Haagerup Property for subgroups of SL_2 and residually free groups

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Abstract

In this note, we prove that all subgroups of $\mathrm{SL}(2, R)$ have the Haagerup Property if R is a commutative reduced ring. This is based on the case when R is a field, recently established by Guentner, Higson, and Weinberger. As an application, residually free groups have the Haagerup Property.

A locally compact, σ-compact group G is said to have the Haagerup Property if it has a metrically proper isometric action on some Hilbert space. A recent panorama of the Haagerup Property is proposed in [CCJJV]. Here, we only need the following facts: the Haagerup Property is closed under taking closed subgroups, finite direct products, and the two less trivial facts ([CCJJV, chap. 6]):

(1) If Γ is a lattice in G, then G has the Haagerup Property if (and only if) Γ does.

(2) If G is discrete, then G has the Haagerup Property if and only if all its finitely generated subgroups do so.

Motivated by (2), we say that a discrete (not necessarily countable) group has the Haagerup Property if all its finitely generated subgroups do so (equivalently, if all its countable subgroups do so)[1].

The purpose of this short note is to point out a straightforward generalization of the following theorem:

[1]This coincides with the right definition of Haagerup Property for general locally compact groups: existence of a unitary C_0 representation weakly containing the trivial representation, or, equivalently, existence of a net of definite positive normalized C_0 functions, converging to 1, uniformly on compact subsets.

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Theorem 1 (Guentner-Higson-Weinberger [GHW]). *Let K be a field, and G be a subgroup of* $\mathrm{SL}(2,K)$*. Then G has the Haagerup Property (as a discrete group).*

Using Theorem 1, we obtain the following generalization.

Theorem 2. *Let R be a reduced (= without nilpotent elements) commutative ring, and G be a subgroup of* $\mathrm{SL}(2,R)$*. Then G has the Haagerup Property (as a discrete group).*

Proof: 1) Suppose R is a finite direct product of fields. Then it is a immediate consequence of Theorem 1 (since the Haagerup Property is stable under taking finite direct products and (closed) subgroups).

2) General case. We can suppose that G is finitely generated, hence that R is finitely generated as a ring. So R is Noetherian, hence has a finite number of minimal prime ideals $\mathfrak{p}_i$. Since R is reduced, $\bigcap \mathfrak{p}_i = \{0\}$, so that R embeds in $\prod R/\mathfrak{p}_i$, hence in the finite product $\prod K_i$, where $K_i = \mathrm{Frac}(R/\mathfrak{p}_i)$. So Case 1 applies. ∎

Remark 3. The assumption that R is reduced cannot be dropped. For instance, $\mathrm{SL}(2,\mathbf{Z}[t]/t^2)$ does not have the Haagerup Property. Indeed, if H is the kernel of the natural morphism $\mathrm{SL}(2,\mathbf{Z}[t]/t^2) \to \mathrm{SL}(2,\mathbf{Z})$, then H is infinite, while the pair $(\mathrm{SL}(2,\mathbf{Z}[t]/t^2), H)$ has Kazhdan's relative Property (T). This can be seen by embedding it as a lattice in the Lie group $\mathrm{SL}(2,\mathbf{R}[t]/t^2)$, which is isomorphic to $\mathrm{SL}(2,\mathbf{R}) \ltimes \mathfrak{sl}(2,\mathbf{R})$, where the action of $\mathrm{SL}_2(\mathbf{R})$ on the vector space $V = \mathfrak{sl}(2,\mathbf{R})$ is the adjoint action. This is the three-dimensional real irreducible representation of $\mathrm{SL}_2(\mathbf{R})$, so that it is well-known that $(\mathrm{SL}_2(\mathbf{R}) \ltimes V, V)$ has Property (T); see, for instance, Chapter 1 in [BHV].

Remark 4. The commutativity assumption cannot be dropped, even in Theorem 1. Indeed, let $\mathbf{H}$ be the skew-field of Hamilton quaternions. Then $\mathrm{SL}(2,\mathbf{H})$ has infinite subgroups with Kazhdan's Property (T): recall that $\mathrm{SL}(2,\mathbf{H}) \simeq \mathrm{SO}(5,1)$, the latter contains $\mathrm{SO}(5)$ as a subgroup, and it is well-known that $\mathrm{SO}(5)$ has infinite subgroups with Property (T) (for instance, obtained by projecting an irreducible lattice from $\mathrm{SO}(5) \times \mathrm{SO}(2,3)$).

Here is an application of Theorem 2. Recall that a group G is called residually free if it satisfies one of the (clearly) equivalent conditions:

- (i) For all $x \in G\backslash\{1\}$, there exist a (nonabelian) free group F and a morphism $f : G \to F$ such that $f(x) \neq 1$.
- (ii) G embeds in a direct product of free groups.
- (iii) G embeds in a direct product of free groups of finite rank.

Theorem 5. *Let G be a residually free group. Then G has the Haagerup Property.*

Proof: It suffices to show that any product $\prod_{i\in I} F_i$ of free groups of finite rank has the Haagerup Property. But such a product embeds in $\prod_{i\in I} \mathrm{SL}(2,\mathbf{Z}) = \mathrm{SL}(2,\mathbf{Z}^I)$. So this follows from Theorem 2. ∎

Remark 6. The Haagerup Property is not closed under infinite direct products (with the discrete topology). If it were, all residually finite groups would have the Haagerup Property! For instance, the discrete group $\prod_i \mathrm{SL}(n, \mathbf{Z}/p^i\mathbf{Z})$ does not have the Haagerup Property if $n \geq 3$, since it contains the infinite Kazhdan group $\mathrm{SL}(n, \mathbf{Z})$ as a subgroup. On the other hand, I do not know if the class of *torsion-free* groups with the Haagerup Property is closed under infinite products.

Remark 7. V. Guirardel pointed out to me that, using some nontrivial properties of residually free groups, Theorem 5 can directly be deduced from Theorem 1. Indeed, a residually free group can be embedded in $\mathrm{SL}(2, R)$, where R is a *finite* product of fields. The first ingredient is that a residually free group can be embedded in a finite direct product of fully residually free groups. The second ingredient is that a fully residually free group can be embedded in the ultraproduct ${}^*F_2$, which embeds in $\mathrm{SL}(2, {}^*\mathbf{Q})$, and ${}^*\mathbf{Q}$ is a field. For details and and many other interesting properties of (fully) residually free groups, see [CG].

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