of (2) as:

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p < [\frac{\pi}{\sin(\pi/p)}]^p \sum_{n=1}^{\infty} a_n^p. \quad (6)$$

For $p = q = 2$, in (4), one has $0 < \lambda \leq 2$ and

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\frac{\lambda}{2}, \frac{\lambda}{2})\left\{\sum_{n=1}^{\infty} n^{1-\lambda}a_n^2 \sum_{n=1}^{\infty} n^{1-\lambda}b_n^2\right\}^{\frac{1}{2}}. \quad (7)$$

It is obvious that inequality (7) is and extension of (1). We call (7) the extended Hilbert's inequality. Yang [8] proved that (7) is still valid for $0 < \lambda \leq 4$. In 2003, Yang et al. [9] provided and extensive account of the above results. Recently, Yang [10] gave a reversion of the integral analogue of (4) following the assumption of $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$ and $2-p < \lambda < 2-q$.

The main objective of this paper is to build a new Hilbert-type inequality by introducing a parameter λ and the Beta function, which is related to the double series as

$$\sum_{n=n_0}^{\infty}\sum_{m=n_0}^{\infty}\frac{a_m b_n}{(1+u(m)u(n))^{\lambda}} \ (\lambda > 2-\min\{p,q\}).$$

As applications, the equivalent form, the reversions and some particular results are considered. All the theorems provide some new estimates on this type of inequalities.

2 Main results and the equivalent form

First, we need the formula of the Beta function as (cf. Wang et al. [11]):

$$B(u,v)=\int_0^\infty \frac{1}{(1+t)^{u+v}}t^{u-1}dt=B(v,u)\ (u,v>0). \tag{8}$$

LEMMA 2.1. If $p>1, \frac{1}{p}+\frac{1}{q}=1, u(t)$ is a differentiable strict increasing function in (n_0-1,∞) $(n_0\in N)$, such that $u((n_0-1)+)=0$ and $u(\infty)=\infty$, and for $r=p,q,\lambda>2-r,(u(t))^{\frac{\lambda-2}{r}}u'(t)$ $(t\in(n_0-1,\infty))$ is decreasing, define the weight function $\omega_\lambda(r,m)$ as

$$\omega_\lambda(r,m):=\sum_{n=n_0}^\infty \frac{1}{(1+u(m)u(n))^\lambda}\left(\frac{u(m)}{u(n)}\right)^{\frac{2-\lambda}{r}}u'(n)\ (m\in N, m\ge n_0). \tag{9}$$

Then, one has

$$\omega_\lambda(r,m)<B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)(u(m))^{\frac{2(2-\lambda)}{r}-1}\ (r=p,q;m\ge n_0). \tag{10}$$

Proof. By the assumption of the lemma, since $\lambda>2-\min\{p,q\}\ge 0$ and $(u(t))^{\frac{\lambda-2}{r}}u'(t)$ $(t\in(n_0-1,\infty))$ is decreasing, one has

$$\omega_\lambda(r,m)<\int_{n_0-1}^\infty \frac{1}{(1+u(m)u(y))^\lambda}\left(\frac{u(m)}{u(y)}\right)^{\frac{2-\lambda}{r}}u'(y)\ dy.$$

Setting $t=u(m)u(y)$ in the above integral, one finds

$$\begin{aligned}\omega_\lambda(r,m) &< \int_0^\infty \frac{1}{(1+t)^\lambda}\left(\frac{u^2(m)}{t}\right)^{\frac{2-\lambda}{r}}\frac{1}{u(m)}dt\\ &= \int_0^\infty \frac{1}{(1+t)^\lambda}t^{\frac{r+\lambda-2}{r}-1}dt\ (u(m))^{\frac{2(2-\lambda)}{r}-1}.\end{aligned}$$

In view of $\frac{r+\lambda-2}{r}>0$ $(r=p,q)$ and $\frac{p+\lambda-2}{p}+\frac{q+\lambda-2}{q}=\lambda$, then by (8), one has (10). The lemma is proved.

THEOREM 2.2. If $p>1, \frac{1}{p}+\frac{1}{q}=1,$ $u(t)$ is a differentiable strict increasing function in (n_0-1,∞) $(n_0\in N)$, such that $u((n_0-1)+)=0$ and $u(\infty)=\infty$, and for $r=p,q,$ $\lambda>2-r,$ $(u(t))^{\frac{\lambda-2}{r}}u'(t)$ $(t\in(n_0-1,\infty))$ is decreasing; a_n , $b_n\ge 0$, satisfy

$$0<\sum_{n=n_0}^\infty \frac{(u(n))^{\frac{2}{q}(2-\lambda)-1}}{(u'(n))^{p-1}}a_n^p<\infty \quad \text{and} \quad 0<\sum_{n=n_0}^\infty \frac{(u(n))^{\frac{2}{p}(2-\lambda)-1}}{(u'(n))^{q-1}}b_n^q<\infty,$$

then one has

$$\begin{aligned}&\sum_{n=n_0}^\infty\sum_{m=n_0}^\infty \frac{a_m b_n}{(1+u(m)u(n))^\lambda}\\ &< k_\lambda(p)\left\{\sum_{n=n_0}^\infty \frac{(u(n))^{\frac{2}{q}(2-\lambda)-1}}{(u'(n))^{p-1}}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=n_0}^\infty \frac{(u(n))^{\frac{2}{p}(2-\lambda)-1}}{(u'(n))^{q-1}}b_n^q\right\}^{\frac{1}{q}},\end{aligned} \tag{11}$$

where $k_\lambda(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$. In particular,

(i) setting $u(t) = t^\alpha$ $(\alpha > 0; t \in (0,\infty))$, then for $2 - r < \lambda \le 2 + r(\frac{1}{\alpha} - 1)$ $(r = p, q)$, one has

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{[1+(mn)^\alpha]^\lambda} < \frac{k_\lambda(p)}{\alpha}\left\{\sum_{n=1}^{\infty} n^{\frac{2\alpha(2-\lambda)+p}{q}-p\alpha} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{\frac{2\alpha(2-\lambda)+q}{p}-q\alpha} b_n^q\right\}^{\frac{1}{q}}; \quad (12)$$

(ii) setting $u(t) = \ln t$ $(t \in (1,\infty))$, then for $2 - \min\{p, q\} < \lambda \le 2$, one has

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{(1+\ln m \ln n)^\lambda} < k_\lambda(p)\left\{\sum_{n=2}^{\infty}\frac{n^{p-1}a_n^p}{(\ln n)^{1-\frac{2}{q}(2-\lambda)}}\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty}\frac{n^{q-1}\,b_n^q}{(\ln n)^{1-\frac{2}{p}(2-\lambda)}}\right\}^{\frac{1}{q}}. \quad (13)$$

Proof. By *Hölder's* inequality with weight (see Kuang [12]) and (9) , one has

$$\begin{aligned}
&\sum_{n=n_0}^{\infty}\sum_{m=n_0}^{\infty}\frac{a_m b_n}{(1+u(m)u(n))^\lambda} \\
&\sum_{n=n_0}^{\infty}\sum_{m=n_0}^{\infty}\frac{1}{(1+u(m)u(n))^\lambda}\left[\left(\frac{u(m)}{u(n)}\right)^{\frac{2-\lambda}{pq}}\frac{(u'(n))^{1/p}}{(u'(m))^{1/q}}a_m\right] \\
&\times\left[\left(\frac{u(n)}{u(m)}\right)^{\frac{2-\lambda}{pq}}\frac{(u'(m))^{1/q}}{(u'(n))^{1/p}}b_n\right] \\
\le\ &\left\{\sum_{m=n_0}^{\infty}\sum_{n=n_0}^{\infty}\frac{1}{(1+u(m)u(n))^\lambda}\left[\left(\frac{u(m)}{u(n)}\right)^{\frac{2-\lambda}{q}}\frac{u'(n)}{(u'(m))^{p-1}}a_m^p\right]\right\}^{\frac{1}{p}} \\
&\times\left\{\sum_{n=n_0}^{\infty}\sum_{m=n_0}^{\infty}\frac{1}{(1+u(m)u(n))^\lambda}\left[\left(\frac{u(n)}{u(m)}\right)^{\frac{2-\lambda}{p}}\frac{u'(m)}{(u'(n))^{q-1}}b_n^q\right]\right\}^{\frac{1}{q}} \\
=\ &\left\{\sum_{m=n_0}^{\infty}\frac{\omega_\lambda(q,m)}{(u'(m))^{p-1}}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=n_0}^{\infty}\frac{\omega_\lambda(p,n)}{(u'(n))^{q-1}}b_n^q\right\}^{\frac{1}{q}}. \quad (14)
\end{aligned}$$

Hence by (10), one has (11). The theorem is proved.

THEOREM 2.3. If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $u(t)$ is a differentiable strict increasing function in $(n_0 - 1, \infty)$ $(n_0 \in N)$, such that $u((n_0 - 1)+) = 0$ and $u(\infty) = \infty$, and for $r = p, q$, $\lambda > 2 - r$, $(u(t))^{\frac{\lambda-2}{r}}u'(t)$ $(t \in (n_0 - 1, \infty))$ is decreasing; $a_n \ge 0$, satisfy

$$0 < \sum_{n=n_0}^{\infty}\frac{(u(n))^{\frac{2}{q}(2-\lambda)-1}}{(u'(n))^{p-1}}a_n^p < \infty,$$

then, one has the equivalent form of (11) as

$$\sum_{n=n_0}^{\infty}\frac{u'(n)}{(u(n))^{\frac{2(2-\lambda)-p}{q}}}\left[\sum_{m=n_0}^{\infty}\frac{a_m}{(1+u(m)u(n))^\lambda}\right]^p < [k_\lambda(p)]^p\sum_{n=n_0}^{\infty}\frac{(u(n))^{\frac{2}{q}(2-\lambda)-1}}{(u'(n))^{p-1}}a_n^p, \quad (15)$$

where $[k_\lambda(p)]^p = [B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)]^p$. In particular,

(i) setting $u(t) = t^{\alpha}$ ($\alpha > 0; t \in (0, \infty)$), then for $2 - r < \lambda \leq 2 + r(\frac{1}{\alpha} - 1)$ ($r = p, q$), one has the equivalent form of (12) as

$$\sum_{n=1}^{\infty} n^{\alpha[p - \frac{2(2-\lambda)}{q}]-1} \left[\sum_{m=1}^{\infty} \frac{a_m}{[1+(mn)^{\alpha}]^{\lambda}}\right]^p < \left[\frac{k_\lambda(p)}{\alpha}\right]^p \sum_{n=1}^{\infty} n^{\frac{2\alpha(2-\lambda)+p}{q} - p\alpha} a_n^p; \tag{16}$$

(ii) setting $u(t) = \ln t$ ($t \in (1, \infty)$), then for $2 - \min\{p, q\} < \lambda \leq 2$, one has the equivalent form of (13) as

$$\sum_{n=2}^{\infty} \frac{(\ln n)^{\frac{p-2(2-\lambda)}{q}}}{n} \left[\sum_{m=2}^{\infty} \frac{a_m}{(1+\ln m \ln n)^{\lambda}}\right]^p < [k_\lambda(p)]^p \sum_{n=2}^{\infty} \frac{n^{p-1}}{(\ln n)^{1-\frac{2}{q}(2-\lambda)}} a_n^p. \tag{17}$$

Proof. Set b_n as

$$b_n := \frac{u'(n)}{(u(n))^{\frac{2(2-\lambda)-p}{q}}} \left[\sum_{m=n_0}^{\infty} \frac{a_m}{(1+u(m)u(n))^{\lambda}}\right]^{p-1},$$

and use (11) to obtain

$$\begin{aligned} 0 &< \sum_{n=n_0}^{\infty} \frac{(u(n))^{\frac{2}{p}(2-\lambda)-1}}{(u'(n))^{q-1}} b_n^q = \sum_{n=n_0}^{\infty} \frac{u'(n)}{(u(n))^{\frac{2(2-\lambda)-p}{q}}} \left[\sum_{m=n_0}^{\infty} \frac{a_m}{(1+u(m)u(n))^{\lambda}}\right]^p \\ &= \sum_{n=n_0}^{\infty} \sum_{m=n_0}^{\infty} \frac{a_m b_n}{(1+u(m)u(n))^{\lambda}} \\ &\leq k_\lambda(p) \left\{\sum_{n=n_0}^{\infty} \frac{(u(n))^{\frac{2}{q}(2-\lambda)-1}}{(u'(n))^{p-1}} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=n_0}^{\infty} \frac{(u(n))^{\frac{2}{p}(2-\lambda)-1}}{(u'(n))^{q-1}} b_n^q\right\}^{\frac{1}{q}}; \end{aligned} \tag{18}$$

$$\begin{aligned} \left\{\sum_{n=n_0}^{\infty} \frac{(u(n))^{\frac{2}{p}(2-\lambda)-1}}{(u'(n))^{q-1}} b_n^q\right\}^{\frac{1}{p}} &= \left\{\sum_{n=n_0}^{\infty} \frac{u'(n)}{(u(n))^{\frac{2(2-\lambda)-p}{q}}} \left[\sum_{m=n_0}^{\infty} \frac{a_m}{(1+u(m)u(n))^{\lambda}}\right]^p\right\}^{\frac{1}{p}} \\ &\leq k_\lambda(p) \left\{\sum_{n=n_0}^{\infty} \frac{(u(n))^{\frac{2}{q}(2-\lambda)-1}}{(u'(n))^{p-1}} a_n^p\right\}^{\frac{1}{p}} < \infty. \end{aligned} \tag{19}$$

It follows that (18) takes the form of strict inequality by using (11); so does (19). Then (15) holds.

On the other hand, suppose that (15) holds. By *Hölder's* inequality, one has

$$\begin{aligned} &\sum_{n=n_0}^{\infty} \sum_{m=n_0}^{\infty} \frac{a_m b_n}{(1+u(m)u(n))^{\lambda}} \\ &= \sum_{n=n_0}^{\infty} \left\{\left[\frac{(u'(n))^{q-1}}{(u(n))^{\frac{2(2-\lambda)}{p}-1}}\right]^{\frac{1}{q}} \sum_{m=n_0}^{\infty} \frac{a_m}{(1+u(m)u(n))^{\lambda}}\right\} \left\{\left[\frac{(u(n))^{\frac{2(2-\lambda)}{p}-1}}{(u'(n))^{q-1}}\right]^{\frac{1}{q}} b_n\right\} \\ &\leq \left\{\sum_{n=n_0}^{\infty} \frac{u'(n)}{(u(n))^{\frac{2(2-\lambda)-p}{q}}} \left[\sum_{m=n_0}^{\infty} \frac{a_m}{(1+u(m)u(n))^{\lambda}}\right]^p\right\}^{\frac{1}{p}} \left\{\sum_{n=n_0}^{\infty} \frac{(u(n))^{\frac{2}{p}(2-\lambda)-1}}{(u'(n))^{q-1}} b_n^q\right\}^{\frac{1}{q}} \end{aligned} \tag{20}$$

Then by (15), one has (11). Hence (15) and (11) are equivalent. The theorem is proved.

REMARK 2. 4. For $\alpha = \lambda = 1$ in (12) and (16), one has the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{1+mn} < \frac{\pi}{\sin(\frac{\pi}{p})}\left\{\sum_{n=1}^{\infty} n^{\frac{2}{q}-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{\frac{2}{p}-1}b_n^q\right\}^{\frac{1}{q}}; \tag{21}$$

$$\sum_{n=1}^{\infty} n^{p-\frac{2}{q}-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{1+mn}\right)^p < \left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^p \sum_{n=1}^{\infty} n^{\frac{2}{q}-1}a_n^p, \tag{22}$$

which are similar to (2) and (6).

3 Some reversions

THEOREM 3.1. If $0 < p < 1, \frac{1}{p}+\frac{1}{q} = 1,\ u(t)$ is a differentiable strict increasing function in (n_0-1,∞) $(n_0 \in N)$, such that $u((n_0-1)+) = 0$ and $u(\infty) = \infty$, and $u'(t)$ $(t \in (n_0-1,\infty))$ is decreasing; $a_n,\ b_n \geq 0$, satisfy

$$0 < \sum_{n=n_0}^{\infty}\frac{(u'(n))^{1-p}}{u(n)}a_n^p < \infty \quad \text{and} \quad 0 < \sum_{n=n_0}^{\infty}\frac{(u'(n))^{1-q}}{u(n)}b_n^q < \infty,$$

then one has

$$\sum_{n=n_0}^{\infty}\sum_{m=n_0}^{\infty}\frac{a_m b_n}{(1+u(m)u(n))^2} > \left\{\sum_{n=n_0}^{\infty}\frac{(u'(n))^{1-p}a_n^p}{(1+u(n_0)u(n))u(n)}\right\}^{\frac{1}{p}}\left\{\sum_{n=n_0}^{\infty}\frac{(u'(n))^{1-q}b_n^q}{u(n)}\right\}^{\frac{1}{q}}. \tag{23}$$

In particular,

(i) setting $u(t) = t^{\alpha}$ $(0 < \alpha \leq 1; t \in (0,\infty))$, one has

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{[1+(mn)^{\alpha}]^2} > \frac{1}{\alpha}\left\{\sum_{n=1}^{\infty}\frac{a_n^p}{(1+n^{\alpha})n^{p(\alpha-1)+1}}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{b_n^q}{n^{q(\alpha-1)+1}}\right\}^{\frac{1}{q}}; \tag{24}$$

(ii) setting $u(t) = \ln t$ $(t \in (1,\infty))$, one has

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{(1+\ln m \ln n)^2} > \left\{\sum_{n=2}^{\infty}\frac{n^{p-1}a_n^p}{(1+\ln 2\ln n)\ln n}\right\}^{\frac{1}{p}}\left\{\sum_{n=2}^{\infty}\frac{n^{q-1}b_n^q}{\ln n}\right\}^{\frac{1}{q}}. \tag{25}$$

Proof. By the reverse *Hölder's* inequality with weight (see [12]), using the reverse (14) for $\lambda = 2$, one has

$$\sum_{n=n_0}^{\infty}\sum_{m=n_0}^{\infty}\frac{a_m b_n}{(1+u(m)u(n))^2} \geq \left\{\sum_{m=n_0}^{\infty}\frac{\omega(m)}{(u'(m))^{p-1}}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=n_0}^{\infty}\frac{\omega(n)}{(u'(n))^{q-1}}b_n^q\right\}^{\frac{1}{q}}, \tag{26}$$

where $\omega(m) := \omega_2(r,m) = \sum_{n=n_0}^{\infty}\frac{1}{(1+u(m)u(n))^2}u'(n)$ $(r = p,q)$. Since $u'(t)$ $(t \in (n_0-1,\infty))$ is decreasing, one has

$$\begin{aligned}\frac{1}{u(m)}\left(\frac{1}{1+u(n_0)u(m)}\right) &= \int_{n_0}^{\infty}\frac{u'(t)}{(1+u(m)u(t))^2}dt<\omega(m)\\ &< \int_{n_0-1}^{\infty}\frac{u'(t)}{(1+u(m)u(t))^2}dt=\frac{1}{u(m)}.\end{aligned}$$

then by (26), since $q<0$, one has (23). The theorem is proved.

THEOREM 3.2. If $p<1, p\neq 0, \frac{1}{p}+\frac{1}{q}=1,$ $u(t)$ is a differentiable strict increasing function in (n_0-1,∞) $(n_0\in N)$, such that $u((n_0-1)+)=0$ and $u(\infty)=\infty$, and $u'(t)(t\in(n_0-1,\infty))$ is decreasing; $a_n\geq 0$, satisfy $0<\sum_{n=n_0}^{\infty}\frac{a_n^p}{u(n)(u'(n))^{p-1}}<\infty$, then

(i) for $0<p<1$, one has the equivalent form of (23) as

$$\sum_{n=n_0}^{\infty}\frac{u'(n)}{(u(n))^{1-p}}\left[\sum_{m=n_0}^{\infty}\frac{a_m}{(1+u(m)u(n))^2}\right]^p>\sum_{n=n_0}^{\infty}\frac{(u'(n))^{1-p}a_n^p}{(1+u(n_0)u(n))\ u(n)}; \tag{27}$$

(ii) for $p<0$, one has the reversion of (27), which is equivalent to (23) for the same $p(<0)$.

Proof. Set b_n as

$$b_n:=\frac{u'(n)}{(u(n))^{1-p}}\left[\sum_{m=n_0}^{\infty}\frac{a_m}{(1+u(m)u(n))^2}\right]^{p-1},$$

and use (23) to obtain

$$\begin{aligned}0 &< \sum_{n=n_0}^{\infty}\frac{b_n^q}{(u'(n))^{q-1}u(n)}=\sum_{n=n_0}^{\infty}\frac{u'(n)}{(u(n))^{1-p}}\left[\sum_{m=n_0}^{\infty}\frac{a_m}{(1+u(m)u(n))^2}\right]^p\\ &= \sum_{n=n_0}^{\infty}\sum_{m=n_0}^{\infty}\frac{a_mb_n}{(1+u(m)u(n))^2}\\ &\geq \left\{\sum_{n=n_0}^{\infty}\frac{(u'(n))^{1-p}a_n^p}{(1+u(n_0)u(n))u(n)}\right\}^{\frac{1}{p}}\left\{\sum_{n=n_0}^{\infty}\frac{(u'(n))^{1-q}b_n^q}{u(n)}\right\}^{\frac{1}{q}};\end{aligned} \tag{28}$$

$$\begin{aligned}\left\{\sum_{n=n_0}^{\infty}\frac{b_n^q}{(u'(n))^{q-1}u(n)}\right\}^{\frac{1}{p}} &= \left\{\sum_{n=n_0}^{\infty}\frac{u'(n)}{(u(n))^{1-p}}\left[\sum_{m=n_0}^{\infty}\frac{a_m}{(1+u(m)u(n))^2}\right]^p\right\}^{\frac{1}{p}}\\ &\geq \left\{\sum_{n=n_0}^{\infty}\frac{(u'(n))^{1-p}a_n^p}{(1+u(n_0)u(n))u(n)}\right\}^{\frac{1}{p}};\end{aligned} \tag{29}$$

(i) For $0<p<1$, if $\sum_{n=n_0}^{\infty}\frac{b_n^q}{(u'(n))^{q-1}u(n)}<\infty$, then (28) takes strict inequality by using (23); so does (29); if $\sum_{n=n_0}^{\infty}\frac{b_n^q}{(u'(n))^{q-1}u(n)}=\infty$, then (29) takes strict inequality. Hence (27) holds.

(ii) For $p<0, 0<q<1$, by (29), one has

$$0<\sum_{n=n_0}^{\infty}\frac{b_n^q}{(u'(n))^{q-1}u(n)}\leq\sum_{n=n_0}^{\infty}\frac{(u'(n))^{1-p}a_n^p}{(1+u(n_0)u(n))u(n)}<\infty.$$

Hence (28) takes strict inequality by using (23); so does (29). Hence the reversion of (27) is valid.

On the other hand , (i) for $0 < p < 1$, suppose that (27) holds. By the reverse *Hölder's* inequality, one has

$$\begin{aligned}
&\sum_{n=n_0}^{\infty}\sum_{m=n_0}^{\infty}\frac{a_m b_n}{(1+u(m)u(n))^2}\\
= &\sum_{n=n_0}^{\infty}\left\{\left[\frac{(u'(n))^{q-1}}{(u(n))^{-1}}\right]^{\frac{1}{q}}\sum_{m=n_0}^{\infty}\frac{a_m}{(1+u(m)u(n))^2}\right\}\left\{\left[\frac{(u(n))^{\ -1}}{(u'(n))^{q-1}}\right]^{\frac{1}{q}} b_n\right\}\\
\geq &\left\{\sum_{n=n_0}^{\infty}\frac{u'(n)}{(u(n))^{1-p}}\left[\sum_{m=n_0}^{\infty}\frac{a_m}{(1+u(m)u(n))^2}\right]^p\right\}^{\frac{1}{p}}\left\{\sum_{n=n_0}^{\infty}\frac{b_n^q}{(u'(n))^{q-1}u(n)}\right\}^{\frac{1}{q}}. \quad (30)
\end{aligned}$$

Then by (27), one has (23). Hence (27) and (23) are equivalent. (ii) For $p < 0$, suppose that the reversion of (27) holds. By (30), one still has (23). Hence the reversion of (27) and inequality (23) are equivalent for $p < 0$. The theorem is proved.

REMARK 3.3. For $p < 1, p \neq 0, \alpha = 1$ in (24) , one has

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(1+mn)^2} > \left\{\sum_{n=1}^{\infty}\frac{a_n^p}{(1+n)n}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{b_n^q}{n}\right\}^{\frac{1}{q}}, \quad (31)$$

which is a reversion of (12) for $\lambda = 2$ and $\alpha = 1$ $(p > 1)$ as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(1+mn)^2} < \left\{\sum_{n=1}^{\infty}\frac{a_n^p}{n}\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\frac{b_n^q}{n}\right\}^{\frac{1}{q}}. \quad (32)$$

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