Optimal energy decay rate for Rayleigh beam equation with dynamical boundary controls

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Abstract

We consider a Rayleigh beam equation with two dynamical boundary controls. First, by a multiplier method, we show that the smooth solution has a polynomial energy decay rate. Next, using a spectrum method, we justify that the polynomial energy decay rate is optimal.

1 Introduction and main result

In this paper, we consider the equation of Rayleigh beam, which is clamped at one end and subjected to dynamical boundary controls at the other end :

$$\begin{cases} y_{tt} - \gamma y_{xxtt} + y_{xxxx} = 0, & 0 < x < 1, \quad t > 0, \\ y(0,t) = y_x(0,t) = 0, & t > 0, \\ y_{xx}(1,t) + \eta(t) = 0, & t > 0, \\ y_{xxx}(1,t) - \gamma y_{xtt}(1,t) = \xi(t), & t > 0 \end{cases} \tag{1.1}$$

where $\gamma > 0$ is a physical constant, η, ξ designate respectively the boundary feedback controls.

In the case of static feedbacks: $\eta(t) = y_{xt}(1,t)$, $\xi(t) = y_t(1,t)$ the stabilization of the system (1.1) was well studied by Rao in [14]. In this work, we propose a dynamical boundary force control $\xi(t)$ and a dynamical boundary moment control

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$\eta(t)$ applied at the right end of the beam. The dynamical controls $\eta(t)$ and $\xi(t)$ are given by the following integral system :

$$\begin{cases} \eta_t(t) - y_{xt}(1,t) + \eta(t) = 0, \\ \xi_t(t) - y_t(1,t) + \xi(t) = 0. \end{cases} \tag{1.2}$$

The concept of dynamical controls has been introduced by the automaticians in the finite dimensional case (see Francis [4]). In the infinite dimensional case, the concept of dynamical controls is considered as indirect damping mechanisms proposed by Russell [19].

Let y be a smooth solution of the system (1.1)-(1.2). We define the associated energy $E(t)$ by the following formula :

$$E(t) = \frac{1}{2}\left\{ \int_0^1 (y_t^2 + \gamma y_{xt}^2 + y_{xx}^2)dx + \eta^2 + \xi^2 \right\}. \tag{1.3}$$

By a direct computation we have :

$$\frac{dE(t)}{dt} = -\eta^2(t) - \xi^2(t) \leq 0, \qquad \forall\, t \geq 0. \tag{1.4}$$

Then the system (1.1)-(1.2) is dissipative in the sense that the energy $E(t)$ is a nonincreasing function of the time variable t.

Now, let

$$V = \{y \in H^1(0,1) \;:\; y(0) = 0\}, \quad \| y \|_V^2 = \int_0^1 (y^2 + \gamma y_x^2)dx,$$

$$W = \{y \in H^2(0,1) \;:\; y(0) = y_x(0) = 0\}, \quad \| y \|_W^2 = \int_0^1 y_{xx}^2 dx.$$

We define the energy space

$$\mathcal{H} = W \times V \times \mathbb{R} \times \mathbb{R}$$

endowed with the usual inner product. Let y be a smooth solution of the system (1.1)-(1.2). We multiply the equation (1.1) by a function $\phi \in W$ and integrate by parts :

$$\int_0^1 (y_{tt}\phi + \gamma y_{xtt}\phi_x)dx + \int_0^1 y_{xx}\phi_{xx}dx + \xi\phi(1) + \eta\phi_x(1) = 0. \tag{1.5}$$

Now we define the linear operators $A \in \mathcal{L}(W; W')$, $B \in \mathcal{L}(\mathbb{R}; V')$, $C \in \mathcal{L}(V, V')$ and $D \in \mathcal{L}(\mathbb{R}; W')$ by the following way:

$$< Ay, \phi >_{W' \times W} = (y, \phi)_W, \quad < D\eta, \phi >_{W' \times W} = \eta\phi_x(1), \quad \forall\, y,\ \phi \in W,\ \eta \in \mathbb{R},$$

$$< Cy, \phi >_{V' \times V} = (y, \phi)_V, \quad < B\xi, \phi >_{V' \times V} = \xi\phi(1), \quad \forall\, y,\ \phi \in V,\ \xi \in \mathbb{R}.$$

Then we can formulate the variational equation (1.5) as :

$$Cy_{tt} + Ay + D\eta + B\xi = 0, \quad \text{in } W'.$$

Assume that $Ay + D\eta \in V'$, then we obtain that :

$$y_{tt} + C^{-1}(Ay + D\eta + B\xi) = 0, \text{ in } V.$$

We next introduce the linear bounded operator $\mathcal{B}$ in $\mathcal{H}$ and the linear unbounded operator $\mathcal{A}$ as follows :

$$\mathcal{B}u = \begin{pmatrix} 0 \\ 0 \\ \eta \\ \xi \end{pmatrix}^T, \quad \mathcal{A}u = \begin{pmatrix} -z \\ C^{-1}(Ay + D\eta + B\xi) \\ -z_x(1) \\ -z(1) \end{pmatrix}^T,$$

$$D(\mathcal{A}) = \Big\{ (y, z, \eta, \xi) \in \mathcal{H} \;:\; z \in W \text{ and } Ay + D\eta \in V' \Big\}.$$

Setting $u = (y, y_t, \eta, \xi)$, we rewrite (1.1)-(1.2) as a first-order system :

$$u_t + (\mathcal{A} + \mathcal{B})u = 0, \quad u(0) = u_0 \in \mathcal{H}. \tag{1.6}$$

It is easy to prove that $\mathcal{A}$ is maximal monotone operator and $\mathcal{B}$ is monotone operator. Then $\mathcal{A} + \mathcal{B}$ generates a C_0 semigroup $S_{\mathcal{A}+\mathcal{B}}(t)$ of contractions on the energy space $\mathcal{H}$ (see Brezis [2]). Moreover, since $\mathcal{A}$ is skew adjoint and $\mathcal{B}$ is compact, then using the compact perturbation theory of Russell [18] the system (1.1)-(1.2) is not be uniformly stable (see Rao[14]). Moreover, using the decomposition theory of Sz-Nagy-Foias and Foguel we can prove that the energy $E(t)$ decreases asymptotically to zero (see Benchimol [1]) for all $u_0 \in \mathcal{H}$.

In this paper, we will prove that for any $u_0 \in D(\mathcal{A})$ the energy of the system (1.1)-(1.2) has a polynomial decay rate :

$$E(t) \leq \; E(0)\frac{2M}{M+t}, \quad \forall\, t \geq 0 \tag{1.7}$$

where $M > 0$, depending on $u_0 \in D(\mathcal{A})$. To this end, we employ a nonlinear technique (see Rao [15]). Moreover we prove that the polynomial energy decay rate (1.7) is optimal in the sense that for any $\varepsilon > 0$ there exists $u_0^\varepsilon \in D(\mathcal{A})$ such that the associated energy satisfies the estimate :

$$E^\varepsilon(t) \geq \; \frac{C_\varepsilon}{t^{1+\varepsilon}}, \qquad t \to +\infty.$$

Our approach is based on the theory of Riesz basis and earlier results of Littman and Markus [11] on the hybrid system.

To our knowledge, the estimate (1.7) and the optimality are new. In fact, there were several works on the energy decay rate for smooth solutions of the wave equation [9], [10]. In [10] Lebeau and Robbiano considered the boundary stabilization for a wave equation. In particular, it was shown that the energy has a decay rate just like $\dfrac{1}{(\ln t)^{2-\delta}}$, $\delta > 0$ for any $u_0 \in D(A)$. Moreover, in [12] Morgül considered the Euler-Bernoulli equation with two dynamical boundary controls : the dynamic boundary moment control $\eta(t) + d_1 y_{xt}(1,t)$ and the dynamic boundary force control $\xi(t) + d_2 y_t(1,t)$ where $d_1 > 0$ and $d_2 \geq 0$. The stability of system (1.1)-(1.2) was an open problem in the case $d_1 = d_2 = 0$.

Unlike the spectrum method, the multiplier method does not necessitate any knowledge of the spectrum of the system. It is simple and can be adapted to the study of other problems in any spatial dimension (see Rao-Wehbe [16]). Because the essential difficulty intervening in the determination of the spectrum of the system, the spectrum method is obviously limited to one-dimensional problems.

2 Polynomial energy decay rate

In this section, using a multiplier method we establish the polynomial energy decay rate for the smooth solution of the system (1.1)-(1.2).

Theorem 2.1. *For any $u_0 \in D(\mathcal{A})$, let*

$$M = (\gamma+1)\frac{\| u_0 \|^2_{D(\mathcal{A})}}{\| u_0 \|^2_{\mathcal{H}}} + 9\gamma + 17.$$

Then the following polynomial energy decay rate holds

$$E(t) \leq \ E(0)\frac{2M}{M+t}, \quad \forall\, t \geq 0 \tag{2.1}$$

for the solution u of the system (1.6).

The idea of the proof is based on an earlier work of Rao [15]. We proceed in several steps. We first recall the following result

Lemma 2.1. *(i) Let $u = (y, z, \eta, \xi) \in D(\mathcal{A})$. Then we have*

$$y \in H^3(0,1) \cap W, \quad z \in W \quad \textit{such that} \quad y_{xx}(1) + \eta = 0. \tag{2.2}$$

Moreover, the resolvent $(I+\mathcal{A})^{-1}$ is compact in $\mathcal{H}$. In particular, $\mathcal{A}$ is skew adjoint.

(ii) Let $u = (y, z, \eta, \xi) \in D\Big((\mathcal{A}+\mathcal{B})^2\Big)$. Then we have

$$y \in H^4(0,1) \cap W, \quad z \in H^3(0,1) \cap W \quad \textit{such that}$$

$$y_{xx}(1) + \eta = 0 \quad \textit{and} \quad y_{xxx}(1) - \gamma v_x(1) - \xi = 0 \tag{2.3}$$

where $v = C^{-1}(Ay + D\eta + B\xi) \in V$.

The proof is the same as in Rao [14]. We omit the details here.

Lemma 2.2. *Assume that $u_0 \in D\Big((\mathcal{A}+\mathcal{B})^2\Big)$. Let $0 \leq S \leq T < +\infty$. Then the solution u of the problem (1.6) satisfies :*

$$\frac{1}{2}\int_S^T \int_0^1 (y_t^2 - \gamma y_{xt}^2 + \frac{11}{4} y_{xx}^2) E(t)\, dx dt \leq M_0 E(S) E(0) \tag{2.4}$$

where M_0 is given by :

$$M_0 = (\gamma+1)\frac{\| u_0 \|^2_{D(\mathcal{A})}}{\| u_0 \|^2_{\mathcal{H}}} + 7\gamma + 8.$$

Proof. Assume that $u_0 \in D(\mathcal{A}+\mathcal{B})$, then using (i) Lemma 2.1 we have $y \in H^3(0,1)$ and $y_{xx}(1,t) = -\eta(t) \in L^2(S;T)$. Multiplying equation (1.1)-(1.2) by $xy_xE(t)$ and integrating by parts we obtain that :

$$\frac{1}{2}\int_S^T\int_0^1\Big(y_t^2-\gamma y_{xt}^2+3y_{xx}^2\Big)E(t)dxdt = -\Big[\int_0^1 y_txy_xE(t)dx\Big]_S^T+$$
$$\int_S^T\int_0^1 y_txy_xE_t(t)dxdt-\gamma\Big[\int_0^1 y_{xt}y_xE(t)dx+\int_0^1 y_{xt}xy_{xx}E(t)dx\Big]_S^T$$
$$+\gamma\int_S^T\int_0^1 y_{xt}y_xE_t(t)dxdt+\gamma\int_S^T\int_0^1 y_{xt}xy_{xx}E_t(t)dxdt+$$
$$\frac{1}{2}\int_S^T\Big(y_t^2(1,t)+\gamma y_{xt}^2(1,t)+y_{xx}^2(1,t)\Big)E(t)dt-\int_S^T\Big(\eta+\xi\Big)y_x(1,t)E(t)dt. \quad (2.5)$$

Using poincaré's inequality, we have :

$$\int_0^1 y_txy_xdx \le E(t), \quad \forall t\ge 0.$$

This implies that

$$-\Big[\int_0^1 y_txy_xE(t)dx\Big]_S^T+\int_S^T\int_0^1 y_txy_xE_t(t)dxdt \le 3E(S)E(0), \quad \forall T\ge S\ge 0.$$

Similarly we obtain that

$$-\gamma\Big[\int_0^1 y_{xt}y_xE(t)dx\Big]_S^T+\gamma\int_S^T\int_0^1 y_{xt}y_xE_t(t)dxdt \le 3\gamma E(S)E(0), \quad \forall T\ge S\ge 0,$$

$$-\gamma\Big[\int_0^1 y_{xt}xy_{xx}E(t)dx\Big]_S^T+\gamma\int_S^T\int_0^1 y_{xt}xy_{xx}E_t(t)dxdt \le 3\gamma E(S)E(0), \quad \forall T\ge S\ge 0.$$

Then using (2.5) we deduce that :

$$\frac{1}{2}\int_S^T\int_0^1\Big(y_t^2-\gamma y_{xt}^2+3y_{xx}^2\Big)E(t)dxdt \le 3(2\gamma+1)E(0)E(S)$$
$$+\frac{1}{2}E(S)\int_S^T\Big(y_t^2(1,t)+\gamma y_{xt}^2(1,t)+y_{xx}^2(1,t)\Big)dt-E(S)\int_S^T(\eta+\xi)y_x(1,t)dt. \quad (2.6)$$

On the other hand, from (1.4) we deduce that :

$$\int_S^T\Big(\xi^2(t)+\eta^2(t)\Big)dt = -\int_S^T E'(t)dt \le E(S). \quad (2.7)$$

Now assume that $u_0 \in D\Big((\mathcal{A}+\mathcal{B})^2\Big)$ then differentiating the system (1.6) with respect to the variable t gives :

$$\int_S^T\Big(\xi_t^2(t)+\eta_t^2(t)\Big)dt \le E_1(S)$$

where the energy of high order $E_1(t)$ is defined by :

$$E_1(t) = \frac{1}{2} \parallel S_{\mathcal{A}+\mathcal{B}}(t)(\mathcal{A}+\mathcal{B})u_0 \parallel_{\mathcal{H}}^2 = \frac{1}{2} \parallel u'(t) \parallel_{\mathcal{H}}^2, \quad \forall t \geq 0.$$

Then we deduce that :

$$\int_S^T \left(y_t^2(1,t) + \gamma y_{xt}^2(1,t) + y_{xx}^2(1,t) \right) dt =$$
$$\int_S^T \left[\left(\xi_t(t) + \xi(t) \right)^2 + \gamma \left(\eta_t(t) + \eta(t) \right)^2 + \eta^2(t) \right] dt$$
$$\leq 2(\gamma+1) \left(\frac{E_1(0)}{E(0)} + 1 \right) E(0). \quad (2.8)$$

Choosing $\varepsilon > 0$ then we have :

$$-\int_S^T (\eta + \xi) y_x(1,t) E(t) dt \leq \frac{1}{\varepsilon} \int_S^T (\eta^2 + \xi^2) E(t) dt + \frac{\varepsilon}{2} \int_S^T y_x^2(1,t) E(t) dt$$
$$\leq \frac{1}{\varepsilon} E(0)E(S) + \frac{\varepsilon}{2} \int_S^T \int_0^1 y_{xx}^2 E(t) dx dt. \quad (2.9)$$

Let $\varepsilon = \frac{1}{4}$. Use (2.8) and (2.9) in (2.6) gives (2.4). The proof is thus complete.

Lemma 2.3. *Assume that $u_0 \in D(\mathcal{A}+\mathcal{B})$. Let $0 \leq S \leq T < +\infty$. Then the solution u of the problem (1.6) satisfies :*

$$\int_S^T \int_0^1 (y_t^2 + \gamma y_{xt}^2 - \frac{9}{8} y_{xx}^2) E(t) dx dt \leq (3\gamma + 7) E(S) E(0). \quad (2.10)$$

Proof. Since $u_0 \in D(\mathcal{A}+\mathcal{B})$, then using (i) Lemma 2.1 we have $y \in H^3(0,1)$ and $y_{xx}(1,t) = -\eta(t) \in L^2(S;T)$. Multiplying the equation (1.1)-(1.2) by $yE(t)$ and integrating by parts we obtain that :

$$\int_S^T \int_0^1 \left(y_t^2 + \gamma y_{xt}^2 - y_{xx}^2 \right) E(t) dx dt = \left[\int_0^1 y_t y E(t) dx + \gamma \int_0^1 y_{xt} y_x E(t) dx \right]_S^T -$$
$$\int_S^T \int_0^1 y_t y E_t(t) dx dt - \gamma \int_S^T \int_0^1 y_{xt} y_x E_t(t) dx dt +$$
$$\int_S^T \left(\eta(t) y_x(1,t) + \xi(t) y(1,t) \right) E(t) dt. \quad (2.11)$$

Using poincarré's inequality, we have :

$$-\left[\int_0^1 y_t y E(t) dx \right]_S^T + \int_S^T \int_0^1 y_t y E_t(t) dx dt \leq 3E(S)E(0), \quad \forall T \geq S \geq 0,$$

and

$$-\gamma \left[\int_0^1 y_{xt} y_x E(t) dx \right]_S^T + \gamma \int_S^T \int_0^1 y_{xt} y_x E_t(t) dx dt \leq 3\gamma E(S)E(0), \quad \forall T \geq S \geq 0.$$

This implies that

$$\int_S^T \int_0^1 \Big(y_t^2 + \gamma y_{xt}^2 - y_{xx}^2 \Big) E(t) dx dt \le 3(\gamma+1)E(0)E(S) +$$

$$\int_S^T \Big(\eta y_x(1,t) + \xi(t) y(1,t) \Big) E(t) dt. \tag{2.12}$$

Now, choosing $\varepsilon > 0$. Then we have

$$\int_S^T \Big(\eta y_x(1,t) + \xi(t) y(1,t) \Big) E(t) dt \le$$

$$\frac{1}{2\varepsilon} \int_S^T \Big(\eta^2 + \xi^2 \Big) E(t) dt + \varepsilon \int_S^T \int_0^1 y_{xx}^2 E(t) dx dt. \tag{2.13}$$

Let $\varepsilon = \frac{1}{8}$. Use (2.7) and (2.13) in (2.12) gives (2.10).

Proof of the theorem 2.1 Assume that $u_0 \in D\Big((\mathcal{A}+\mathcal{B})^2 \Big)$. Using (2.4), (2.7) and (2.10) we obtain that

$$\int_S^T E^2(t) dt \le M E(0) E(S), \quad \forall 0 \le S \le T < +\infty$$

where we have put :

$$M = (\gamma+1) \frac{\| u_0 \|^2_{D(\mathcal{A})}}{\| u_0 \|^2_{\mathcal{H}}} + 10\gamma + 16.$$

Thanks to a classic result of Haraux (see [6]) we deduce that :

$$E(t) \le \ E(0) \frac{2M}{M+t}, \quad \forall\, t \ge 0. \tag{2.14}$$

Now, let $u_0 \in D(\mathcal{A}+\mathcal{B})$, by density (see Pazy [13]) there exists a sequence $u_0^n \in D\Big((\mathcal{A}+\mathcal{B})^2 \Big)$ such that $u_0^n \to u_0$ for the graph norm in $D(\mathcal{A}+\mathcal{B})$. Then we have :

$$E_n(t) = \frac{1}{2} \| S_{\mathcal{A}+\mathcal{B}}(t) u_0^n \|^2 \le \frac{1}{2} \| u_0^n \|^2 \, \frac{2M_n}{M_n+t}, \quad \forall\, t \ge 0$$

where

$$M_n = (\gamma+1) \frac{\| u_0^n \|^2_{D(\mathcal{A})}}{\| u_0^n \|^2_{\mathcal{H}}} + 10\gamma + 16.$$

Since $u_0^n \to u_0$ in $D(\mathcal{A}+\mathcal{B})$ we get : $M_n \to M$ and $S_{\mathcal{A}+\mathcal{B}}(t) u_0^n \to S_{\mathcal{A}+\mathcal{B}}(t) u_0$ in $\mathcal{H}$. Therefor we get

$$E(t) = \frac{1}{2} \| S_{\mathcal{A}+\mathcal{B}}(t) u_0 \|^2_{\mathcal{H}} \le \frac{1}{2} \| u_0 \|^2_{\mathcal{H}} \, \frac{2M}{M+t}, \quad \forall\, t \ge 0.$$

Then we prove (2.14) for any $u_0 \in D(\mathcal{A})$.

3 Optimal energy decay rate

Let $u_0 \in D(\mathcal{A})$, we define the energy decay rate ω by :

$$\omega(u_0) = \sup\left\{\alpha \in \mathbb{R} : \quad E(t) = \frac{1}{2} \parallel S_{\mathcal{A}+\mathcal{B}}(t)u_0 \parallel_{\mathcal{H}}^2 \leq \frac{C}{t^\alpha}\right\}.$$

From Theorem 2.1, we have $\omega(u_0) \geq 1$ for any $u_0 \in D(\mathcal{A})$. In the following, we will prove that this upper-bound is optimal in the sense that for any $\varepsilon > 0$ there exists $u_0^\varepsilon \in D(\mathcal{A})$ such that $\omega(u_0^\varepsilon) = 1 + \varepsilon$. We first recall the following result (see Littman-Markus [11]) :

Lemma 3.1. *Consider a C^0-semigroup $S_{\mathcal{A}}(t)$ acting on a real or complex Hilbert space $\mathcal{H}$, with infinitesimal generator $\mathcal{A}$. Assume that :*
(i) The eigenvalues λ_n of the operator $\mathcal{A}$ has the following form $\lambda_n = -\sigma_n + i\tau_n$ such that

$$\sigma_n > \frac{a}{n^\delta}, \qquad a > 0,\ \delta > 0.$$

(ii) The system of root vectors $\{\Phi_n\}_{n\geq 1}$ associated to the eigenvalues λ_n form a Riesz basis in $\mathcal{H}$.
(iii) Let $u_0 \in \mathcal{H}$ such that

$$u_0 = \sum_{n\geq 1} a_n \Phi_n, \qquad | a_n | \leq \frac{b}{n^q},\ b > 0,\ q > \frac{1}{2}.$$

Then there exists a constant $C > 0$ depending on u_0 such that

$$\parallel S_{\mathcal{A}}(t)u_0 \parallel_{\mathcal{H}} \leq \frac{C}{t^{(q-1/2)/\delta}}, \quad \forall t > 0.$$

Remark 3.1. *In fact, the Lemma is proved by mean of Riesz basis property. Moreover the corresponding series is evaluated by an equivalent improper integral. So if we assume that*

$$\sigma_n \sim \frac{a}{n^\delta} \quad \textit{and} \quad a_n \sim \frac{b}{n^q}.$$

Then we have

$$\parallel S_{\mathcal{A}}(t)u_0 \parallel_{\mathcal{H}} \sim \frac{C}{t^{(q-1/2)/\delta}}, \quad \forall t > 0.$$

Now we define the function logarithm by :

$$\ln(z) = \ln|z| + i\arg(z), \quad \text{where} \ \frac{\pi}{4} < \arg(z) < \frac{\pi}{4} + 2\pi. \tag{3.1}$$

Let $\lambda \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}+\mathcal{B}$ and $\Phi = (y, z, \eta, \xi)$ the corresponding eigenvector. Then we have

$$(\mathcal{A}+\mathcal{B})\Phi = \lambda\Phi.$$

This gives that

$$\begin{cases} y_{xxxx} - \gamma\lambda^2 y_{xx} + \lambda^2 y = 0, & 0 < x < 1\ , \\ y(0) = y_x(0) = 0, \\ y_{xx}(1) + \dfrac{\lambda}{\lambda - 1} y_x(1) = 0, \\ y_{xxx}(1) - \gamma\lambda^2 y_x(1) - \dfrac{\lambda}{\lambda - 1} y(1) = 0. \end{cases} \tag{3.2}$$

Since $\mathcal{A}$ is skew adjoint and $\mathcal{B}$ is compact, using Lemma 10.1 in Gohberg and Krein [5], for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that the spectrum of the operator $\mathcal{A} + \mathcal{B}$ is contained in the union of the disc $|\lambda| \leq r_\varepsilon$ and the two sectors :

$$\frac{\pi}{2} - \varepsilon < \arg(\lambda) < \frac{\pi}{2} + \varepsilon, \quad \frac{3\pi}{2} - \varepsilon < \arg(\lambda) < \frac{3\pi}{2} + \varepsilon.$$

On the other hand, the eigenvalues of $\mathcal{A} + \mathcal{B}$ are in conjugate pairs. Then we only the spectrum in the sector $\mathcal{D}_0$:

$$|\lambda| >> 1, \quad \text{and} \quad \frac{\pi}{2} - \varepsilon < \arg(\lambda) < \frac{\pi}{2} + \varepsilon. \tag{3.3}$$

Next for all $\lambda \in \mathcal{D}_0$, we define two holomorphic functions $\theta(\lambda)$ and $\omega(\lambda)$ by :

$$\theta(\lambda) = \sqrt{\frac{\gamma\lambda^2 + \sqrt{\gamma^2\lambda^4 - 4\lambda^2}}{2}}, \quad \omega(\lambda) = \sqrt{\frac{\gamma\lambda^2 - \sqrt{\gamma^2\lambda^4 - 4\lambda^2}}{2}}. \tag{3.4}$$

Then, we find that a general solution y of (3.2) is given by

$$y(x) = C_1\Big(\omega\sinh(\theta x) - \theta\sinh(\omega x)\Big) + C_2\omega\Big(\cosh(\theta x) - \cosh(\omega x)\Big) \tag{3.5}$$

where C_1 and C_2 are complex constants. Notice that $\lambda = 0$ is not be an eigenvalue of $\mathcal{A} + \mathcal{B}$. Hence by writing the bounded conditions at $x = 1$ in (3.2) in matrix form and taking the determinant of the coefficient matrix, We deduce that λ is an eigenvalue of $\mathcal{A} + \mathcal{B}$ if and only if λ is zero of the function :

$$f(\lambda) = P_2 e^{2\theta} + P_1 e^{\theta} + P_0, \tag{3.6}$$

where P_0, P_1 and P_2 are given by :

$$\begin{aligned} P_0(\lambda) = &\left(-\gamma\omega\theta + \frac{\omega\sqrt{\gamma^2\lambda^4 - 4\lambda^2}}{\lambda(\lambda-1)} + \frac{\omega\theta\sqrt{\gamma^2\lambda^4 - 4\lambda^2}}{\lambda^2(\lambda-1)^2} - \frac{\gamma\omega\theta^2}{\lambda(\lambda-1)} \right)\sinh(\omega) + \\ &\left(-\gamma\lambda^2 + 2 + \frac{2}{(\lambda-1)^2} - \frac{(\omega^2 - \lambda^2)\theta\sqrt{\gamma^2\lambda^4 - 4\lambda^2}}{\lambda^3(\lambda-1)} \right)\cosh(\omega), \end{aligned} \tag{3.7}$$

$$P_1(\lambda) = 2\left(2 - \frac{1 - \cosh(2\omega)}{(\lambda-1)^2} - \frac{\omega\sinh(2\omega)}{\lambda(\lambda-1)}\right), \tag{3.8}$$

$$\begin{aligned} P_2(\lambda) = &\left(\gamma\omega\theta + \frac{\omega\sqrt{\gamma^2\lambda^4 - 4\lambda^2}}{\lambda(\lambda-1)} + \frac{\omega\theta\sqrt{\gamma^2\lambda^4 - 4\lambda^2}}{\lambda^2(\lambda-1)^2} + \frac{\gamma\omega\theta^2}{\lambda(\lambda-1)} \right)\sinh(\omega) + \\ &\left(-\gamma\lambda^2 + 2 + \frac{2}{(\lambda-1)^2} + \frac{(\omega^2 - \lambda^2)\theta\sqrt{\gamma^2\lambda^4 - 4\lambda^2}}{\lambda^3(\lambda-1)} \right)\cosh(\omega). \end{aligned} \tag{3.9}$$

3.1 In what follows we analyze the spectrum of $\mathcal{A}+\mathcal{B}$

In this subsection we give a good asymptotic expansion of the eigenvalues of $\mathcal{A}+\mathcal{B}$.

Theorem 3.1. *Let* $a=\sqrt{\gamma}\tanh(\frac{1}{\sqrt{\gamma}})-\gamma+\frac{1}{2\gamma}$ *and* $b=\frac{1}{2\gamma^2}\tanh(\frac{1}{\sqrt{\gamma}})+\gamma$. *Then we have the following asymptotic expansion of the eigenvalues of* $\mathcal{A}+\mathcal{B}$

$$\sqrt{\gamma}\lambda_n=i\left(n\pi+\frac{\pi}{2}-\frac{a}{n\pi}+\frac{a}{2n^2\pi}+2(-1)^n\cosh^{-1}(\frac{1}{\sqrt{\gamma}})\frac{1}{n^2\pi^2}\right)+\frac{b\sqrt{\gamma}}{n^2\pi^2}+O\left(\frac{1}{n^3}\right) \quad (3.10)$$

for sufficiently large $n\in\mathbb{N}$.

Proof. If λ is an eigenvalue of $\mathcal{A}+\mathcal{B}$, then λ is a root of $f(\lambda)=0$. Then using (3.6) we have

$$e^{\theta}=-\frac{P_1}{2P_2}\pm\frac{\Delta^{1/2}}{2P_2}$$

where Δ is given by :

$$\Delta=P_1^2-4P_2P_0.$$

By using asymptotic analysis, (3.4) can be given as :

$$\theta(\lambda)=\sqrt{\gamma}\lambda\left(1-\frac{1}{2\gamma^2\lambda^2}-\frac{5}{8\gamma^4\lambda^4}+O\left(\frac{1}{\lambda^6}\right)\right), \quad (3.11)$$

$$\omega(\lambda)=\frac{1}{\sqrt{\gamma}}\left(1+\frac{1}{2\gamma^2\lambda^2}+O\left(\frac{1}{\lambda^4}\right)\right). \quad (3.12)$$

Then using (3.7), (3.8) and (3.9) we obtain that :

$$P_0=\lambda^2\Bigg[-\gamma\cosh(\frac{1}{\sqrt{\gamma}})+\left(-\gamma\sinh(\frac{1}{\sqrt{\gamma}})+\gamma^{3/2}\cosh(\frac{1}{\sqrt{\gamma}})\right)\frac{1}{\lambda}$$

$$+\left((\gamma^{3/2}+\sqrt{\gamma}+\frac{1}{2\gamma^{3/2}})\sinh(\frac{1}{\sqrt{\gamma}})+(2+\gamma^{3/2})\cosh(\frac{1}{\sqrt{\gamma}})\right)\frac{1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right)\Bigg], \quad (3.13)$$

$$P_1=4-2\left(1+\cosh(\frac{2}{\sqrt{\gamma}})+\frac{1}{\sqrt{\gamma}}\sinh(\frac{2}{\sqrt{\gamma}})\right)\frac{1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right)\Bigg), \quad (3.14)$$

and

$$P_2=\lambda^2\Bigg[-\gamma\cosh(\frac{1}{\sqrt{\gamma}})+\left(\gamma\sinh(\frac{1}{\sqrt{\gamma}})-\gamma^{3/2}\cosh(\frac{1}{\sqrt{\gamma}})\right)\frac{1}{\lambda}$$

$$+\left((\gamma^{3/2}+\sqrt{\gamma}-\frac{1}{2\gamma^{3/2}})\sinh(\frac{1}{\sqrt{\gamma}})+(2-\gamma^{3/2})\cosh(\frac{1}{\sqrt{\gamma}})\right)\frac{1}{\lambda^2}+O\left(\frac{1}{\lambda^3}\right)\Bigg]. \quad (3.15)$$

It follows from (3.13)-(3.15) that

$$e^{\theta}=\frac{-P_1}{2P_2}\pm\frac{\Delta^{1/2}}{2P_2}=\pm i\left(1+\frac{a_1}{a_0\lambda}+\frac{a_2a_0+a_1^2-a_4a_0\pm 4ia_0}{2a_0^2\lambda^2}+O\left(\frac{1}{\lambda^3}\right)\right), \quad (3.16)$$

where a_i are given by :

$$a_0=-\gamma\cosh(\frac{1}{\sqrt{\gamma}}),$$

$$a_1 = -\gamma \sinh(\frac{1}{\sqrt{\gamma}}) + \gamma^{3/2} \cosh(\frac{1}{\sqrt{\gamma}}),$$

$$a_2 = (\gamma^{3/2} + \sqrt{\gamma} + \frac{1}{2\gamma^{3/2}}) \sinh(\frac{1}{\sqrt{\gamma}}) + (2 + \gamma^{3/2}) \cosh(\frac{1}{\sqrt{\gamma}}),$$

$$a_3 = 1 + \cosh(\frac{2}{\sqrt{\gamma}}) + \frac{1}{\sqrt{\gamma}} \sinh(\frac{2}{\sqrt{\gamma}}),$$

and

$$a_4 = (\gamma^{3/2} + \sqrt{\gamma} - \frac{1}{2\gamma^{3/2}}) \sinh(\frac{1}{\sqrt{\gamma}}) + (2 - \gamma^{3/2}) \cosh(\frac{1}{\sqrt{\gamma}}).$$

Since $|\lambda_n|$ goes to infinity then, using (3.16), we obtain the first asymptotic expansion :

$$\theta_{2n} = 2in\pi + i\frac{\pi}{2} + \frac{a_1}{a_0\lambda_{2n}} + \frac{a_2 - a_4 + 4i}{2a_0\lambda_{2n}^2} + O\left(\frac{1}{\lambda_{2n}^3}\right), \tag{3.17}$$

$$\theta_{2n+1} = i(2n+1)\pi + i\frac{\pi}{2} + \frac{a_1}{a_0\lambda_{2n+1}} + \frac{a_2 - a_4 - 4i}{2a_0\lambda_{2n+1}^2} + O\left(\frac{1}{\lambda_{2n+1}^3}\right) \tag{3.18}$$

for sufficiently large $n \in \mathbb{N}$. Inserting (3.11) into (3.17) and (3.18), we obtain that :

$$\sqrt{\gamma}\lambda_{2n} = 2in\pi + i\frac{\pi}{2} + \left(\frac{a_1}{a_0} + \frac{1}{2\gamma^{3/2}}\right)\frac{1}{\lambda_{2n}} + \frac{a_2 - a_4 + 4i}{2a_0\lambda_{2n}^2} + O\left(\frac{1}{\lambda_{2n}^3}\right), \tag{3.19}$$

$$\sqrt{\gamma}\lambda_{2n+1} = i(2n+1)\pi + i\frac{\pi}{2} + \left(\frac{a_1}{a_0} + \frac{1}{2\gamma^{3/2}}\right)\frac{1}{\lambda_{2n+1}} + \frac{a_2 - a_4 - 4i}{2a_0\lambda_{2n+1}^2} + O\left(\frac{1}{\lambda_{2n+1}^3}\right)$$

for sufficiently large $n \in \mathbb{N}$. Then we deduce that

$$\sqrt{\gamma}\lambda_{2n} = 2in\pi + i\frac{\pi}{2} + O\left(\frac{1}{n}\right).$$

This implies

$$\frac{1}{\sqrt{\gamma}\lambda_{2n}} = \frac{1}{2in\pi} - \frac{1}{8in^2\pi} + O\left(\frac{1}{n^3}\right), \tag{3.20}$$

$$\frac{1}{\sqrt{\gamma}\lambda_{2n}^2} = -\frac{\sqrt{\gamma}}{4n^2\pi^2} + O\left(\frac{1}{n^3}\right) \tag{3.21}$$

for sufficiently large $n \in \mathbb{N}$.

Finally inserting (3.20) and (3.21) into (3.19) we obtain (3.10) for sufficiently large $n \in 2\mathbb{N}$. The same analysis can be done for all λ_{2n+1}. The proof is thus complete.

3.2 System of root vectors

We first recall the following result (see Rao [17]).

Lemma 3.2. *Let $\{\widetilde{\Phi}_n\}_{-\infty}^{+\infty}$ be a Riesz basis in a Hilbert space X, and let $\{\Phi_n\}_{|n|\geq N}$ be a $\omega -$ linearly independent system. Assume that*

$$\sum_{|n|\geq N} \| \Phi_n - \widetilde{\Phi}_n \|_X^2 < +\infty.$$

Then $\{\Phi_n\}_{|n|\geq N}$ is a Riesz basis in the subspace X_0 spanned by itself in X.

Lemma 3.3. *The system of root vectors of $\mathcal{A}+\mathcal{B}$ is complete in the energy space $\mathcal{H}$.*

Proof. Since the operator $i\mathcal{A}$ is selfadjoint with compact resolvent, then the spectrum of $\mathcal{A}$ consists entirely of isolated eigenvalues with finite multiplicities (see Brezis [2]). On the other hand, since $\mathcal{B}$ is a finite-dimensional and nonselfadjoint operator, then the s-numbers of $\mathcal{B}$ are given by (see Gohberg and Krein [5]):

$$s_1(\mathcal{B}) = 1,\ s_2(\mathcal{B}) = 1,\ s_j(\mathcal{B}) = 0,\ \forall j \geq 2.$$

We deduce that

$$\sum_{j=1}^{\infty} s_j(\mathcal{B}) = 2 < \infty.$$

It follows that the order of the compact operator $\mathcal{B}$ is *one*. From Theorem 10.1 in Gohberg and Krein [5], we conclude that the system of root vectors of $i(\mathcal{A}+\mathcal{B})$ is complete in the energy space $\mathcal{H}$. The proof is thus complete.

Now, let $\lambda_n \in \mathbb{C}$ be an eigenvalue of $\mathcal{A}+\mathcal{B}$. We will numerate the eigenvalues λ_n of high frequencies ($|n| \geq N$) following the asymptotic form (3.6). We denote by $\mu_l,\ 1 \leq l \leq L$, the eigenvalues of low frequencies with algebraic multiplicity $m_l \geq 1$. Let

$$K = \sum_{l=1}^{L} m_l$$

the total number of eigenvalues corresponding to the low frequency. Accordinly, we denote by Φ_n the eigenvector associated to the eigenvalue λ_n of high frequency, and by Ψ_k, $1 \leq k \leq K$ the eigenvector associated to the eigenvalue μ_k of low frequency. Thus we obtain a system of root vectors of $\mathcal{A}+\mathcal{B}$:

$$\{\Phi_n;\quad |n| \geq N\} \bigcup \{\Psi_k;\quad 1 \leq 1 \leq\ \ K\} \tag{3.22}$$

$$\Phi_n = \begin{pmatrix} y_n(x) \\ -\lambda_n y_n(x) \\ -y_{nxx}(1) \\ \frac{\lambda_n}{\lambda_n - 1} y_n(1) \end{pmatrix}, \quad |n| \geq N$$

where the function $y_n(x)$ is given by :

$$y_n(x) = C_1^n \bigg(\omega_n \sinh(\theta_n x) - \theta_n \sinh(\omega_n x) \bigg) + C_2^n \omega_n \bigg(\cosh(\theta_n x) - \cosh(\omega_n x) \bigg). \tag{3.23}$$

Using the boundary conditions in (3.2) we obtain that

$$C_1^n = -\nu_n C_2^n$$

where ν_n is given by :

$$\nu_n = \frac{\theta_n^2 \cosh(\theta_n) - \omega_n^2 \cosh(\omega_n) + \lambda_n(\lambda_n - 1)^{-1} \bigg(\theta_n \sinh(\theta_n) - \omega_n \sinh(\omega_n) \bigg)}{\theta_n^2 \sinh(\theta_n) + \theta_n \omega_n \sinh(\omega_n) + \lambda_n(\lambda_n - 1)^{-1} \theta_n \bigg(\cosh(\theta_n) - \cosh(\omega_n) \bigg)}.$$

On the other hand, using (3.10), (3.11) and (3.12) we deduce that

$$\sqrt{\gamma}\lambda_n = in\pi + i\frac{\pi}{2} + O\left(\frac{1}{n}\right), \quad \theta_n = in\pi + i\frac{\pi}{2} + O\left(\frac{1}{n}\right), \quad \omega_n = \frac{1}{\sqrt{\gamma}} + O\left(\frac{1}{n^2}\right). \quad (3.24)$$

This implies

$$\sinh(\theta_n) = i(-1)^n + O\left(\frac{1}{n^2}\right), \quad \cosh(\theta_n) = O\left(\frac{1}{n}\right),$$

$$\sinh(\omega_n) = \sinh\left(\frac{1}{\sqrt{\gamma}}\right) + O\left(\frac{1}{n^2}\right), \quad \cosh(\omega_n) = \cosh\left(\frac{1}{\gamma}\right) + O\left(\frac{1}{n^2}\right).$$

As consequence we have

$$\nu_n = O\left(\frac{1}{n}\right). \quad (3.25)$$

Theorem 3.2. *The system of root vectors (3.22) of $\mathcal{A} + \mathcal{B}$ is a Riesz basis in the energy space $\mathcal{H}$.*

Proof. Let $\sqrt{\gamma}\widetilde{\lambda}_n = i(n\pi + \frac{\pi}{2})$. We will defined the following function :

$$\widetilde{y}_n(x) = \frac{1}{\gamma\widetilde{\lambda}_n^2}\cos(n\pi + \frac{\pi}{2})x.$$

Taking $C_2^n = \dfrac{1}{\sqrt{\gamma}\lambda_n^2}$ and inseting (3.24) and (3.25) in (3.23) then we have

$$y_n(x) = \widetilde{y}_n(x) + O\left(\frac{1}{n^2}\right), \quad y_{nxx}(x) = \widetilde{y}_{nxx}(x) + O\left(\frac{1}{n}\right), \quad |n| \geq N. \quad (3.26)$$

As consequence we have

$$\Phi_n = \widetilde{\Phi}_n + O\left(\frac{1}{n}\right), \quad |n| \geq N$$

where $\tilde{\Phi}_n$ is given by

$$\tilde{\Phi}_n = \begin{pmatrix} \widetilde{y}_n(x) \\ -\widetilde{\lambda}\widetilde{y}_n(x) \\ \frac{1}{n} \\ \frac{1}{n} \end{pmatrix}. \quad (3.27)$$

It easy to see that the system (3.27) is a Riesz basis in the energy space $\mathcal{H}$.
On the other hand, we have

$$\| \Phi_n - \widetilde{\Phi}_n \|_{\mathcal{H}}^2 = \int_0^1 |y_{nxx}(x) - \widetilde{y}_{nxx}(x)|^2 \, dx + \int_0^1 \left|\lambda_n y_n(x) - \widetilde{\lambda}_n \widetilde{y}_n(x)\right|^2 dx$$

$$+\gamma \int_0^1 \left|\lambda_n y_{nx}(x) - \widetilde{\lambda}_n \widetilde{y}_{nx}(x)\right|^2 dx + \left|\frac{\lambda_n}{\lambda_n - 1} y_{nx}(1) - \frac{1}{n}\right|^2 + \left|\frac{\lambda_n}{\lambda_n - 1} y_n(1) - \frac{1}{n}\right|^2. \quad (3.28)$$

Inserting (3.26) into (3.28) we obtain that

$$\sum_{|n|\geq N} \| \Phi_n - \widetilde{\Phi}_n \|^2 < +\infty.$$

On the other hand, we know that the system (3.22) is ω-linearly independent (see Lemma A.6 in [3]), then applying Lemma 3.2 we conclude that the system (3.24) is Riesz basis in the subspace spanned by itself in $\mathcal{H}$, therefore in the whole space $\mathcal{H}$, since it is complete in $\mathcal{H}$ (Lemma 3.3). This achieves the proof.

Theorem 3.3. *The polynomial energy decay rate (2.1) is optimal:*

$$\inf_{u_0\in D(\mathcal{A})} \omega(u_0) = 1.$$

Proof. Let $n \in \mathbb{Z}^*$ by $|n| \geq N$ everywhere. Now let $\varepsilon > 0$, we define u_0^ε by

$$u_0^\varepsilon = \sum_{n\in\mathcal{Z}} \frac{1}{n^{3/2+\varepsilon}} \Phi_n.$$

Then we have

$$(\mathcal{A}+\mathcal{B})u_0^\varepsilon = \sum_{n\in\mathcal{Z}} \frac{\lambda_n}{n^{3/2+\varepsilon}} \Phi_n.$$

Using Theorem 3.1 we deduce that $\left|\frac{\lambda_n}{n^{3/2+\varepsilon}}\right| \sim \frac{1}{n^{1/2+\varepsilon}}$. Since $(\Phi_n)_{n\in\mathcal{Z}}$ is a Riesz basis in $\mathcal{H}$, then we have :

$$\| (\mathcal{A}+\mathcal{B})u_0^\varepsilon \|_{\mathcal{H}}^2 \sim \sum_{n\in\mathcal{Z}} \left|\frac{\lambda_n}{n^{3/2+\varepsilon}}\right|^2 \sim \sum_{n\in\mathcal{Z}} \frac{C}{n^{1+2\varepsilon}} < \infty.$$

This implies that $u_0^\varepsilon \in D(\mathcal{A})$. Finally, thanks to Lemma 3.1 and Remark 3.1 (with $q = 3/2+\varepsilon$ and $\delta = 2$) we deduce that

$$E_\varepsilon(t) = \frac{1}{2} \| S_{\mathcal{A}+\mathcal{B}}(t)u_0^\varepsilon \|_{\mathcal{H}}^2 \sim \frac{C_\varepsilon}{t^{1+\varepsilon}}, \quad \forall t > 0$$

where C_ε is a constant depending on u_0^ε. It follows that:

$$\omega(u_0^\varepsilon) = 1+\varepsilon.$$

On the other hand, from Theorem 2.1 we deduce that

$$\omega(u_0) \geq 1, \quad \forall u_0 \in D(\mathcal{A}).$$

Then we deduce that

$$1 \leq \inf_{u_0\in D(\mathcal{A})} \omega(u_0) \leq 1+\varepsilon.$$

Since ε can be arbitrarily small, we obtain the result. This achieves the proof.

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