Positive elements of left amenable Lau algebras

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Abstract

In the present paper, we deal with a large class of Banach algebras known as Lau algebras. It is well-known that if $\mathfrak{A}$ is a left amenable Lau algebra, then any $f \in \mathfrak{A}$ such that $|fg| = |f|g$ for all $g \in \mathfrak{A}$ with $g \geq 0$ is a scalar multiple of a positive element in $\mathfrak{A}$. We show that this result remains valid for the group algebra $\ell^1(G)$ of any, not necessarily amenable, discrete group G. We also give an example which shows that the result is, in general, not true without the hypothesis of left amenability of $\mathfrak{A}$. This resolves negatively an open problem raised by F. Ghahramani and A. T. Lau.

1 Introduction

Throughout this paper, $\mathfrak{A}$ will denote a *Lau algebra*; that is a complex Banach algebra which is the unique predual of a W^*-algebra $\mathfrak{M}$ and the identity element of $\mathfrak{M}$ is a multiplicative linear functional on $\mathfrak{A}$. Note that $\mathfrak{M}$ need not be unique; we shall endow the continuous dual $\mathfrak{A}^*$ of $\mathfrak{A}$ with a fixed W^*-algebra structure whose identity element 1 is multiplicative on $\mathfrak{A}$.

The subject of this class of Banach algebras originated with a paper published in 1983 by Lau [3] in which he referred to them as "F-algebras". Later on, in his useful monograph Pier [8] introduced the name "Lau algebra". As pointed out in [3], the wide range of Lau algebras includes the Fourier algebra $A(G)$, the Fourier-Stieltjes algebra $B(G)$, the group algebra $L^1(G)$ of a locally compact group G and

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the measure algebra of a locally compact topological semigroup; in particular it includes the convolution semigroup algebra $\ell^1(S)$ of a discrete semigroup S.

The Lau algebra $\mathfrak{A}$ is called *left amenable* if for each two-sided Banach $\mathfrak{A}$-module X with

$$f.x = f(1)\, x \qquad (f \in \mathfrak{A},\ x \in X),$$

every bounded derivation $D : \mathfrak{A} \to X^*$ is inner.

The notion of left amenability for Lau algebras was introduced by Lau [3]. In the same paper he extended several characterizations of amenable locally compact groups to left amenable Lau algebras; see also Ghahramani and Lau [1], Lashkarizadeh-Bami [2], Lau [3, 4], Lau and Wong [5], and the second author [6].

Let us recall from [3] that any commutative Lau algebra is left amenable; in particular, $A(G)$ and $B(G)$ of locally compact groups G are always left amenable. Also $L^1(G)$ (resp. $M(G)$) is left amenable if and only if G is amenable; see Pier [7] for details on amenable locally compact groups.

Denote by $P(\mathfrak{A})$ the positive cone of $\mathfrak{A}$ consisting of all elements f in $\mathfrak{A}$ that induce positive functionals on $\mathfrak{A}^*$ and by $P_1(\mathfrak{A})$ the set of all element in $P(\mathfrak{A})$ with norm one. Let us remark from Lemmas I.9.9 and III.3.2 of Takesaki [9] that

$$P(\mathfrak{A}) = \{f \in \mathfrak{A} :\ \|f\| = f(1)\}.$$

Also, for any $f \in \mathfrak{A}$, let $|f| \in P(\mathfrak{A})$ denote the absolute value of f regarded as an element in the predual of the W^*-algebra $\mathfrak{A}^*$. Indeed, the absolute value $h = |f|$ of f is uniquely determined by the properties

$$\|f\| = \|h\| \quad \text{and} \quad |f(y)|^2 \le \|f\|\, h(yy^*) \quad \text{for all} \quad y \in \mathfrak{A}^*.$$

In particular, $f \in P(\mathfrak{A})$ if and only if $|f| = f$; see [9], Proposition III.4.6.

It is shown in Proposition 2.4 of [1] that

Proposition 1.1. (Ghahramani-Lau) *Let $\mathfrak{A}$ be a left amenable Lau algebra. If $f \in \mathfrak{A}$ and $|fg| = |f|\, g$ for all $g \in P(\mathfrak{A})$, then f is a scalar multiple of an element in $P(\mathfrak{A})$.*

An open problem arising from [1] is that does Proposition 1.1 remain valid when $\mathfrak{A}$ is not left amenable?

Our main purpose in this paper is to give an example which shows that the answer to the Ghahramani-Lau problem is negative; i.e., Proposition 1.1 does not remain valid for all Lau algebras. Moreover, we show that the problem still has a positive answer for all group algebra $\ell^1(G)$ of discrete groups G. We also study multiples of positive elements of left amenable Lau algebras.

2 Solution to the Ghahramani-Lau problem

First, let us recall the following characterization of left amenable Lau algebras obtained by Lau [3], Corollary 4.8.

Theorem 2.1. *Let $\mathfrak{A}$ be a Lau algebra. A necessary and sufficient condition for that $\mathfrak{A}$ to be left amenable is that*

$$|f(1)| = \inf\{\|fg\| : g \in P_1(\mathfrak{A})\ \} \qquad\qquad (f \in \mathfrak{A}).$$

We are now ready to present an example which gives a negative answer to the Ghahramani-Lau problem.

Example 2.2. Let $\mathfrak{M}$ be a W^*-algebra and $\mathfrak{A}$ be its predual. Then $\mathfrak{A}$ with the multiplication defined by

$$fg = g(1)\, f \quad \text{for all} \quad f, g \in \mathfrak{A}$$

is a Lau algebra for which the following statements are equivalent.

(a) $\mathfrak{A}$ is left amenable.

(b) Any $f \in \mathfrak{A}$ is a scalar multiple of an element in $P(\mathfrak{A})$.

(c) $\mathfrak{A}$ is one dimensional.

Indeed, if $\mathfrak{A}$ is left amenable, then Theorem 2 implies that for every $f \in \mathfrak{A}$ we have

$$|f(1)| = \inf\{\|fg\| : g \in P_1(\mathfrak{A})\ \}.$$

But $\|fg\| = |g(1)|\ \|f\|$ for all $f, g \in \mathfrak{A}$ and hence $|f(1)| = \|f\|$ which is equivalent to (b); this is because that $f(1)^{-1}f \in P(\mathfrak{A})$ if $f(1) \neq 0$. That is (a) implies (b).

Now, if (b) holds, then for each two nonzero elements $f, g \in \mathfrak{A}$, we have $f(1)$ and $g(1)$ are nonzero, and therefore $f(1)^{-1}f - g(1)^{-1}g$ is a scalar multiple of an element in $P(\mathfrak{A})$. Thus

$$\left\|f(1)^{-1}f - g(1)^{-1}g\right\| = \left|\left(f(1)^{-1}f - g(1)^{-1}g\right)(1)\right|$$

whence $f = f(1)\ g(1)^{-1}g$; that is (c) holds. That (c) implies (a) is trivial.

Now, if moreover $\mathfrak{A}$ has dimension more than one, then $\mathfrak{A}$ is not left amenable and there is an element $f_0 \in \mathfrak{A}$ which is not a scalar multiple of an element in $P(\mathfrak{A})$; but

$$|f_0\, g| = |f_0|\ g \quad \text{for all} \quad g \in P(\mathfrak{A}).$$

This shows that the conclusion of Proposition 1.1 is not true for $\mathfrak{A}$.

Remark 2.3. It should be mentioned that Proposition 2.4 of [1] is stated as follows.

($\circledast$): Let $\mathfrak{A}$ be a left amenable Lau algebra. If $f \in \mathfrak{A}$ and $|fg| = |f|\ g$ for all $g \in \mathfrak{A}$, then f is a scalar multiple of an element in $P(\mathfrak{A})$.

It seems that there is a misprint in the statement there; $g \in \mathfrak{A}$ is stated instead of $g \in P(\mathfrak{A})$. In fact, a careful reading of Proposition 2.4 in [1] shows that the proof uses only "$g \in P(\mathfrak{A})$" and this condition was intended as part of this proposition.

On the other hand, a more general statement of ($\circledast$) is trivially true without the left amenability assumption of $\mathfrak{A}$. Indeed, let $\mathfrak{A}$ be an arbitrary Lau algebra. If $f \in \mathfrak{A}$ and $|fg| = |f|\, g$ for all $g \in \mathfrak{A}$, then with $g := -|f|$ we conclude that

$$|f|f|\,| = |f(-g)| = |f|g = -|f|^2$$

from which it follows that $f = 0$.

3 Positive elements of certain Lau algebras

We commence this section by a characterization of multiples of positive element in a left amenable Lau algebras.

Theorem 3.1. *Let $\mathfrak{A}$ be a left amenable Lau algebra and $f \in \mathfrak{A}$. Then the following are equivalent.*

(a) $|fg| = |f|\, g$ *for all* $g \in P(\mathfrak{A})$.
(b) $\|fg\| = \|f\|$ *for all* $g \in P_1(\mathfrak{A})$.
(c) $|f(1)| = \|f\|$.
(d) $f(1)^{-1} f \in P(\mathfrak{A})$.

Proof. That (a) implies (b) follows by evaluating at the identity 1 of $\mathfrak{A}^*$.

Now, if $\|fg\| = \|f\|$ for all $g \in P_1(\mathfrak{A})$, then since $\mathfrak{A}$ is left amenable, we conclude from Theorem 2.1 that

$$\begin{aligned} |f(1)| &= \inf\{\|fg\| : g \in P_1(\mathfrak{A})\} \\ &= \|f\|. \end{aligned}$$

That is (b) implies (c).

That (c) yields (d) will follow if we note that $f = 0$ when $f(1) = 0$, and $f(1)^{-1} f \in P(\mathfrak{A})$ otherwise. The rest implication is trivial. $\square$

To prepare the setting for the next result, let $\mathbf{C}P(\mathfrak{A})$ denote the set of all scalar multiples of elements in $P(\mathfrak{A})$, and note that

$$\mathbf{C}P(\mathfrak{A}) = \{f \in \mathfrak{A} : |f(1)| = \|f\|\,\}.$$

Let us remark that clearly for any Lau algebra $\mathfrak{A}$ we have

$$\mathbf{C}P(\mathfrak{A}) \subseteq \{f \in \mathfrak{A} : |fg| = |f|\, g \;\text{ for all }\; g \in P(\mathfrak{A})\,\}$$

This together with Theorem 3.1 imply that

$$\mathbf{C}P(\mathfrak{A}) = \{f \in \mathfrak{A} : |fg| = |f|\, g \;\text{ for all }\; g \in P(\mathfrak{A})\,\}$$

if $\mathfrak{A}$ is left amenable. The next result shows that this is not an “if and only if” statement. It proves that the conclusion of Proposition 1.1 holds for the Lau algebra $\ell^1(G)$ of an arbitrary discrete group G. Let us remark that left amenability of $\ell^1(G)$ is equivalent to amenability of G; see Corollary 4.3 of Lau [3]. In particular, $\ell^1(G)$ need not be left amenable in general; for example, if F_2 is the free group on two generators, then $\ell^1(F_2)$ is not left amenable; see Proposition 14.1 of Pier [7].

Proposition 3.2. *Let G be a discrete group and $\ell^1(G)$ be its convolution algebra. If $f \in \ell^1(G)$ and $|f * g| = |f| * g$ for all $g \in P(\ell^1(G))$, then f is a scalar multiple of an element in $P(\ell^1(G))$.*

Proof. Let $f \in \ell^1(G)$ be nonzero and $|f * g| = |f| * g$ for all $g \in P(\ell^1(G))$. Then by evaluating at the identity element of $\ell^\infty(G))$, we have

$$\|f * g\|_1 = \|f\|_1 \, \|g\|_1.$$

Now, let a be an element of G with $f(a) \neq 0$. Then, for every $x \in G$ we obtain

$$\|f * (\delta_x + \delta_a)\|_1 = 2\|f\|_1$$

because of $\delta_x + \delta_a \in P(\ell^1(G))$; here δ_t denotes the Dirac measure at $t \in G$. This implies that

$$\begin{aligned} \sum_{t \in G} |f(tx) + f(ta)| &= 2 \sum_{t \in G} |f(t)| \\ &= \sum_{t \in G} (|f(tx)| + |f(ta)|). \end{aligned}$$

We therefore have

$$|f(tx) + f(ta)| = |f(tx)| + |f(ta)| \quad \text{for all} \quad t, x \in G.$$

In particular,

$$|f(x) + f(a)| = |f(x)| + |f(a)| \quad \text{for all} \quad x \in G,$$

and hence there is a positive number $h(x)$ such that $f(x) = f(a)\ h(x)$. Since the function

$$h := \sum_{x \in G} h(x)\ \delta_x$$

is in $P(\ell^1(G))$, the result follows from that $f = f(a)\ h$. $\square$

The next example shows that Proposition 3.2 is not true for discrete semigroups.

Example 3.3. Let S be a discrete semigroup with binary operation $xy = x$ for all $x, y \in S$. Then the convolution semigroup algebra $\ell^1(S)$ of S is an example of Lau algebras considered in Example 2.2. Clearly, $\ell^1(S)$ is left amenable if and only if S is singleton. So, if S has at least two distinct elements, then $\ell^1(S)$ has an element f_0 with $|f_0 * g| = |f_0| * g$ for all $g \in P(\ell^1(S))$ which is not a scalar multiple of an element in $P(\ell^1(S))$. Thus Proposition 3.2 does not remain valid for discrete semigroups.

Finally, let us remark from Proposition 1.1 that the conclusion of Proposition 3.2 for discrete groups is also valid for amenable locally compact groups. This leads us to the following natural question.

Question. Does Proposition 3.2 remains true for an arbitrary locally compact group G ?

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