On the Hochschild cohomology of Beurling Algebras

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Abstract

Let G be a locally compact group and let ω be a weight function on G. Under a very mild assumption on ω, we show that $L^1(G,\omega)$ is (2n+1)-weakly amenable for every $n \in \mathbb{Z}^+$. Also for every odd $n \in \mathbb{N}$ we show that $\mathcal{H}^2(L^1(G,\omega),(L^1(G,\omega))^{(n)})$ is a Banach space.

1 introduction

In this paper we shall be concerned with the structure of the first and second cohomology group of $L^1(G,\omega)$ with coefficients in the nth dual space $(L^1(G,\omega))^{(n)}$. We begin by recalling some terminology.

Let $\mathcal{A}$ be a Banach algebra, and X be a Banach $\mathcal{A}$-bimodule. The dual space X' is a Banach $\mathcal{A}$-bimodule where the products $a \cdot \lambda$ and $\lambda \cdot a$ are specified by

$$a \cdot \lambda(x) = \lambda(x \cdot a), \qquad \lambda \cdot a(x) = \lambda(a \cdot x) \tag{1.1}$$

for all $a \in \mathcal{A}$, $x \in X$ and $\lambda \in X'$. The canonical embedding of X in X'' is denoted by $\imath$ or $\widehat{\ }$. We denote higher duals by $X^{(n+1)} = X^{(n)'}$ for all $n \in \mathbb{N}$; with the convention $X^{(0)} = X$. Then $X^{(n)}$ is also a Banach $\mathcal{A}$- bimodule; the definitions are consistent in the sense that $\widehat{a \cdot x} = a \cdot \hat{x}$. So that $X^{(n)}$ is a submodule of $X^{(n+2)}$. If X is symmetric, then so is $X^{(n)}$. If X is unital, then so is X'. The adjoint of the

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injective map $\imath : X^{(n-1)} \to X^{(n+1)}$ is the projective map $P : X^{(n+2)} \to X^{(n)}$, defined by $P(\Lambda) = \Lambda|_{\imath(X^{(n-1)})}$. Then P is a $\mathcal{A}$-bimodule morphism, and so we may write

$$X^{(n+2)} = X^{(n)} \oplus \operatorname{Ker} P = X^{(n)} \oplus \imath(X^{(n-1)})^{\perp},$$

as Banach $\mathcal{A}$-bimodules. We shall also consider the second dual $\mathcal{A}''$ of a Banach algebra $\mathcal{A}$ as a Banach algebra; indeed, two products are defined on $\mathcal{A}''$ as follows. Let $a \in \mathcal{A}$, $\lambda \in \mathcal{A}'$ and $m, n \in \mathcal{A}''$. Then $m \cdot \lambda$ and $\lambda \cdot m$ are defined by

$$m \cdot \lambda(a) = m(\lambda \cdot a), \qquad \lambda \cdot m(a) = m(a \cdot \lambda),$$

where $\lambda \cdot a$ and $a \cdot \lambda$ are defined by (1.1). Next $m \Box n$ and $m \diamond n$ are defined in $\mathcal{A}''$ by

$$m \Box n(\lambda) = m(n \cdot \lambda), \qquad m \diamond n(\lambda) = n(\lambda \cdot m). \tag{1.2}$$

Then $\mathcal{A}''$ is a Banach algebra with respect to each of the products $\Box$ and $\diamond$, which are called the first and second Arens products on $\mathcal{A}''$, respectively. For fixed n in $\mathcal{A}''$, the map $m \to m \Box n$ is weak* weak* continuous, but map $m \to n \diamond m$ in general is not weak* weak* continuous unless m is in $\mathcal{A}$.

The cohomology complex is

$$0 \longrightarrow X \xrightarrow{\delta^0} \mathcal{C}^1(\mathcal{A}, X) \xrightarrow{\delta^1} \mathcal{C}^2(\mathcal{A}, X) \xrightarrow{\delta^2} \cdots,$$

where for $n \in \mathbb{Z}^+$, $\mathcal{C}^n(\mathcal{A}, X)$ is the set of all bounded n-linear maps from $\mathcal{A}$ to X. The map $\delta^0 : X \longrightarrow \mathcal{C}^1(\mathcal{A}, X)$ is given by $\delta^0(x)(a) = a \cdot x - x \cdot a$ and for $n \in \mathbb{Z}^+$, the map $\delta^n : \mathcal{C}^n(\mathcal{A}, X) \longrightarrow \mathcal{C}^{n+1}(\mathcal{A}, X)$ is given by

$$\begin{aligned} \delta^n T(a_1, \ldots, a_{n+1}) \;=\; & a_1 \cdot T(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i T(a_1, \ldots, a_i a_{i+1}, \ldots a_{n+1}) \\ & + (-1)^{n+1} T(a_1, \ldots, a_n) \cdot a_{n+1}, \end{aligned}$$

where $T \in \mathcal{C}^n(\mathcal{A}, X)$ and $a_1, \ldots, a_{n+1} \in \mathcal{A}$. The space $\ker \delta^n$ of bounded n-cocycle is denoted by $\mathcal{Z}^n(\mathcal{A}, X)$ and the space $\operatorname{Im} \delta^{n-1}$ of bounded n-coboundary is denoted by $\mathcal{B}^n(\mathcal{A}, X)$. We recall that $\mathcal{B}^n(\mathcal{A}, X)$ is a subspace of $\mathcal{Z}^n(\mathcal{A}, X)$ and that the nth cohomology group $\mathcal{H}^n(\mathcal{A}, X)$ is defined by the quotient

$$\mathcal{H}^n(\mathcal{A}, X) = \frac{\mathcal{Z}^n(\mathcal{A}, X)}{\mathcal{B}^n(\mathcal{A}, X)},$$

which is called the nth Hochschild (continuous) cohomology of $\mathcal{A}$ with coefficients in X.

The n-cochain T is called *cyclic* if

$$T(a_1, a_2, \ldots, a_n)(a_0) = (-1)^n T(a_0, a_1, \ldots, a_{n-1})(a_n),$$

and we denote the linear space of all cyclic n-cochains by $\mathcal{C}^n_\lambda(\mathcal{A}, \mathcal{A}')$. It is well known (see [9]) that the cyclic cochains $\mathcal{C}^n_\lambda(\mathcal{A}, \mathcal{A}')$ form a subcomplex of $C^n(\mathcal{A}, \mathcal{A}')$, that is $\delta^n : \mathcal{C}^n_\lambda(\mathcal{A}, \mathcal{A}') \to \mathcal{C}^{n+1}_\lambda(\mathcal{A}, \mathcal{A}')$, and so we have cyclic versions of the spaces defined above, which we denote by $\mathcal{B}^n_\lambda(\mathcal{A}, \mathcal{A}')$, $\mathcal{Z}^n_\lambda(\mathcal{A}, \mathcal{A}')$ and $\mathcal{H}^n_\lambda(\mathcal{A}, \mathcal{A}')$. Note that it is usual to denote the cyclic cohomology group by $\mathcal{H}^n_\lambda(\mathcal{A})$, as there is only one bimodule used, namely $\mathcal{A}'$.

To show that $\mathcal{H}^n(\mathcal{A}, X) = 0$, we must show that every n-cocycle from $\mathcal{A}$ to X is an n-coboundary. In particular case for $n = 1$, $\mathcal{Z}^1(\mathcal{A}, X)$ is the space of all continuous derivations from $\mathcal{A}$ to X, and $\mathcal{B}^1(\mathcal{A}, X)$ is the space of all inner derivations from $\mathcal{A}$ to X. Thus $\mathcal{H}^1(\mathcal{A}, X) = 0$ if and only if each continuous derivation from $\mathcal{A}$ to X is inner.

The space $\mathcal{Z}^n(\mathcal{A}, X)$ is a Banach space, but in general $\mathcal{B}^n(\mathcal{A}, X)$ is not closed; we regard $\mathcal{H}^n(\mathcal{A}, X)$ as a complete seminormed space with respect to the quotient seminorm. This seminorm is a norm if and only if $\mathcal{B}^n(\mathcal{A}, X)$ is a closed subspace of $\mathcal{C}^n(\mathcal{A}, X)$, which means that $\mathcal{H}^n(\mathcal{A}, X)$ is a Banach space.

There have been very extensive studies devoted to calculation of the cohomology group $\mathcal{H}^1(\mathcal{A}, X)$ and the higher dimensional groups $\mathcal{H}^n(\mathcal{A}, X)$ for various classes of Banach algebras $\mathcal{A}$ and Banach $\mathcal{A}$-bimodules X. Our purpose here, being particularly concerned with the cohomology groups $\mathcal{H}^1(\mathcal{A}, X^{(n)})$ and $\mathcal{H}^2(\mathcal{A}, X^{(n)})$ for $n \in \mathbb{N}$.

A Banach algebra $\mathcal{A}$ is called n-weakly amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^{(n)}) = 0$. Note that 1-weakly amenable Banach algebras are called weakly amenable.

It was shown in [13] that $L^1(G)$ is weakly amenable for every locally compact group G; see also [6] for a shorter proof. Dales, Ghahramani and Grønbæk [5] showed that $L^1(G)$ is always $(2n+1)$-weakly amenable for $n \in \mathbb{Z}^+$. Johnson [14] for the free group on two generators, proved that $\mathcal{H}^1(\ell^1(\mathbb{F}_2), (\ell^1(\mathbb{F}_2))^{(n)}) = 0$ for every $n \in \mathbb{N}$ and in [12] he proved that $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \mathbb{C}) \neq 0$ which by [19, Theorem 8.3.1] implies that $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \ell^1(\mathbb{F}_2)) \neq 0$ and $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \ell^\infty(\mathbb{F}_2)) \neq 0$.

In [11] Ivanov and in [15] Matsumoto and Morita showed that $\mathcal{H}^2(\ell^1(G), \mathbb{C})$ is a Banach space for every discrete group G with trivial action on $\mathbb{C}$. A. Pourabbas [18] showed that the second cohomology group of $L^1(G)$ with coefficients in $L^1(G)^{(2n+1)}$ is a Banach space for every locally compact group G and every $n \in \mathbb{Z}^+$. Meanwhile Soma [20] showed that $\mathcal{H}^3(\ell^1(\mathbb{F}_2), \mathbb{R})$ is not a Banach space. In [4] Burger and Monod showed that for a compactly generated locally compact second countable group G, the second continuous cohomology $\mathcal{H}^2_{cb}(G, F)$ is a Banach space, where F is a separable coefficient module.

In this paper for every locally compact group G and every $n \in \mathbb{Z}^+$, first we show that $\mathcal{H}^1(L^1(G, \omega), L^1(G, \omega)^{(2n+1)}) = 0$. Next we show that the second cohomology group of $L^1(G, \omega)$ with coefficients in $L^1(G, \omega)^{(2n+1)}$ is a Banach space, where ω is a weight function with $\sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$. At the end we will give examples which show dependence of cohomology on the weight ω.

2 The first cohomology group

Let G be a locally compact group. A weight on G is a continuous function $\omega : G \to (0, \infty)$ satisfying $\omega(e) = 1$, $\omega(xy) \leq \omega(x)\omega(y)$ for all $x, y \in G$. We say that the weight ω is diagonally bounded if $\sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$. Throughout for a diagonally bounded weight ω we set $Db(\omega) = \sup\{\omega(g)\omega(g^{-1}) : g \in G\}$. The

Beurling algebra $L^1(G,\omega)$ is defined as below,

$$L^1(G,\omega) = \left\{ f : G \to \mathbb{C} \, : \, f \text{ is measurable and } \|f\|_1^\omega = \int |f(x)|\,\omega(x)d(x) < \infty \right\}.$$

$L^1(G,\omega)$ is a Banach algebra with convolution product and norm $\|\cdot\|_1^\omega$. The dual space $L^\infty(G,\omega^{-1}) = L^1(G,\omega)'$ consists of all measurable functions φ on G with

$$\|\varphi\|_\omega^\infty = \text{ess sup} \left\{ \frac{|\varphi(g)|}{\omega(g)} : g \in G \right\} < \infty.$$

$L^1(G,\omega)$ has a bounded approximate identity $\{e_\alpha\}$, and by [2, Proposition 28.7], the Banach algebra $(L^1(G,\omega)'', \Box)$ has a right identity element E such that $\|E\| \leq M$, where $M = \sup_\alpha \|e_\alpha\|_1^\omega$.

The space $M(G,\omega)$ of all complex, regular Borel measures μ on G such that $\mu \cdot \omega \in M(G)$ with the convolution product and norm

$$\|\mu\|_\omega = \int \omega(x) d\,|\mu|\,(x)$$

is a Banach algebra. The weighted measure algebra $M(G,\omega)$ has a unit element δ_e and contains $L^1(G,\omega)$ as a closed two sided ideal. Also $M(G,\omega)_* = C_0(G,\omega^{-1})$ consists of all continuous functions on G such that $\frac{f}{\omega} \in C_0(G)$.

Lemma 2.1. *The multiplier algebra of $L^1(G,\omega)$ is isometrically isomorphic with $M(G,\omega)$.*

Proof. The proof is similar to the proof $\Delta(L^1(G)) = M(G)$ [10, p. 276]. ■

Let $\{\mu_\alpha\}$ be a net in $M(G,\omega)$ and $\mu \in M(G,\omega)$. We say that (μ_α) tends to μ in so-topology if for every $f \in L^1(G,\omega)$, we have

$$\mu_\alpha * f \to \mu * f \quad \text{and} \quad f * \mu_\alpha \to f * \mu.$$

Lemma 2.2. *Let G be a locally compact group. Then the so-closed convex span of*

$$\left\{ \frac{\lambda}{\omega(g)} \delta_g : g \in G, \lambda \in \mathbb{C}, |\lambda| = 1 \right\}$$

is the unit ball in $M(G,\omega)$.

Proof. The proof is the same as the unweighted case [8, 1.1.1-1.1.3]. ■

Note. By the previous Lemma every measure μ in $M(G,\omega)$ is the so-limit of a net $\{\mu_\alpha\}$, where each μ_α is a linear combination of point masses.

Now for every $n \in \mathbb{Z}^+$ we will show that $L^1(G,\omega)_{\mathbb{R}}^{(2n+1)}$, the real-valued functions in $L^1(G,\omega)^{(2n+1)}$, is a complete lattice in the sense that every non-empty subset of $L^1(G,\omega)^{(2n+1)}$ which is bounded above has a supremum.

Proposition 2.3. *The Banach space $L^\infty(G,\omega^{-1})$ with the product*

$$f \cdot g(x) = \frac{f(x)g(x)}{\omega(x)}, \qquad f,g \in L^\infty(G,\omega^{-1})$$

and complex conjugate as involution is a commutative C-algebra.*

Proof. Define $\varphi : L^\infty(G,\omega^{-1}) \to L^\infty(G)$ by $\varphi(f) = f\omega^{-1}$. Then φ is a $*$-isometrical isomorphism from $L^\infty(G,\omega^{-1})$ onto $L^\infty(G)$. Thus $L^\infty(G,\omega^{-1})$ is a commutative C*-algebra. ∎

Remark 2.4. Set $X = L^1(G,\omega)^{(2n)}$ $(n \geq 1)$. We note that $L^1(G,\omega)' = L^\infty(G,\omega^{-1})$ is a commutative C*-algebra. Because the second dual of a commutative C*-algebra is a commutative von Neumann algebra, then $X' = L^1(G,\omega)^{(2n+1)}$ is the underlying space of a commutative von Neumann algebra, and hence it is an L^∞-space. The space $X'_{\mathbb{R}}$ of real-valued functions in X' forms a complete lattice.

Throughout the rest of this section we set $\mathcal{A} = L^1(G,\omega)$ and $X = \mathcal{A}^{(2n+2)}$, where $n \in \mathbb{Z}^+$. The map

$$\theta : M(G,\omega) \to (\mathcal{A}'', \Box), \qquad \mu \mapsto E\Box\mu$$

is a continuous embedding. In fact for all $\mu \in M(G,\omega)$ we have

$$\|\theta(\mu)\| \leq \|\mu\|_\omega \|E\| \leq \|\mu\|_\omega M.$$

We write E_s for $E\Box\delta_s$, where $s \in G$ and E is a right identity for $(\mathcal{A}'', \Box)$. If $D : \mathcal{A} \longrightarrow X'$ is a continuous derivation, then by [5, Proposition 1.7] $D'' : (\mathcal{A}'', \Box) \longrightarrow X'''$ is a continuous derivation.

Lemma 2.5. *Let ω be a diagonally bounded weight on G. Then*

(i) *For every subset B of $X'_{\mathbb{R}}$, and for every $r \in G$, we have*

$$E \cdot \sup\{E_r \cdot \Lambda : \Lambda \in B\} = E_r \cdot \sup\{E \cdot \Lambda : \Lambda \in B\}$$

and

$$\sup\{E_r \cdot \Lambda : \Lambda \in B\} \cdot E = \sup\{E \cdot \Lambda : \Lambda \in B\} \cdot E_r.$$

(ii) *The set $\{E_{s^{-1}} \cdot \operatorname{Re} D''(E_s) : s \in G\}$ is a bounded subset of $X'_{\mathbb{R}}$.*

Proof. (i) Let $\alpha = \sup\{E \cdot \Lambda : \Lambda \in B\}$ and $\gamma = \sup\{E_r \cdot \Lambda : \Lambda \in B\}$. For all $\Lambda \in B$ we have $E_r \cdot \Lambda = E_r \cdot (E \cdot \Lambda) \leq E_r \cdot \alpha$. So

$$E \cdot \sup\{E_r \cdot \Lambda : \Lambda \in B\} \leq E_r \sup\{E \cdot \Lambda : \Lambda \in B\}.$$

Conversely

$$\alpha = \sup\{E \cdot \Lambda : \Lambda \in B\} = \sup\{E_{r^{-1}}(E_r \cdot E \cdot \Lambda) : \Lambda \in B\} \leq E_{r^{-1}} \cdot E \cdot \gamma.$$

Thus $E_r \cdot \alpha \leq E \cdot \gamma$. By the same method we have

$$\sup\{E_r \cdot \Lambda : \Lambda \in B\} \cdot E = \sup\{E \cdot \Lambda : \Lambda \in B\} \cdot E_r.$$

(ii) Since $\|E_s\| \leq \omega(s)M$ for every $s \in G$, then

$$\begin{aligned}\|E_{s^{-1}} \cdot \operatorname{Re} D''(E_s)\| &= \|Re(E_{s^{-1}} \cdot D''(E_s))\| \\ &\leq \|E_{s^{-1}} \cdot D''(E_s)\| \leq \|E_{s^{-1}}\| \, \|D''\| \, \|E_s\| \\ &\leq \omega(s)\omega(s^{-1}) \, \|D''\| \, M^2 \leq Db(\omega) \, \|D''\| \, M^2.\end{aligned}$$

Thus $\{E_{s^{-1}} \cdot \operatorname{Re}(D''(E_s)) : s \in G\}$ is a bounded subset of $X'_{\mathbb{R}}$. ■

Theorem 2.6. *Let G be a locally compact group. Then $L^1(G,\omega)$ is a $(2n+1)$-weakly amenable for every $n \in \mathbb{Z}^+$, whenever ω is a diagonally bounded weight on G.*

Proof. Set $\mathcal{A} = L^1(G,\omega)$ and $X = L^1(G,\omega)^{(2n)}$. The result in [17] establishes the case $n = 1$ and we may suppose that $n \in \mathbb{N}$. Let $\{e_\alpha\}$ be a bounded approximate identity for $\mathcal{A}$. Then there exists a right identity E for $(\mathcal{A}'', \Box)$ such that $\|E\| \leq M$.

Since $\mathcal{A}$ is a closed ideal of $M(G,\omega)$, then by [7] $(\mathcal{A}'', \Box)$ is a closed ideal of $(M(G,\omega)'', \Box)$. Let $D \in \mathcal{Z}^1(A, X')$. Then $D'' : (\mathcal{A}'', \Box) \to X'''$ is a continuous derivation. For $r, s \in G$ we have

$$D''(E_{st}) = D''(E_s) \cdot E_t + E_s \cdot D''(E_t)$$

and so

$$E_{(st)^{-1}} \cdot D''(E_{st}) = E_{t^{-1}} \cdot (E_{s^{-1}} \cdot D''(E_s)) \cdot E_t + E_{t^{-1}} \cdot D''(E_t). \tag{2.1}$$

By Lemma 2.5(ii) the set $\{E_{s^{-1}} \cdot \operatorname{Re} D''(E_s) : s \in G\}$ is bounded in $X'''_{\mathbb{R}}$. Since $X'''_{\mathbb{R}}$ is a complete lattice, then

$$\phi_r = \sup\{E_{s^{-1}} \cdot \operatorname{Re}(D''(E_s)) : s \in G\} \tag{2.2}$$

exists in $X'''_{\mathbb{R}}$. Let $t \in G$. Then from (2.1), (2.2) and Lemma 2.5(i) we have

$$E \cdot \phi_r \cdot E = E_{t^{-1}} \cdot \phi_r \cdot E_t + E_{t^{-1}} \cdot \operatorname{Re} D''(E_t) \cdot E.$$

Hence

$$E \cdot \operatorname{Re} D''(E_t) \cdot E = E_t \cdot \phi_r \cdot E - E \cdot \phi_r \cdot E_t.$$

Similarly, by considering imaginary parts we obtain $\phi_i \in X'''_{\mathbb{R}}$ such that

$$E \cdot \operatorname{Im} D''(E_t) \cdot E = E_t \cdot \phi_i \cdot E - E \cdot \phi_i \cdot E_t.$$

Thus if we define $\phi = \phi_r + \phi_i$, then $\phi \in X'''$ and for all $t \in G$,

$$E \cdot D''(E_t) \cdot E = E_t \cdot \phi \cdot E - E \cdot \phi \cdot E_t.$$

If ν is a linear combination of point masses and $f, g \in \mathcal{A}$, then we have

$$f \cdot D''(E\Box\nu) \cdot g = (f * \nu) \cdot \phi \cdot g - f \cdot \phi \cdot (\nu * g). \tag{2.3}$$

Now take $h \in \mathcal{A}$. Then there is a net $\{\nu_\alpha\}$ of linear combination of point masses such that $\nu_\alpha \to h$ in the strong operator topology on $\mathcal{A}$, that is, $\lim_\alpha(f * \nu_\alpha) = f * h$ and $\lim_\alpha(\nu_\alpha * g) = h * g$ for every $f, g \in \mathcal{A}$.

Let $f, g \in \mathcal{A}$. Then

$$\begin{aligned}\lim_\alpha f \cdot D''(E\Box\nu_\alpha) \cdot g &= \lim_\alpha (D''(f * \nu_\alpha) \cdot g - D''(f) \cdot (\nu_\alpha * g)) \\ &= D''(f * h) \cdot g - D''(f) \cdot (h * g) \\ &= f \cdot D''(h) \cdot g.\end{aligned}$$

So, from (2.3) we have

$$\begin{aligned}f \cdot D''(h) \cdot g &= (f * h) \cdot \phi \cdot g - f \cdot \phi \cdot (h * g) \\ &= f \cdot (h \cdot \phi - \phi \cdot h) \cdot g.\end{aligned}$$

Let $P : X''' \to X' = \mathcal{A}^{(2k+1)}$ be the natural projection, so that P is an $\mathcal{A}$-bimodule morphism. We have $D = P \circ D''$. Set $\phi_0 = P(\phi)$. Then

$$f \cdot D(h) \cdot g = f \cdot (h \cdot \phi_0 - \phi_0 \cdot h) \cdot g$$

for every $f, g, h \in \mathcal{A}$, and so

$$D(h)(f \cdot x \cdot g) = (h \cdot \phi_0 - \phi_0 \cdot h)(f \cdot x \cdot g)$$

for every $f, g, h \in \mathcal{A}$ and $x \in X$. Now by [5, proposition 1.17] we have $D(h)(x) = (h\cdot\phi_0 - \phi_0\cdot h)(x)$. Then D is an inner derivation and so $\mathcal{A}$ is $(2k+1)$- weak amenable. ∎

3 The second cohomology group

In this section firstly we prove that $\mathcal{H}^2(\ell^1(G,\omega), \ell^1(G,\omega)^{(2n+1)})$ is a Banach space for every discrete group G. Secondly we will generalize this method to show that $\mathcal{H}^2(L^1(G,\omega), (L^1(G,\omega))^{(2n+1)})$ is a Banach space for every locally compact group G. Recall that we set $Db(\omega) = \sup\{\omega(g)\omega(g^{-1}) : g \in G\}$.

Theorem 3.1. *$\mathcal{H}^2(\ell^1(G,\omega), \ell^1(G,\omega)^{(2n+1)})$ is a Banach space for every discrete group G and for every diagonally bounded weight ω.*

Proof. Set $X = \ell^1(G,\omega)^{(2n)}$. Let $\psi \in \mathcal{C}^1(\ell^1(G,\omega), X')$. Then for every $g, h \in G$ and $s \in X$ with $\|s\| \le 1$ we have

$$|\delta\psi(g,h)(s)| = |\psi(g)(hs) - \psi(gh)(s) + \psi(h)(sg)| \le \|\delta\psi\|\, \omega(g)\omega(h). \tag{3.1}$$

Since the set $\{\operatorname{Re}\psi(g) \cdot g^{-1} : g \in G\}$ is bounded above by $\|\psi\|\, Db(\omega)$ in $X'_{\mathbb{R}}$. Then

$$f_r(s) = \sup_{g \in G} \left\{ \operatorname{Re}\psi(g)(g^{-1}s) \right\},$$

exists in $X'_{\mathbb{R}}$. For every $h \in G$ by (3.1) we have

$$\begin{aligned} f_r(hs) &= \sup_{g\in G}\left\{\operatorname{Re}\psi(g)(g^{-1}hs)\right\} \\ &= \sup_{g\in G}\left\{\operatorname{Re}\psi(hg)(g^{-1}s)\right\} \\ &\le \sup_{g\in G}\left\{\operatorname{Re}\psi(h)(s) + \operatorname{Re}\psi(g)(g^{-1}sh) + \|\delta\psi\|\,\omega(g)\omega(g^{-1})\omega(h)\right\} \\ &= \operatorname{Re}\psi(h)(s) + \sup_{g\in G}\left\{\operatorname{Re}\psi(g)(g^{-1}sh)\right\} + \|\delta\psi\|\,\omega(h)Db(\omega) \\ &= \operatorname{Re}\psi(h)(s) + f_r(sh) + \|\delta\psi\|\,\omega(h)Db(\omega). \end{aligned} \tag{3.2}$$

On the other hand

$$\begin{aligned} f_r(hs) &= \sup_{g\in G}\left\{\operatorname{Re}\psi(g)(g^{-1}hs)\right\} \\ &\ge \operatorname{Re}\psi(h)(s) + f_r(sh) - \|\delta\psi\|\,\omega(h)Db(\omega). \end{aligned} \tag{3.3}$$

¿From (3.2) and (3.3) we have

$$|h\cdot f_r(s) - f_r\cdot h(s) + \operatorname{Re}\psi(h)(s)| \le \|\delta\psi\|\,\omega(h)Db(\omega).$$

Similarly, by considering imaginary parts we have

$$|h\cdot f_i(s) - f_i\cdot h(s) + \operatorname{Im}\psi(h)(s)| \le \|\delta\psi\|\,\omega(h)Db(\omega).$$

By putting $f = f_r + if_i$ we obtain

$$|h\cdot f(s) - f\cdot h(s) + \psi(h)(s)| \le 2\,\|\delta\psi\|\,\omega(h)Db(\omega).$$

Now let us define

$$\bar\psi(h)(s) = (\delta f)(h)(s) + \psi(h)(s),$$

so $\delta\bar\psi = \delta\psi$ and $\left|\bar\psi(h)(s)\right| \le 2\,\|\delta\psi\|\,\omega(h)Db(\omega)||s||$ for every $h \in G$ and $s \in X$. Thus $\left\|\bar\psi\right\| \le 2\,\|\delta\psi\|\,Db(\omega)$ and this finishes the proof. ■

Lemma 3.2. *The cyclic cohomology group $\mathcal{H}^2_\lambda(\ell^1(G,\omega))$ is a Banach space for every discrete group G and for every diagonally bounded weight ω.*

Proof. Let $\psi \in \mathcal{C}^1(\ell^1(G,\omega), \ell^\infty(G,\omega^{-1}))$ such that $\psi(h)(g) = -\psi(g)(h)$ for $g, h \in G$, and let us consider $\bar\psi(h)(g) = (\delta f)(h)(g)+\psi(h)(g)$ as in Theorem 3.1. Then $\delta\bar\psi = \delta\psi$ and $\left\|\bar\psi\right\| \le 2\,\|\delta\psi\|\,Db(\omega)$, further

$$\begin{aligned} \bar\psi(h)(g) &= (\delta f)(h)(g) + \psi(h)(g) \\ &= -(\delta f)(g)(h) - \psi(g)(h) \\ &= -\bar\psi(g)(h). \end{aligned}$$

Hence $\mathcal{H}^2_\lambda(\ell^1(G),\omega)$ is a Banach space. ■

We can now state the final result of this paper, we show that the cohomology group $\mathcal{H}^2(L^1(G,\omega), L^1(G,\omega)^{(2n+1)})$ is a Banach space for every locally compact group G and every diagonally bounded weight ω.

We recall a construction that shows that $L^\infty(G,\omega^{-1})$ is an $M(G,\omega)$-bimodule. For $f \in L^\infty(G,\omega^{-1}), a \in L^1(G,\omega)$ and $\mu \in M(G,\omega)$ define the module actions by

$$(f\mu)(a) = f(\mu * a) \quad \text{and} \quad (\mu f)(a) = f(a * \mu).$$

Throughout this section the notations lim sup and lim inf are frequently simplified to $\overline{\lim}$ and $\underline{\lim}$. We denote by so-lim μ_α the limits of measures in the strong operator topology.

Proposition 3.3. *Set $X = L^1(G,\omega)^{(2n)}$. Let $\psi \in \mathcal{C}^1(L^1(G,\omega), X')$. Then there is a $\tilde{\psi} \in \mathcal{C}^1(M(G,\omega), X')$ with*

(i) $\tilde{\psi}|_{L^1(G,\omega)} = \psi$ *and* $\delta\tilde{\psi}|_{L^1(G,\omega)\times L^1(G,\omega)} = \delta\psi$.

(ii) *Let μ be in $M(G,\omega)$ with $\|\mu\|_\omega \le 1$, and let x be in X with $\|x\| \le 1$ and $a, b \in L^1(G,\omega)$ with $\|a\|_1^\omega \le 1$ and $\|b\|_1^\omega \le 1$. If $\{\mu_\alpha\}$ is a net in $M(G,\omega)$ with $\|\mu_\alpha\|_\omega \le 1$ such that* so-lim $\mu_\alpha = \mu$, *then*

$$\left|(\overline{\lim}_\alpha \operatorname{Re}\tilde{\psi}(\mu_\alpha)(a\cdot x\cdot b) + i\,\overline{\lim}_\alpha \operatorname{Im}\tilde{\psi}(\mu_\alpha)(a\cdot x\cdot b)) - \tilde{\psi}(\mu)(a\cdot x\cdot b)\right| \le 3\left\|\delta\tilde{\psi}\right\|.$$

Proof. (i) We follow the proof of [12, Lemma 1.10] for this particular case. Let $\mu \in M(G,\omega)$ and let $\{e_\alpha\}$ be a bounded approximate identity for $L^1(G,\omega)$ with bound M. Defining

$$\psi_\alpha(\mu) = \psi(\mu * e_\alpha)$$

we see that ψ_α is a bounded net in $\mathcal{C}^1(M(G,\omega), X')$ and so has a cofinal subnet ψ_β convergent to a limit $\tilde{\psi}$ in the weak*-topology induced by identifying $\mathcal{C}^1(M(G,\omega), X')$ with $\mathcal{C}_1(M(G,\omega), X)'$. Thus

$$\lim_\beta \psi(\mu * e_\beta)(x) = \tilde{\psi}(\mu)(x)$$

for all $\mu \in M(G,\omega)$, $x \in X$. Since for all $a \in L^1(G,\omega)$, $\psi(a * e_\beta) \to \psi(a)$ in norm, $\tilde{\psi}|_{L^1(G,\omega)} = \psi$. Also $\delta\tilde{\psi}|_{L^1(G,\omega)\times L^1(G,\omega)} = \delta\psi$.

To prove (ii) let us consider $\mu, \nu \in M(G,\omega)$ with $\|\mu\|_\omega, \|\nu\|_\omega \le 1$ and $x \in X$ with $\|x\| \le 1$. Then

$$\left|\delta\tilde{\psi}(\mu,\nu)(x)\right| = \left|\mu\cdot\tilde{\psi}(\nu)(x) - \tilde{\psi}(\mu*\nu)(x) + \tilde{\psi}(\mu)\cdot\nu(x)\right| \le \left\|\delta\tilde{\psi}\right\|. \tag{3.4}$$

For $a, b \in L^1(G,\omega)$ with $\|a\|_1^\omega \le 1, \|b\|_1^\omega \le 1$ and $x \in X$ with $\|x\| \le 1$ by (3.4)

$$\begin{aligned} -\operatorname{Re}\tilde{\psi}(\mu_\alpha)(a\cdot x\cdot b) &= -\operatorname{Re}\tilde{\psi}(\mu_\alpha)\cdot a(x\cdot b) \\ &\le \operatorname{Re}\mu_\alpha\cdot\psi(a)(x\cdot b) - \operatorname{Re}\psi(\mu_\alpha * a)(x\cdot b) + \left\|\delta\tilde{\psi}\right\| \end{aligned}$$

and so

$$\begin{aligned} -\overline{\lim}\operatorname{Re}\tilde{\psi}(\mu_\alpha)(a\cdot x\cdot b) &\le \underline{\lim}\left\{\operatorname{Re}\mu_\alpha\cdot\psi(a)(x\cdot b) - \operatorname{Re}\psi(\mu_\alpha * a)(x\cdot b) + \left\|\delta\tilde{\psi}\right\|\right\} \\ &= \operatorname{Re}\mu\cdot\psi(a)(x\cdot b) - \operatorname{Re}\psi(\mu * a)(x\cdot b) + \left\|\delta\tilde{\psi}\right\|. \end{aligned}$$

On the other hand

$$-\overline{\lim}\,\mathrm{Re}\,\tilde{\psi}(\mu_\alpha)(a\cdot x\cdot b)\geq \mathrm{Re}\,\mu\cdot\psi(a)(x\cdot b)-\mathrm{Re}\,\psi(\mu*a)(x\cdot b)-\left\|\delta\tilde{\psi}\right\|.$$

Hence

$$\left|\mu\cdot\mathrm{Re}\,\psi(a)(x\cdot b)-\mathrm{Re}\,\psi(\mu*a)(x\cdot b)+\overline{\lim}\,\mathrm{Re}\,\tilde{\psi}(\mu_\alpha)(a\cdot x\cdot b)\right|\leq\left\|\delta\tilde{\psi}\right\|.$$

Similarly for imaginary parts we have

$$\left|\mu\cdot\mathrm{Im}\,\psi(a)(x\cdot b)-\mathrm{Im}\,\psi(\mu*a)(x\cdot b)+\overline{\lim}\,\mathrm{Im}\,\tilde{\psi}(\mu_\alpha)(a\cdot x\cdot b)\right|\leq\left\|\delta\tilde{\psi}\right\|.$$

Therefore

$$\begin{aligned}&\Big|\mu\cdot\psi(a)(x\cdot b)-\psi(\mu*a)(x\cdot b)\\&\qquad+\left(\overline{\lim}\,\mathrm{Re}\,\tilde{\psi}(\mu_\alpha)+i\,\overline{\lim}\,\mathrm{Im}\,\tilde{\psi}(\mu_\alpha)\right)(a\cdot x\cdot b)\Big|\leq 2\left\|\delta\tilde{\psi}\right\|.\end{aligned}\tag{3.5}$$

but from (3.4) we also have

$$\left|\mu\cdot\psi(a)(x\cdot b)-\psi(\mu*a)(x\cdot b)+\tilde{\psi}(\mu)(a\cdot x\cdot b)\right|\leq\left\|\delta\tilde{\psi}\right\|.\tag{3.6}$$

Hence (3.5) and (3.6) imply that

$$\left|\left(\overline{\lim}\,\mathrm{Re}\,\tilde{\psi}(\mu_\alpha)(a)+i\,\overline{\lim}\,\mathrm{Im}\,\tilde{\psi}(\mu_\alpha)\right)(a\cdot x\cdot b)-\tilde{\psi}(\mu)(a\cdot x\cdot b)\right|\leq 3\left\|\delta\tilde{\psi}\right\|.$$

■

Proposition 3.4. *[18, Proposition 3.1] Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity, and let X be a Banach $\mathcal{A}$-bimodule. Let $\psi\in\mathcal{C}^1(\mathcal{A},X')$ such that $|\psi(a)(b\cdot x\cdot c)|\leq\|\delta\psi\|$ for every $x\in X$ with $\|x\|\leq 1$ and $a,b,c\in\mathcal{A}$ with $\|a\|\leq 1$, $\|b\|\leq 1$ and $\|c\|\leq 1$. Then there exists $\widehat{\psi}\in X'$ such that*

$$\left|\psi(a)(x)-\delta\widehat{\psi}(a)(x)\right|\leq 5\left\|\delta\psi\right\|.$$

Theorem 3.5. *Let G be a locally compact group, and let ω be a diagonally bounded weight on G. Then $\mathcal{H}^2(L^1(G,\omega),L^1(G,\omega)^{(2n+1)})$ is a Banach space for every $n\in\mathbb{Z}^+$.*

Proof. Set $X=L^1(G,\omega)^{(2n)}$. Let $\phi\in\mathcal{C}^1(L^1(G,\omega),X')$ and let us consider $\tilde{\phi}\in\mathcal{C}^1(M(G,\omega),X')$ as in Proposition 3.3. Set

$$S=\left\{\mathrm{Re}\,\delta_{g^{-1}}\tilde{\phi}(\delta_g):g\in G\right\},$$

Since S is bounded above by $\left\|\tilde{\phi}\right\|Db(\omega)$ in $X'_{\mathbb{R}}$, the complete vector lattice of real valued functions in X', then $\psi_r=\sup_{g\in G}S$ exists in $X'_{\mathbb{R}}$.

For every $h\in G$ and $x\in X$ with $\|x\|\leq 1$ by (3.4) we have

$$\begin{aligned}\delta_h\cdot\psi_r(x)&=\sup_{k\in G}\left\{\mathrm{Re}(\delta_h*\delta_{k^{-1}})\cdot\tilde{\phi}(\delta_k)(x)\right\}=\sup_{g\in G}\left\{\mathrm{Re}\,\delta_{g^{-1}}\cdot\tilde{\phi}(\delta_g*\delta_h)(x)\right\}\\&\leq\sup_{g\in G}\left\{\mathrm{Re}(\delta_{g^{-1}}*\delta_g)\cdot\tilde{\phi}(\delta_h)(x)+\mathrm{Re}\,\delta_{g^{-1}}\cdot\tilde{\phi}(\delta_g)\cdot\delta_h(x)\right\}+\left\|\delta\tilde{\phi}\right\|Db(\omega)\omega(h)\\&\leq\mathrm{Re}\,\tilde{\phi}(\delta_h)(x)+\psi_r\cdot\delta_h(x)+\left\|\delta\tilde{\phi}\right\|Db(\omega)\omega(h),\end{aligned}$$

where $hk^{-1} = g^{-1}$. On the other hand,

$$\delta_h \cdot \psi_r(x) \geq \operatorname{Re}\tilde{\phi}(\delta_h)(x) + \psi_r \cdot \delta_h(x) - \left\|\delta\tilde{\phi}\right\| Db(\omega)\omega(h).$$

Therefore,

$$\left|\delta_h \cdot \psi_r(x) - \psi_r \cdot \delta_h(x) - \operatorname{Re}\tilde{\phi}(\delta_h)(x)\right| \leq \left\|\delta\tilde{\phi}\right\| Db(\omega)\omega(h). \tag{3.7}$$

Now if $\mu_\alpha = \sum_{i=1}^n \alpha_i \delta_{h_i}$, then by (3.7)

$$\begin{aligned} &|\mu_\alpha \cdot \psi_r(x) - \psi_r \cdot \mu_\alpha(x) - \operatorname{Re}\tilde{\phi}(\mu_\alpha)(x)| \\ &\qquad \leq \sum_{i=1}^n |\alpha_i| \left|\delta_{h_i} \cdot \psi_r(x) - \psi_r \cdot \delta_{h_i}(x) - \operatorname{Re}\tilde{\phi}(\delta_{h_i})(x)\right| \\ &\qquad \leq \sum_{i=1}^n |\alpha_i| \left\|\delta\tilde{\phi}\right\| Db(\omega)\omega(h_i) \leq \left\|\delta\tilde{\phi}\right\| Db(\omega) \left\|\mu_\alpha\right\|_\omega . \end{aligned} \tag{3.8}$$

Similarly, by considering imaginary parts we obtain ψ_i such that

$$\left|\mu_\alpha \cdot \psi_i(x) - \psi_i \cdot \mu_\alpha(x) - \operatorname{Im}\tilde{\phi}(\mu_\alpha)(x)\right| \leq \left\|\delta\tilde{\phi}\right\| Db(\omega) \left\|\mu_\alpha\right\|_\omega . \tag{3.9}$$

Since every h in $L^1(G,\omega)$ with $\|h\|_1^\omega \leq 1$ is the so-limit of a net $\{\mu_\alpha\}$ with $\|\mu_\alpha\|_\omega \leq 1$, where every μ_α is a linear combination of point masses, then by (3.8) and (3.9) for every $x \in X$ with $\|x\| \leq 1$ and $a, b \in L^1(G,\omega)$ with $\|a\|_1^\omega \leq 1$ and $\|b\|_1^\omega \leq 1$ we have

$$\left|(h \cdot \psi - \psi \cdot h)\,(a \cdot x \cdot b) - \left(\overline{\lim}\operatorname{Re}\tilde{\phi}(\mu_\alpha) + i\,\overline{\lim}\operatorname{Im}\tilde{\phi}(\mu_\alpha)\right)(a \cdot x \cdot b)\right| \leq 2\left\|\delta\tilde{\phi}\right\| Db(\omega)$$

where $\psi = \psi_r + i\,\psi_i$. Now by Proposition 3.3 (ii), we have

$$\left|(\overline{\lim}_\alpha \operatorname{Re}\tilde{\phi}(\mu_\alpha)(a \cdot x \cdot b) + i\,\overline{\lim}_\alpha \operatorname{Im}\tilde{\phi}(\mu_\alpha)(a \cdot x \cdot b)) - \phi(h)(a \cdot x \cdot b)\right| \leq 3\left\|\delta\tilde{\phi}\right\|.$$

Thus

$$\begin{aligned} &\left|(h \cdot \psi - \psi \cdot h)(a \cdot x \cdot b) - \phi(h)(a \cdot x \cdot b)\right| \\ &\quad \leq \left|(h \cdot \psi - \psi \cdot h)(a \cdot x \cdot b) - \left(\overline{\lim}\operatorname{Re}\tilde{\phi}(\mu_\alpha) + i\,\overline{\lim}\operatorname{Im}\tilde{\phi}(\mu_\alpha)\right)(a \cdot x \cdot b)\right| \\ &\qquad + \left|\left(\overline{\lim}\operatorname{Re}\tilde{\phi}(\mu_\alpha) + i\,\overline{\lim}\operatorname{Im}\tilde{\phi}(\mu_\alpha)\right)(a \cdot x \cdot b) - \phi(h)(a \cdot x \cdot b)\right| \\ &\quad \leq \left\|\delta\tilde{\phi}\right\| (2Db(\omega) + 3). \end{aligned}$$

Now by Proposition 3.4 there exist $\hat{\phi} \in X'$ such that

$$\left|(h \cdot \psi - \psi \cdot h)(x) - \delta\hat{\phi}(h)(x) - \phi(h)(x)\right| \leq 5\left\|\delta\tilde{\phi}\right\| (2Db(\omega) + 3)$$

Define

$$\bar{\psi}(h)(x) = -\delta\psi(h)(x) - \delta\hat{\phi}(h)(x) + \phi(h)(x).$$

Then $\delta\bar{\psi} = \delta\tilde{\phi}$ and $\left|\bar{\psi}(h)(x)\right| \leq 5\left\|\delta\tilde{\phi}\right\| (2Db(\omega) + 3)$ for every $h \in L^1(G,\omega)$ with $\|h\|_1^\omega \leq 1$ and $x \in X$ with $\|x\| \leq 1$. So $\left\|\bar{\psi}\right\| \leq 5\left\|\delta\tilde{\phi}\right\| (2Db(\omega) + 3)$ and this completes the proof. ∎

Theorem 3.6. *$\mathcal{H}^2_\lambda(L^1(G,\omega))$ is a Banach space for every locally compact group G and for every diagonally bounded weight ω.*

Proof. Let $\phi \in \mathcal{C}^1(L^1(G,\omega), L^\infty(G,\omega^{-1}))$ be such that for $a, b \in L^1(G,\omega)$

$$\phi(a)(b) = -\phi(b)(a).$$

By the proof of Theorem 3.5 there exists $\bar{\psi} \in \mathcal{C}^1(L^1(G,\omega), L^\infty(G,\omega^{-1}))$ defined by $\bar{\psi}(b)(a) = -\delta\psi(b)(a) + \phi(b)(a)$ such that $\delta\bar{\psi} = \delta\phi$ and for a constant M, $\left\|\bar{\psi}\right\| \leq M\left\|\delta\phi\right\|$ and obviously $\bar{\psi}(b)(a) = -\bar{\phi}(a)(b)$. ■

Example 3.7. [17, Example 3.15] It is well known that for $\mathbb{F}_2$, the free group on two generators, the second unbounded cohomology $H^2(\mathbb{F}_2, \mathbb{R})$ is trivial [3, Example 4.3 and Example 1 on page 58]. So all bounded 2-cocycles have the form $\phi(g,h) = \psi(g) - \psi(gh) + \psi(h)$ for some possibly unbounded ψ. We define

$$\omega(g) = \begin{cases} \exp(K - \psi(g)) & \text{if } g \neq e \\ 1 & \text{otherwise,} \end{cases}$$

where K is a bound for ϕ, we get a weight on $\mathbb{F}_2$ such that $\sup\{\omega(g)\omega(g^{-1})\} < \infty$. Thus $\mathcal{H}^2(\ell^1(\mathbb{F}_2,\omega), \ell^\infty(\mathbb{F}_2,\omega^{-1}))$ is a Banach space. In the case $\omega = 1$ as noted in the Introduction $\mathcal{H}^2(\ell^1(\mathbb{F}_2), \ell^\infty(\mathbb{F}_2)) \neq 0$ and by [18] it is a Banach space.

Example 3.8. Bade et al. [1] studied the Beurling algebra $\ell^1(\mathbb{Z},\omega_\alpha)$. They defined a weight ω_α on $\mathbb{Z}$ by $\omega_\alpha(n) = (1+|n|)^\alpha$ and they proved

(i) If $\alpha > 0$, then $\ell^1(\mathbb{Z},\omega_\alpha)$ is not amenable.
(ii) If $0 \leq \alpha < 1/2$, then $\ell^1(\mathbb{Z},\omega_\alpha)$ is weakly amenable.
(iii) If $\alpha \geq 1/2$, then $\ell^1(\mathbb{Z},\omega_\alpha)$ is not weakly amenable.

Note that if $\alpha = 0$, then $\omega = 1$ and $\ell^1(\mathbb{Z},\omega_\alpha) = \ell^1(\mathbb{Z})$ is an amenable algebra [2, §43.3]. Thus by [12] $\mathcal{H}^n(\ell^1(\mathbb{Z}), X') = 0$ for every Banach $\ell^1(\mathbb{Z})$-bimodule X and every $n \geq 1$. In [16] the second author showed that $\mathcal{H}^2(\ell^1(\mathbb{Z},\omega_\alpha), \mathbb{C}) \neq 0$ for every $\alpha > 0$, then by [19] $\mathcal{H}^2(\ell^1(\mathbb{Z},\omega_\alpha), \ell^\infty(\mathbb{Z},\omega_\alpha)) \neq 0$. Note that ω_α is not diagonally bounded. So Theorem 3.5 is not applicable. We do not know whether $\mathcal{H}^2(\ell^1(\mathbb{Z},\omega_\alpha), \ell^\infty(\mathbb{Z},\omega_\alpha))$ is a Banach space or not.

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