Improvement on the Bound of Intransitive Permutation Groups with Bounded Movement

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Abstract

Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a positive integer. Then we define the movement of G as, $m := move(G) := sup_{\Gamma}\{|\Gamma^g \setminus \Gamma||g \in G\}$. Let p be a prime, $p \geq 5$, and let $move(G) = m$. We show that if G is not a 2-group and p is the least odd prime dividing $|G|$, then $n := |\Omega| \leq 4m - p$.

Moreover for an infinite family of groups the maximum bound $n = 4m - p$ is attained.

1 Introduction

Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a positive integer. If for a subset Γ of Ω the size $|\Gamma^g - \Gamma|$ is bounded, for $g \in G$, we define the *movement* of Γ as $\text{move}(\Gamma) = \max_{g \in G} |\Gamma^g - \Gamma|$. If $\text{move}(\Gamma) \leq m$ for all $\Gamma \subseteq \Omega$, then G is said to have *bounded movement* and the *movement* of G is defined as the maximum of $\text{move}(\Gamma)$ over all subsets Γ, that is,

$$m := move(G) := sup\{|\Gamma^g \setminus \Gamma||\Gamma \subseteq \Omega, g \in G\}.$$

This notion was introduced in [4]. By [4, Theorem 1], if G has bounded movement m, then Ω is finite. Moreover both the number of G-orbits in Ω and the length of each G-orbit are bounded above by linear functions of m. In particular, it was

Received by the editors August 2004.
Communicated by H. Van Maldeghem.
1991 *Mathematics Subject Classification :* 20B05.
Key words and phrases : Permutation group, Bounded movement, Transitive, Cycle, Semidirect product.

proved that each G-orbit has length at most $3m$ and $n := |\Omega| \leq 5m - 2$. In [1] it was shown that $n = 5m - 2$ if and only if $n = 3$ and G is transitive. But in [3], this bound was refined further and it was shown that $n \leq \frac{1}{2}(9m - 3)$. Moreover, if $n = \frac{1}{2}(9m - 3)$ then either $n = 3$ and $G = S_3$ or G is an elementary abelian 3-group and all its orbits have length 3.

Now suppose that G is not a 2-group. Let p be the least odd prime dividing $|G|$ and suppose that $p \geq 5$. Then by [4, Lemma 2.2], $n \leq (9m - 3)/2$. In this paper we aim to improve the bound as follows:

Theorem 1.1 Let m be a positive integer, and let G be a finite permutation group on a set Ω with movement m such that G has no fixed points in Ω. If $|G|$ is not a 2-power and $|G|$ is not divisible by 3, then $n := |\Omega| \leq 4m - p$, where p is the smallest odd prime dividing $|G|$

The maximum bound in Theorem 1.1 is attained for an infinite family of groups (see example 2.4).

We denote by $K.P$ a semi-direct product of K by P with normal subgroup K.

2 Examples and preliminaries

Let $1 \neq g \in G$ and suppose that g in its disjoint cycle representation has t nontrivial cycles of lengths $l_1, ..., l_t$ say. We might represent g as

$$g = (a_1 a_2 ... a_{l_1})(b_1 b_2 ... b_{l_2})...(z_1 z_2 ... z_{l_t}).$$

Let $\Gamma(g)$ denote a subset of Ω consisting of $\lfloor l_i/2 \rfloor$ points from the i^{th} cycle, for each i , chosen in such a way that $\Gamma(g)^g \bigcap \Gamma(g) = \emptyset$. For example, we could choose

$$\Gamma(g) = \{a_2, a_4, \ldots, a_{k_1}, b_2, b_4, \ldots, b_{k_2}, ..., z_2, z_4, \ldots, z_{k_t}\},$$

where $k_i = l_i - 1$ if l_i is odd and $k_i = l_i$ if l_i is even. Note that $\Gamma(g)$ is not uniquely determined as it depends on the way each cycle is written. For any set $\Gamma(g)$ of this kind we say that $\Gamma(g)$ *consists of every second point of every cycle of* g . From the definition of $\Gamma(g)$ we see that

$$|\Gamma(g)^g \backslash \Gamma(g)| = |\Gamma(g)| = \sum_{i=1}^{t} \lfloor l_i/2 \rfloor.$$

The next lemma shows that this quantity is an upper bound for $|\Gamma^g \backslash \Gamma|$ for an arbitrary subset Γ of Ω.

Lemma 2.1 [2, Lemma 2.1] Let G be a permutation group on a set Ω and suppose that $\Gamma \subseteq \Omega$. Then for each $g \in G$, $|\Gamma^g \backslash \Gamma| \leq \Sigma_{i=1}^{t} \lfloor l_i/2 \rfloor$ where l_i is the length of the i^{th} cycle of g and t is the number of nontrivial cycles of g in its disjoint cycle representation. This upper bound is attained for $\Gamma = \Gamma(g)$ defined above.

Now we have the following lemma which is a classification of all transitive permutation groups G of degree p where p is the least odd prime dividing $|G|$.

Lemma 2.2 Let G be a transitive permutation group on a set Ω of size p, where p is the least odd prime dividing $|G|$. Then $G = Z_p.Z_{2^a}$, where $a \geq 0$, and $2^a|(p-1)$.

Proof. Let G be a transitive permutation group on a set Ω of size p. Then G is isomorphic to a transitive subgroup of S_p and so p is the largest prime divisor of $|G|$. Since p is also the least odd prime dividing $|G|$, we have $|G| = p.2^a$ for some $a \geq 0$. By Burnside's" pq theorem" (see [6, Theorem 2.10.17]) G is soluble, and hence by a theorem of Galois [6,Theorem 3.6.1] G is isomorphic to a subgroup of the group $\mathrm{AGL}(1,p)$ of affine transformations of a finite field consisting of p elements. Thus $G = Z_p.Z_{2^a}$ as asserted.

Corollary 2.3 Let G be a permutation group on a set Ω, and suppose that Δ is a G-orbit of length p in Ω where p is the least odd prime dividing $|G|$. Then the induced permutation group G^Δ is $Z_p.Z_{2^a}$ where $2^a|p-1$.

Let d be a positive integer, p a prime, $G := Z_p^d$, $t := (p^d-1)/(p-1)$, and let $H_1, \ldots, H_t$ be all subgroups of index p in G. Define Ω_i to be the right coset space $\{H_ig|g \in G\}$ of H_i and $\Omega := \Omega_1 \cup \cdots \cup \Omega_t$. Consider G as a permutation group on Ω by the right multiplication, that is $x \in G$ is identified with the composite of permutation $H_ig \longmapsto H_igx$ $(i = 1, \ldots, t)$ on Ω_i for $i = 1, \ldots, t$. If $g \in G - \{1\}$, then g lies in $(p^{d-1}-1)/(p-1)$ groups H_i and therefore acts on Ω as a permutation with $p(p^{d-1}-1)/(p-1)$ fixed points and p^{d-1} orbits of length p. Taking every second point from each of these p-cycles to form a set Γ we see that $\mathrm{move}(g) = m \geq p^{d-1}(p-1)/2$ if p is odd or 2^{d-1} if $p = 2$, and it is not hard to prove that in fact move $(g) = m = p^{d-1}(p-1)/2$ if p is odd or 2^{d-1} if $p = 2$. Since g is non-trivial, all non-identity elements of G have the same movement m.

Now we will show that there certainly is an infinite family of groups for which equality in Theorem 1.1 holds, for any prime $p \geq 5$.

Example 2.4 For a positive integer d and a prime $p \geq 5$, let $G_1 := \langle (12 \ldots p) \rangle \cong Z_p$ be a permutation group on $\Omega_1 := \{1, 2, \ldots, p\}$. Moreover, suppose that $G_2 := Z_2^d$, and $H_1, \ldots, H_t$ denote the groups defined in the above for the prime number 2 on $\Omega_2 := \bigcup_{i=1}^{2^d-1} \Omega_{2i}$, where Ω_{2i} denotes the set of two cosets of H_i in G_2, $1 \leq i \leq t = 2^d - 1$. Then G_2 has movement equal to 2^{d-1} and also (2^d-1) nontrivial orbits in Ω_2. Now we consider the direct product $G := G_1 \times G_2$ as a permutation on Ω which is the disjoint union of Ω_1 and Ω_2, and G_1 and G_2 act trivially on Ω_2 and Ω_1, respectively. Then G has movement $m = (p-1)/2 + 2^{d-1}$. The set Ω splits into $2^d = 2m - (p-1)$orbits under G, which are Ω_1 and $2^d - 1$ orbits of length 2 in Ω_2. In particular, none of them is trivial. Furthermore,

$$4m - p = 2(p-1) + 2^{d+1} - p = p + 2(2^d - 1) = |\Omega_1| + |\Omega_2| = |\Omega|.$$

3 The maximum degree of bounded movement groups

Suppose that $G \leq$Sym(Ω) and that G is not a 2-group and move$(G) = m$, and that $p \geq 5$ is the least odd prime dividing $|G|$. In this section we find an upper bound for $|\Omega|$ that is a linear function of m.

To prove the main theorems, we introduce the following notation.
$r_p(a) :=$ number of G-orbits of length p on which G acts as $Z_p.Z_{2^a}$ with $0 \leq a \leq a_0$ and set $r_p := \sum_{a=0}^{a_1} r_p(a)$;
$\Phi :=$ union of G-orbits of lengths 2^b, where $1 \leq b \leq \log_2 p$; and u is the number of orbits in Φ;
$s :=$ number of G-orbits of length $> p$.

The orbits are labeled accordingly: thus $\Omega_1, ..., \Omega_{r_p}$ are those of length p on which G acts as $Z_p.Z_{2^a}$ for some $a \geq 0$; $\Omega_{r_p+1}, ..., \Omega_{r_p+u}$ are those of length 2^b where $1 \leq b \leq \log_2 p$, which the group induced by G on each orbit in Φ is a 2-group; and etc. Define $t := r_p + u + s$, $t_1 := r_p + u$. So t is the total number of G-orbits.

For $1 \leq i \leq r_p$ define K_i to be the kernel of the action of G on Ω_i and for $g \in G$ define $k(g)$ to be the number of i in that range for which g is not in K_i. For $g \in G$ and a G-invariant set Δ we denote by $\text{fix}_\Delta(g) = \{\alpha \in \Delta | \alpha^g = \alpha\}$ and $\text{supp}_\Delta(g) = \{\alpha \in \Delta | \alpha^g \neq \alpha\}$ the set of fixed points of g in Δ and the support of g in Δ , respectively (so that $|\text{fix}_\Delta(\text{g})| + |\text{supp}_\Delta(\text{g})| = |\Delta|$), and define $\text{odd}_\Delta(g) :=$ the number non-trivial cycles of g in Δ that have odd length.

Lemma 3.1 With the above notation, let $\Delta := \bigcup_{i=t_1+1}^{t} \Omega_i$ be the union of G-orbits of length $> p$, and let $g \in G$. Then

$$\frac{p-1}{2}k(g) + \frac{1}{2}|supp_\Phi(g)| + \frac{1}{2}(|supp_\Delta(g)| - odd_\Delta(g)) \leq m.$$

Proof. For each i such that $1 \leq i \leq t_0$ and g is not in K_i, since $|\Omega_i| = p$ then g^{Ω_i} is a p-cycle or a 2-element with one fixed point and we may choose a subset Γ_i of $\frac{p-1}{2}$ points of Ω_i such that $\Gamma_i^g \bigcap \Gamma_i = \emptyset$. Let Γ_0 be the set of chosen points from all the Γ_i for $1 \leq i \leq r_p$, and so by definition $\Gamma_0^g \bigcap \Gamma_0 = \emptyset$.

For each of the non-trivial cycles $(b_1...b_{2l})$ and $(a_1a_2...a_k)$ of g in Φ and Δ respectively, adjoin the points $b_1, b_3, ..., b_{2l-1}$ and also $a_1, a_3, ..., a_{k'}$ to Γ_0, where k' is odd and $k - 2 \leq k' \leq k - 1$.
Let Γ be the resulting set. It has been constructed so that $\Gamma^g \bigcap \Gamma = \emptyset$.
Therefore $|\Gamma| \leq m$. Since

$$|\Gamma| = \frac{p-1}{2}k(g) + \frac{1}{2}|supp_\Phi(g)| + \frac{1}{2}(|supp_\Delta(g)| - odd_\Delta(g)),$$

we have the stated inequality.

To prove Theorem 1.1 we first prove the following lemma.

Lemma 3.2

$$\sum_{a=0}^{a_0} \frac{p-1}{2}.(1-\frac{1}{2^a p})r_p(a) + \frac{|\Phi|-u}{2} + \frac{p-1}{2p}(|\Delta|-s) < m,$$

.

Proof. Suppose that $1 \leq i \leq r_p$. Then the group induced by G on Ω_i is $Z_p Z_{2^a}$ for some $a \geq 0$, such that $2^a|(p-1)$, and since $|G:K_i| = 2^a p$, there are

$$|G| - |K_i| = (2^a p - 1)|K_i|$$

elements g which act nontrivially on Ω_i. It follows that

$$\sum_{g\in G} \frac{p-1}{2} k(g) = \frac{p-1}{2} \sum_{a=0}^{a_0} (\frac{2^a p - 1}{2^a p}|G|) r_p(a).$$

For $r_p + 1 \leq i \leq t_1$, the group induced by G on Ω_i is a 2-group. The union of these sets Ω_i is Φ, and since by Burnside's Lemma [5, Theorem 3.26] the average number of fixed points of elements of G in Φ is the number u of G-orbits in Φ, we have

$$\sum_{g\in G} \frac{1}{2}|supp_\Phi(g)| = \frac{1}{2}\sum_{g\in G}(|\Phi| - |fix_\Phi(g)|) = \frac{1}{2}|\Phi||G| - \frac{|G|}{2}u.$$

Similarly,

$$\sum_{g\in G} \frac{1}{2}|supp_\Delta(g)| = \frac{1}{2}|\Delta|.|G| - \frac{s|G|}{2},$$

and since $\text{odd}_\Delta(g) < \frac{1}{p}|\text{supp }_\Delta(g)|$, we have

$$\sum_{g\in G} \frac{1}{2}(|supp_\Delta(g)| - odd_\Delta(g)) > \frac{p-1}{2p}(|\Delta|.|G| - s|G|).$$

Thus adding the inequality of Lemma 3.1 over all $g \in G$, we obtain

$$m|G| > |G|(\sum_{a=0}^{a_0} \frac{p-1}{2}.(1-\frac{1}{2^a p})r_p(a) + \frac{|\Phi|-u}{2} + \frac{p-1}{2p}(|\Delta|-s))$$

where the last inequality recognizes the fact the inequality of Lemma 3.1 is strict for the identity element of G. This completes the proof of Lemma 3.2.

Recall that in general the movement $move(g)$ of an element g of a permutation group G on a set Ω is defined as

$$move_\Omega(g) := max\{|\Gamma^g \setminus \Gamma||\Gamma \subseteq \Omega\}.$$

Thus the movement m of G is given as $m = max\{move_\Omega(g)|g \in G\}$. Assume that Ω is the disjoint union of G-invariant sets Ω_1 and Ω_2. Then every subset Γ of Ω is a disjoint union of subsets $\Gamma_i := \Gamma \cap \Omega_i$ for $i = 1, 2$. Let g_i be the permutation on Ω_i induced by g for $i = 1, 2$. Since $|\Gamma^g \setminus \Gamma| = |\Gamma_1^{g_1} \setminus \Gamma_1| + |\Gamma_2^{g_2} \setminus \Gamma_2|$, we have

$$move_\Omega(g) = \sum_{i=1}^{2} \max\{|\Gamma_i^{g_i} \setminus \Gamma_i||\Gamma_i \subseteq \Omega_i\} = move_{\Omega_1}(g_1) + move_{\Omega_2}(g_2).$$

Now

$$n = (\Sigma_{a=0}^{a_0} r_p(a))p + |\Phi| + |\Delta|.$$

Also we have $|\Phi| \geq 2u$, and so

$$\frac{|\Phi| - u}{2} \geq \frac{|\Phi|}{4}.$$

By above statement and since G is intransitive, thus the inequality in Lemma 3.2 implies that

$$m - 1 \geq \frac{n}{4} + \sum_{a=0}^{a_0} r_p(a)(\frac{p-1}{2} - \frac{p-1}{2^{a+1}p} - \frac{p}{4}) + |\Delta|(\frac{p-1}{2p} - \frac{1}{4}) - \frac{p-1}{2p}s$$

$$= \frac{n}{4} + \sum_{a=0}^{a_0} r_p(a)(\frac{p-2}{4} - \frac{p-1}{2^{a+1}p}) + |\Delta|(\frac{p-2}{4p}) - \frac{p-1}{2p}s.$$

Since G is not a 2-group, we have either $r_p(a) > 0$ for some a or $s > 0$. If some $r_p(a) > 0$, then

$$m - 1 \geq \frac{n}{4} + \frac{p-2}{4} - \frac{p-1}{2^{a+1}p}. \qquad (*)$$

But we note that since $p \geq 5$, for each $a \geq 0$,

$$\frac{p-2}{4} - \frac{p-1}{2^{a+1}p} \geq \frac{p-2}{4} - \frac{p-1}{2p} > 0.$$

Hence,

$$m - 1 \geq \frac{n}{4} + \frac{p-2}{4} - \frac{p-1}{2p} = \frac{n}{4} + \frac{p^2 - 4p + 2}{4p}.$$

On the other hand if $s > 0$, then $|\Delta| \geq (p+1)s \geq p+1$. Thus,

$$m - 1 \geq \frac{n}{4} + |\Delta|(\frac{p-2}{4p}) - \frac{p-1}{2p}s \geq \frac{n}{4} + s(\frac{(p+1)(p-2)}{4p} - \frac{p-1}{2p}) \geq \frac{n}{4} + \frac{p^2 - 3p}{4p}.$$

So in either case we must have,

$$m - 1 \geq \frac{n}{4} + min\{\frac{p^2 - 4p + 2}{4p}, \frac{p^2 - 3p}{4p}\} = \frac{n}{4} + \frac{p^2 - 4p + 2}{4p}.$$

Hence,

$$4m \geq n + p + \frac{2}{p},$$

That is, $n \leq 4m - p$. Hence the proof of Theorem 1.1 is complete.

References

[1] J. R. Cho, P. S. Kim, and C. E. Praeger, The maximal number of orbits of a permutation groups with bounded movement, J. Algebra 214 (1999), 625-630.

[2] A. Hassani, M. Khayaty (Alaeiyan), E. I. Khukhro and C. E. Praeger, Transitive permutation groups with bounded movement having maximal degree, J. Algebra 214(1999), 317-337.

[3] P. M. Neumann and C. E. Praeger, On the movement of a permutation group, J. Algebra 214 (1999) 631-635.

[4] C. E. Praeger, On permutation group with bounded movement, J. Algebra 144 (1991), 436-442.

[5] J. J. Rotman, An Introduction to the theory of groups, 3rd ed., Allyn and Bacon, Boston, 1984.

[6] T. Tsuzuku, Finite groups and finite geometries, Cambridge University Press, 1982.

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