On the Representation theorems of Riesz and Schwartz

Yves Hupperts Michel Willem

Abstract

We give simple proofs of the representation theorems of Riesz and Schwartz.

1 Introduction

According to J.D. Gray in [2]

> *Only rarely does the mathematical community pay a theorem the accolade of transforming it into a tautology. The Riesz representation theorem has received this accolade.*

Indeed for Bourbaki (see [1]) a measure is a continuous linear functional.

In most textbooks, the Riesz representation theorem asserts that continuous linear functionals on $\mathcal{C}(K)$, where K is a Hausdorff compact space, are represented by integrals with respect to a regular Borel measure. It is interesting to note that many variants of Lebesgue integral were defined by F. Riesz in order to avoid the use of measure theory!

But the original theorem in [4] asserts that continuous linear functionals on $\mathcal{C}([0,1])$ are Stieltjes integrals :

> *Etant donné l'opérateur $A(f(x))$, on peut déterminer la fonction à variation bornée $\alpha(x)$, telle que, quelle que soit la fonction continue $f(x)$, on ait*
> $$A(f(x)) = \int_0^1 f(x)\ d\alpha(x).$$

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Thus the original Riesz representation theorem gives a correspondance between continuous linear functionals and functions of bounded variation . Moreover, as observed by Horváth in [3], since, for f regular with compact support in $]0,1[$,

$$\int_0^1 f(x)\ d\alpha(x) = -\int_0^1 f'(x)\ \alpha(x)\ dx,$$

the functional A is the derivative, in the sense of distributions, of α.

In this note we give a short elementary proof of the original Riesz representation theorem and of its extension in two dimensions in [3]. Contrary to [3], we use only two tools : Abel transform and the fact that a bounded function (on $]0,1[$ or $]0,1[\times]0,1[$) is Riemann integrable if and only if it is almost everywhere continuous.

2 The one-dimensional case

We recall some basic definitions. We call $\mathcal{K}(]0,1[)$ the space of continuous functions from $]0,1[$ to $\mathbb{R}$ with compact support in $]0,1[$ and $\mathcal{D}(]0,1[)$ the subset of infinitely derivable functions of $\mathcal{K}(]0,1[)$.

Definition 2.1. *The space of bounded measures on]0,1[is defined by* $(\mathcal{K}(]0,1[))'$ $:= \{\mu : \mathcal{K}(]0,1[) \to \mathbb{R};$ *μ is linear and there exists* $C > 0 : |\int_0^1 u\ d\mu| \leq C\|u\|_\infty$ *for all* $u \in \mathcal{K}(]0,1[)\}$.

Definition 2.2. *A function* $F \in L^1_{loc}(]0,1[)$ *is of bounded variation if*

$$\|DF\| = \|DF\|_{]0,1[} = \sup\left\{\int_0^1 F(t)u'(t)\ dt; u \in \mathcal{D}(]0,1[), \|u\|_\infty \leq 1\right\} < \infty.$$

After these preparations, we can state the representation theorem of F. Riesz in the following form :

Theorem 2.3. *Let μ be a bounded measure on* $]0,1[$. *Then, there exists a function* $F :]0,1[\longrightarrow \mathbb{R}$ *of bounded variation such that, in the sense of distributions,* $\mu = \partial F$. *Moreover,* $\|\mu\| = \|DF\|$.

Proof. Let $F(t) = \mu(]0,t])$. By countable additivity of the measure μ, F is continuous outside a countable set.

Let us take $u \in \mathcal{D}(]0,1[)$ and define $u_n = \sum_{k=1}^{2^n-1} u\left(\frac{k}{2^n}\right)\chi\left(\left]\frac{k}{2^n},\frac{k+1}{2^n}\right]\right)$. It is clear that $u_n \to u$ uniformly on $[0,1]$. Using Abel transform, we have

$$\begin{aligned}
\int_0^1 u\ d\mu &= \lim_{n\to\infty}\int_0^1 u_n\ d\mu \\
&= \lim_{n\to\infty}\sum_{k=1}^{2^n-1} u\left(\frac{k}{2^n}\right)\mu\left(\left]\frac{k}{2^n},\frac{k+1}{2^n}\right]\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{2^n-1} u\left(\frac{k}{2^n}\right)\left[F\left(\frac{k+1}{2^n}\right) - F\left(\frac{k}{2^n}\right)\right] \\
&= -\lim_{n\to\infty}\sum_{k=1}^{2^n-1} F\left(\frac{k}{2^n}\right)\left[u\left(\frac{k}{2^n}\right) - u\left(\frac{k-1}{2^n}\right)\right] \\
&= -\lim_{n\to\infty}\frac{1}{2^n}\sum_{k=1}^{2^n-1} F\left(\frac{k}{2^n}\right)u'\left(\frac{k-\theta_k}{2^n}\right),
\end{aligned}$$

where $0 < \theta_k < 1$.
For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| \leq \delta \Rightarrow |u'(x) - u'(y)| \leq \varepsilon$. Hence

$$\overline{\lim_{n\to\infty}} \left| \int_0^1 u \; d\mu - \frac{1}{2^n} \sum_{k=1}^{2^n-1} F\left(\frac{k}{2^n}\right) u'\left(\frac{k}{2^n}\right) \right| \leq \varepsilon \|F\|_\infty.$$

Since $u'F$ is Riemann integrable on $]0,1[$, it follows that

$$\int_0^1 u \; d\mu = - \lim_{n\to\infty} \frac{1}{2^n} \sum_{k=1}^{2^n-1} F\left(\frac{k}{2^n}\right) u'\left(\frac{k}{2^n}\right) = - \int_0^1 u'(t)F(t)dt.$$

Finally we obtain

$$\begin{aligned} \|DF\| &= \sup\left\{\int_0^1 F(t)u'(t) \; dt; u \in \mathcal{D}(]0,1[), \|u\|_\infty \leq 1\right\} \\ &= \sup\left\{\int_0^1 u \; d\mu; u \in \mathcal{D}(]0,1[), \|u\|_\infty \leq 1\right\} = \|\mu\|, \end{aligned}$$

and the proof is complete. ■

Corollary 2.4 (Schwartz representation theorem)**.** *Let f be a distribution on $]0,1[$ such that ∂f is a bounded measure μ on $]0,1[$. Then f is a function of bounded variation.*

Proof. Let us define the function $F(x) = \mu(]0,x])$. Then $\partial(f - F) = \mu - \mu$ and $f - F = c \in \mathbb{R}$. ■

3 The higher-dimensional case

In order to be readable, we shall consider the two dimensional case.

Definition 3.1. *The space of bounded measures on* $\Omega =]0,1[\times]0,1[$ *is defined by* $(\mathcal{K}(\Omega))' := \{\mu : \mathcal{K}(\Omega) \to \mathbb{R}; \mu$ *is linear and there exists* $C > 0 : |\int_\Omega u \, d\mu| \leq C\|u\|_\infty$ *for all* $u \in \mathcal{K}(\Omega)\}$.

Lemma 3.2. *Let* $u \in \mathcal{D}(\Omega)$. *For* $\varepsilon > 0$ *there exists* $0 < \delta < 1$ *such that*

$$|u(x+h, y+k) - u(x, y+k) - u(x+h, y) + u(x,y) - \partial_x\partial_y u(x,y)hk| \leq \varepsilon|h||k|$$

for all $(x,y) \in \Omega$ *provided that* $|h|, |k| \leq \delta$.

The following lemma is contained in [3]. We give the proof for the sake of completeness.

Lemma 3.3. *The function* $F(x,y) = \mu(]0,x]\times]0,y])$ *is continuous outside a Lebesgue-negligeable set*

Proof. Since μ is the difference of two positive measures, we can assume that μ is positive. Since $\mu(\bar{\Omega}) < +\infty$, by countable additivity of the measure μ, only a countable number of horizontal or vertical segments in $\bar{\Omega}$ can have a strictly positive μ-measure. Let H be the collection of all horizontal segments σ, going from the left boundary to the right boundary of Ω such that $\mu(\sigma) > 0$.Then H is countable. A similar assertion holds for the collection V of vertical segments σ such that $\mu(\sigma) > 0$. Denote by S the set of all points of Ω which lie on a segment belonging to H or to V. Then S has the Lebesgue measure zero.

We claim now that F is continuous at every point (x,y) not belonging to S. We have

$$\bigcap_{h>0}]0, x+h]\times]0, y+h] =]0,x]\times]0,y]$$

and

$$\bigcup_{h>0}]0, x-h]\times]0, y-h] =]0,x[\times]0,y[.$$

By the choice of (x,y) we have $\mu(\{(x,\eta); 0 < \eta \leq y\}) = \mu(\{(\xi, y); 0 < \xi \leq x\}) = 0$, and

$$\lim_{h\to 0} F(x+h, y+h) = \lim_{h\to 0} \mu(]0, x+h]\times]0, y+h]) = \mu(]0,x]\times]0,y]) = F(x,y).$$

For $x - h \leq \xi \leq x + h$ and $y - h \leq \eta \leq x + h$ the value $f(\xi, \eta)$ lies between $f(x-h, y-h)$ and $f(x+h, y+h)$, from which the lemma follows. ∎

Theorem 3.4. *Let* μ *be a bounded measure on* Ω. *Then there exists a function* $F \in L^\infty(\Omega)$ *such that, in the sense of distributions,* $\mu = \partial_x\partial_y F$.

Proof. Let $u \in \mathcal{D}(\Omega)$. Let us define $u_n := \sum_{k,l=1}^{2^n-1} u\left(\frac{k}{2^n}, \frac{l}{2^n}\right) \chi\left(\left]\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right)$ $\chi\left(\left]\frac{l}{2^n}, \frac{l+1}{2^n}\right]\right)$. It is clear that $u_n \to u$ uniformly on Ω. Using Abel transform we have

$$\begin{aligned}
\int_\Omega u \, d\mu &= \lim_{n\to\infty} \int_\Omega u_n d\mu \\
&= \lim_{n\to+\infty} \sum_{k,l=1}^{2^n-1} u\left(\frac{k}{2^n}, \frac{l}{2^n}\right) \mu\left(\left]\frac{k}{2^n}, \frac{k+1}{2^n}\right] \times \left]\frac{l}{2^n}, \frac{l+1}{2^n}\right]\right) \\
&= \lim_{n\to+\infty} \sum_{k,l=1}^{2^n-1} u\left(\frac{k}{2^n}, \frac{l}{2^n}\right) \left[F\left(\frac{k+1}{2^n}, \frac{l+1}{2^n}\right) - F\left(\frac{k}{2^n}, \frac{l+1}{2^n}\right)\right. \\
&\qquad \left. - F\left(\frac{k+1}{2^n}, \frac{l}{2^n}\right) + F\left(\frac{k}{2^n}, \frac{l}{2^n}\right)\right] \\
&= \lim_{n\to+\infty} \sum_{k,l=1}^{2^n-1} F\left(\frac{k}{2^n}, \frac{l}{2^n}\right) \left[u\left(\frac{k}{2^n}, \frac{l}{2^n}\right) - u\left(\frac{k}{2^n}, \frac{l-1}{2^n}\right)\right. \\
&\qquad \left. - u\left(\frac{k-1}{2^n}, \frac{l}{2^n}\right) + u\left(\frac{k-1}{2^n}, \frac{l-1}{2^n}\right)\right].
\end{aligned}$$

By lemma 3.2 we have, for every $\varepsilon > 0$,

$$\overline{\lim_{n\to\infty}} \left| \int_\Omega u \, d\mu - \frac{1}{2^{2n}} \sum_{k,l=1}^{2^n-1} F\left(\frac{k}{2^n}, \frac{l}{2^n}\right) \partial_x \partial_y u \left(\frac{k}{2^n}, \frac{l}{2^n}\right) \right| \le \varepsilon \|F\|_\infty.$$

By lemma 3.3, $F\partial_x\partial_y u$ is Riemann integrable on Ω. Then we obtain

$$\begin{aligned}
\int_\Omega u \, d\mu &= \lim_{n\to+\infty} \frac{1}{2^{2n}} \sum_{k,l=1}^{2^n-1} F\left(\frac{k}{2^n}, \frac{l}{2^n}\right) \partial_x \partial_y u \left(\frac{k}{2^n}, \frac{l}{2^n}\right) \\
&= \int_\Omega F(x,y) \, \partial_x \partial_y u(x,y) \, dxdy.
\end{aligned}$$

■

Remark 3.5. *Let* (ρ_n) *be a Dirac sequence :* $\rho_n \in \mathcal{D}(\mathbb{R}^2)$, $\operatorname{supp} \rho_n \subset B[0, 1/n]$, $\int_{\mathbb{R}^2} \rho_n \, dxdy = 1, \rho_n \ge 0$. *Then for every* $u \in \mathcal{K}(\Omega)$,

$$\int_\Omega u \, d\mu = \lim_{n\to\infty} \int_\Omega u \, \partial_x \partial_y (\rho_n * F) \, dxdy.$$

For a proof see [5] p. 57.

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Université Catholique de Louvain,
Institut de mathématique pure et appliquée,
Chemin du Cyclotron 2,
B-1348 Louvain-la-Neuve, Belgique.
hupperts@math.ucl.ac.be
willem@math.ucl.ac.be