Ulam stability problem for a mixed type of cubic and additive functional equation *

Kil-Woung Jun Hark-Mahn Kim†

Abstract

It is the aim of this paper to obtain the generalized Hyers-Ulam stability result for a mixed type of cubic and additive functional equation

$$f\Big(\Big(\sum_{i=1}^{l} x_i\Big) + x_{l+1}\Big) + f\Big(\Big(\sum_{i=1}^{l} x_i\Big) - x_{l+1}\Big) + 2\sum_{i=1}^{l} f(x_i)$$
$$= 2f\Big(\sum_{i=1}^{l} x_i\Big) + \sum_{i=1}^{l}[f(x_i + x_{l+1}) + f(x_i - x_{l+1})]$$

for all $(x_1, \cdots, x_l, x_{l+1}) \in X^{l+1}$, where $l \geq 2$.

1 Introduction

In 1940, S. M. Ulam [21] posed the following problem concerning the stability of homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

*This work was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) KRF–2005–070–C00009.

†Corresponding author:hmkim@math.cnu.ac.kr

Received by the editors May 2004.

Communicated by F. Bastin.

1991 *Mathematics Subject Classification :* 39A11, 39B72.

Key words and phrases : Hyers-Ulam stability, cubic mapping, Banach module.

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is approximately a homomorphism, then there exists a true homomorphism near it with possible small error.

In 1941 D. H. Hyers [6] considered approximately additive mapping $f : X \to Y$, where X and Y are real Banach spaces and f satisfies

$$\|f(x+y) - f(x) - f(y)\| < \varepsilon$$

for all $x, y \in X$. It was proved that $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence for every $x \in X$ and that $T : X \to Y$ defined by $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ is the unique additive mapping satisfying

$$\|f(x) - T(x)\| \leq \varepsilon.$$

B. E. Johnson [10] proved that given a Banach space X and a number $\delta > 0$ if a continuous mapping $f : X \to \mathbb{R}$ satisfies

$$\Big| f\Big(\sum_{i=1}^{n} t_i x_i\Big) - \sum_{i=1}^{n} t_i f(x_i)\Big| \leq \delta \sum_{i=1}^{n} \|t_i x_i\|$$

for all positive integers n, all $t_i \in \mathbb{R}$ and all $x_i \in X$, then there exists a linear bounded functional $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| \leq 3\delta\|x\|$ for all $x \in X$.

During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors [2, 5, 8, 14, 17, 18]. The terminology generalized Hyers-Ulam stability originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [7, 9, 19].

The quadratic functional equation

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0 \tag{1.1}$$

clearly has $f(x) = cx^2$ as a solution with c an arbitrary constant when f is a real mapping over $\mathbb{R}$. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic mapping, even in more general contexts. A stability problem for the quadratic functional equation (1.1) was solved by a lot of authors [3, 15, 19, 20]. Moreover, Jun and Lee [11] proved the Hyers-Ulam stability of the pexiderized quadratic equation (1.1).

In this paper we consider the following functional equations,

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.2}$$

$$\begin{aligned} &f(x+y+z) + f(x+y-z) + 2f(x) + 2f(y) \\ &\quad = 2f(x+y) + f(x+z) + f(x-z) + f(y+z) + f(y-z), \end{aligned} \tag{1.3}$$

$$\begin{aligned} &f\Big(\Big(\sum_{i=1}^{l} x_i\Big) + x_{l+1}\Big) + f\Big(\Big(\sum_{i=1}^{l} x_i\Big) - x_{l+1}\Big) + 2\sum_{i=1}^{l} f(x_i) \\ &\quad = 2f\Big(\sum_{i=1}^{l} x_i\Big) + \sum_{i=1}^{l}[f(x_i + x_{l+1}) + f(x_i - x_{l+1})], \end{aligned} \tag{1.4}$$

where l is a positive integer with $l \geq 2$. Since the function $f(x) = cx^3$ on real field is a solution of the functional equation (1.2), the equation (1.2) is naturally called a cubic functional equation and every solution of the cubic functional equation (1.2) is said to be a cubic mapping. Let both E_1 and E_2 be real vector spaces. Authors [12] proved that a mapping $f : E_1 \to E_2$ satisfies the functional equation (1.2) if and only if there exists a mapping $B : E_1 \times E_1 \times E_1 \to E_2$ such that $f(x) = B(x, x, x)$ for all $x \in E_1$, where B is symmetric for each fixed one variable and additive for each fixed two variables.

In the present paper, we will show that a mapping $f : E_1 \to E_2$ satisfies the functional equation (1.4) if and only if there exist two mappings $B : E_1 \times E_1 \times E_1 \to E_2$, $A : E_1 \to E_2$ and a constant c in E_2 such that

$$f(x) = B(x, x, x) + A(x) + c$$

for all $x \in E_1$, where A is additive, and B is symmetric for each fixed one variable and additive for each fixed two variables. Additionally we solve the generalized Hyers-Ulam stability problem for the equation (1.4) under the approximately cubic (or additive) condition by the iterative methods and ideas that are analogous to the ones used in [5, 18].

2 General Solution

Before taking up the main subject we seek for the general solution of (1.4) in the class of functions between real vector spaces.

Theorem 2.1. *A mapping $f : E_1 \to E_2$ satisfies the functional equation (1.3) if and only if there exist mappings $B : E_1 \times E_1 \times E_1 \to E_2$, $A : E_1 \to E_2$ and a constant c in E_2 such that $f(x) = B(x, x, x) + A(x) + c$ for all $x \in E_1$, where B is symmetric for each fixed one variable and additive for each fixed two variables, and A is additive.*

Proof. Let $f : E_1 \to E_2$ satisfy the functional equation (1.3). Putting $x = 0 = y$ in (1.3), we get $f(z) + f(-z) - 2f(0) = 0$. Thus, setting $F(x) := f(x) - f(0)$, we obtain that F is odd and F also satisfies the equation (1.3). Therefore, we may assume without loss of generality that $f : E_1 \to E_2$ satisfies the functional equation (1.3), $f(0) = 0$ and f is odd. Putting $z = x$ in (1.3), we get

$$f(2x + y) = 3f(x + y) + f(2x) + f(y - x) - 2f(x) - 3f(y) \tag{2.1}$$

for all $x,\ y \in E_1$. Setting $z = y$ in (1.3), one obtains that

$$f(x + 2y) = 3f(x + y) + f(2y) + f(x - y) - 3f(x) - 2f(y) \tag{2.2}$$

for all $x,\ y \in E_1$. Adding the equation (2.2) to (2.1), we lead to

$$f(x + 2y) + f(2x + y) = 6f(x + y) + f(2x) + f(2y) - 5f(x) - 5f(y) \tag{2.3}$$

for all $x,\ y \in E_1$. Replacing x by $-x$ in (2.1), we get

$$f(-2x + y) = -3f(x - y) - f(2x) + f(x + y) + 2f(x) - 3f(y) \tag{2.4}$$

for all $x,\ y \in E_1$. Putting $-y$ instead of y in (2.2), one obtains that

$$f(x-2y) = 3f(x-y) - f(2y) + f(x+y) - 3f(x) + 2f(y) \tag{2.5}$$

for all $x,\ y \in E_1$. Adding the equation (2.4) to (2.5), we lead to the relation

$$f(x-2y) + f(-2x+y) = 2f(x+y) - f(2x) - f(2y) - f(x) - f(y) \tag{2.6}$$

for all $x,\ y \in E_1$.

Now, define $B : E_1 \times E_1 \times E_1 \mapsto E_2$ by

$$\begin{aligned} &B(x,y,z) \\ &:= \frac{1}{24}[f(x+y+z) + f(x-y-z) - f(x+y-z) - f(x-y+z)] \end{aligned} \tag{2.7}$$

for all $x, y, z \in E_1$. Then B is symmetric for each fixed one variable since f is odd mapping and B is additive for each fixed two variables, which is verified in the proof of [12, Theorem 2.1].

On the other hand, define a mapping $A : E_1 \mapsto E_2$ as

$$A(x) := f(x) - B(x,x,x) \qquad \text{for all } x \in E_1. \tag{2.8}$$

Utilizing (2.3) and (2.6), we get

$$\begin{aligned} &24A(x+y) - 24A(x) - 24A(y) \\ &= 24f(x+y) - 24f(x) - 24B(x+y, x+y, x+y) \\ &\qquad +24B(x,x,x) + 24B(y,y,y) - 24f(y) \\ &= 24f(x+y) - 24f(x) - 24f(y) - 72B(x,x,y) - 72B(x,y,y) \\ &= 24f(x+y) - 24f(x) - 3[f(2x+y) + f(-2x+y) - 2f(y)] \\ &\qquad -3[f(x+2y) + f(x-2y) - 2f(x)] - 24f(y) \\ &= 0 \end{aligned} \tag{2.9}$$

for all $x,\ y \in E_1$. That is, A is additive and $f(x) = B(x,x,x) + A(x)$ for all $x \in E_1$. Since we regard $f(x)$ as $f(x) - f(0)$, we get $f(x) = B(x,x,x) + A(x) + f(0)$ for all $x \in E_1$ and we obtain the desired results.

Conversely, if there exist mapping $B : E_1 \times E_1 \times E_1 \to E_2$, $A : E_1 \to E_2$ and a constant c such that $f(x) = B(x,x,x) + A(x) + c$ for all $x \in E_1$, where A is additive and B is symmetric for each fixed one variable and B is additive for each fixed two variables, then it is obvious that f satisfies the equation (1.3). This completes the proof of the theorem. ■

Theorem 2.2. *The functional equation (1.4) in the class of functions between real vector spaces is identically equivalent to the functional equation (1.3), and thus the general solution of (1.4) is of the form $f(x) = B(x,x,x) + A(x) + c$, where B is such as previous section and A is additive.*

Proof. Letting f be a solution of (1.4), then $f(x) - f(0)$ satisfies also the equation (1.4). Thus we may assume that f satisfies the equation (1.4) and $f(0) = 0$. Putting

$x_i = 0$ for all $i = 1, \cdots, l$ and $x_{l+1} = z$ in the equation (1.4) yields $f(z) = -f(-z)$. Substituting $x_i = 0$ for all $i = 3, \cdots, l$ and $x_{l+1} = z$ in (1.4) we obtain

$$\begin{aligned} & f(x_1 + x_2 + z) + f(x_1 + x_2 - z) + 2f(x_1) + 2f(x_2) \\ & \quad = 2f(x_1 + x_2) + f(x_1 + z) + f(x_1 - z) + f(x_2 + z) + f(x_2 - z), \end{aligned} \tag{2.10}$$

which is in fact the equation (1.3).

Conversely, if f satisfies the equation (1.3), then by Theorem 2.1 f has the form $f(x) = B(x, x, x) + A(x) + c$, and so it is easily verified that f satisfies the equation (1.4) by expanding (1.4) into B and A terms. ∎

3 Stability of Equation (1.4)

From now on, let X be a real vector space and let Y be a real Banach space unless we give any specific reference. Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and $\mathbb{N}$ the set of all positive integers. Given $f : X \to Y$, we define the difference $Df : X^{l+1} \to Y$ by

$$\begin{aligned} Df(x_1, \cdots, x_l, x_{l+1}) \;:=\; & f\Big(\Big(\sum_{i=1}^{l} x_i\Big) + x_{l+1}\Big) + f\Big(\Big(\sum_{i=1}^{l} x_i\Big) - x_{l+1}\Big) + 2\sum_{i=1}^{l} f(x_i) \\ & -2f\Big(\sum_{i=1}^{l} x_i\Big) - \sum_{i=1}^{l}[f(x_i + x_{l+1}) + f(x_i - x_{l+1})] \end{aligned}$$

for all $(x_1, \cdots, x_l, x_{l+1}) \in X^{l+1}$. We consider the following functional inequality

$$\|Df(x_1, \cdots, x_l, x_{l+1})\| \le \phi(x_1, \cdots, x_l, x_{l+1}),$$

where the mapping $\phi : X^{l+1} \to \mathbb{R}^+$ is called the approximate remainder of the functional equation (1.4), which acts as a perturbation of the equation. For convenience in this paper we denote

$$\hat{\phi}(x, y, z) := \min\{\phi_{i,j}(x, y, z) | i, j = 1, \cdots, l \text{ and } \; i \neq j\},$$

where $\phi_{i,j}(x, y, z) := \phi(0, \cdots, 0, x, 0, \cdots, 0, y, 0, \cdots, 0, z)$, $(i \neq j)$ with x, y in i, j entries, respectively, z in the last entry and 0 otherwise.

Theorem 3.1. *Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, \cdots, x_l, x_{l+1})\| \le \phi(x_1, \cdots, x_l, x_{l+1}) \tag{3.1}$$

for all $x_1, \cdots, x_l, x_{l+1} \in X$. If the approximate remainder $\phi : X^{l+1} \to \mathbb{R}^+$ is a mapping such that the series

$$\sum_{i=1}^{\infty} \frac{\phi(2^i x_1, \cdots, 2^i x_l, 2^i x_{l+1})}{2^i}$$

converges for all $x_1, \cdots, x_l, x_{l+1} \in X$, then there exist a cubic mapping $T : X \to Y$ and an additive mapping $A : X \to Y$ which satisfy the equation (1.4) and the inequality

$$\|f(x) - f(0) - A(x) - T(x)\| \le \frac{1}{16} \sum_{i=1}^{\infty} \frac{4^i - 1}{3 \cdot 8^{i-1}} \Phi(2^{i-1} x) \tag{3.2}$$

for all $x \in X$, where the mapping $\Phi : X \to Y$ is given by

$$\Phi(x) := 3\hat{\phi}(x, x, x) + \hat{\phi}(2x, x, x) + \frac{4(l-2)}{l-1}\hat{\phi}(0, 0, x)$$

and the mappings T and A are defined by

$$T(x) + A(x) = \lim_{n\to\infty}\left[\frac{4^{n+1}-1}{3\cdot 8^n}f(2^n x) - \frac{4^n-1}{6\cdot 8^n}f(2^{n+1}x)\right] \tag{3.3}$$

for all $x \in X$.

Proof. Define $g : X \to Y$ by $g(x) := f(x) - f(0)$ for all $x \in X$. Then g satisfies also the functional inequality

$$\begin{aligned} \|Dg(x_1, \cdots, x_l, x_{l+1})\| &\le \phi(x_1, \cdots, x_l, x_{l+1}), \\ \|Dg(0, \cdots, x, 0, \cdots, 0, y, 0, \cdots, z)\| &\le \phi_{i,j}(x, y, z) \end{aligned} \tag{3.4}$$

for all $x_1, \cdots, x_l, x_{l+1} \in X$ and x, y in i, j entries, respectively, z in the last entry. Noting that $Dg(0, \cdots, 0, x, 0, \cdots, 0, y, 0, \cdots, z)$ is invariant for all x, y in i, j entries, where $i, j = 1, 2, \cdots, l$, we may assume without loss of generality that $\hat{\phi}(x, y, z) := \phi_{1,2}(x, y, z) = \phi(x, y, 0, \cdots, 0, z)$. If we replace $(x_1, \cdots, x_l, x_{l+1})$ by $(2x, x, 0, \cdots, 0, x)$ in (3.4), we have

$$\begin{aligned} &\|g(4x) + 2g(2x) + g(x) - 3g(3x) - (l-2)[g(x) + g(-x)]\| \\ &\le \hat{\phi}(2x, x, x) = \phi_{1,2}(2x, x, x) \end{aligned} \tag{3.5}$$

for all $x \in X$. Replacing $(x_1, \cdots, x_l, x_{l+1})$ by $(x, x, 0, \cdots, 0, x)$ in (3.4), we get

$$\|g(3x) + 5g(x) - 4g(2x) - (l-2)[g(x) + g(-x)]\| \le \hat{\phi}(x, x, x) \tag{3.6}$$

for all $x \in X$. Setting $(x_1, \cdots, x_l, x_{l+1})$ by $(0, 0, \cdots, 0, x)$ in (3.4), one obtains the approximately odd condition of g

$$\|g(x) + g(-x)\| \le \frac{1}{l-1}\hat{\phi}(0, 0, x) \tag{3.7}$$

for all $x \in X$. Combining (3.5) and (3.6) to eliminate the term $g(3x)$, and then applying (3.7) to the resulting inequality, we obtain

$$\begin{aligned} &\|g(4x) + 16g(x) - 10g(2x)\| \\ &\le 3\hat{\phi}(x, x, x) + \hat{\phi}(2x, x, x) + \frac{4(l-2)}{l-1}\hat{\phi}(0, 0, x) \\ &:= \Phi(x), \end{aligned} \tag{3.8}$$

which is rewritten by

$$\left\|g(x) - \frac{5}{8}g(2x) + \frac{1}{16}g(4x)\right\| \le \frac{1}{16}\Phi(x) \tag{3.9}$$

for all $x \in X$. From the inequality (3.9) we use iterative methods and induction on n to prove our next relation:

$$\begin{aligned}&\left\|g(x) - \frac{4^{n+1}-1}{3 \cdot 8^n} g(2^n x) + \frac{4^n - 1}{6 \cdot 8^n} g(2^{n+1}x)\right\| \\ &\leq \frac{1}{16} \sum_{i=1}^{n} \frac{4^i - 1}{3 \cdot 8^{i-1}} \Phi(2^{i-1}x)\end{aligned} \tag{3.10}$$

for all $x \in X$. We set a sequence $\{g_n(x)\}$ given by

$$g_n(x) := \frac{4^{n+1}-1}{3 \cdot 8^n} g(2^n x) - \frac{4^n - 1}{6 \cdot 8^n} g(2^{n+1}x), \quad x \in X$$

and prove the convergence of the sequence. Now we figure out by (3.8)

$$\begin{aligned}
&\|g_{n+1}(x) - g_n(x)\| \\
&= \left\| \frac{4^{n+2}-1}{3 \cdot 8^{n+1}} g(2^{n+1}x) - \frac{4^{n+1}-1}{6 \cdot 8^{n+1}} g(2^{n+2}x) - \frac{4^{n+1}-1}{3 \cdot 8^n} g(2^n x) + \frac{4^n - 1}{6 \cdot 8^n} g(2^{n+1}x) \right\| \\
&= \frac{1}{6 \cdot 8^{n+1}} \Big\| 2(4^{n+2}-1) g(2^{n+1}x) + 8(4^n - 1) g(2^{n+1}x) \\
&\qquad -16(4^{n+1}-1) g(2^n x) - (4^{n+1}-1) g(2^{n+2}x) \Big\| \\
&= \frac{1}{6 \cdot 8^{n+1}} \Big\| 4^{n+1}[10 g(2^{n+1}x) - 16 g(2^n x) - g(2^{n+2}x)] \\
&\qquad -[10 g(2^{n+1}x) - 16 g(2^n x) - g(2^{n+2}x)] \Big\| \\
&\leq \frac{4^{n+1}-1}{6 \cdot 8^{n+1}} \Phi(2^n x)
\end{aligned}$$

for all $x \in X$. Hence it follows by the last inequality that for any positive integers m, n with $m > n > 0$,

$$\|g_n(x) - g_m(x)\| \leq \sum_{k=n}^{m-1} \|g_{k+1}(x) - g_k(x)\| \leq \sum_{k=n}^{m-1} \frac{4^{k+1}-1}{6 \cdot 8^{k+1}} \Phi(2^k x)$$

for all $x \in X$. Since $\sum_{i=1}^{\infty} \frac{\phi(2^i x_1, \cdots, 2^i x_l, 2^i x_{l+1})}{2^i} < \infty$ by assumption, the right hand side of the above inequality tends to 0 as n tends to infinity and thus the sequence $\{g_n(x)\}$ is Cauchy in Y. Therefore, we may define a mapping $F : X \to Y$ as

$$\begin{aligned}
F(x) &= \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \left[\frac{4^{n+1}-1}{3 \cdot 8^n} g(2^n x) - \frac{4^n - 1}{6 \cdot 8^n} g(2^{n+1}x) \right] \\
&= \lim_{n\to\infty} \left[\frac{4^{n+1}-1}{3 \cdot 8^n} f(2^n x) - \frac{4^n - 1}{6 \cdot 8^n} f(2^{n+1}x) \right]
\end{aligned}$$

for all $x \in X$ and hence we right now arrive at the formula

$$\|f(x) - f(0) - F(x)\| \leq \frac{1}{16} \sum_{i=1}^{\infty} \frac{4^i - 1}{3 \cdot 8^{i-1}} \Phi(2^{i-1}x)$$

by letting $n \to \infty$ in (3.10).

To show that F satisfies the equation (1.4), we calculate the following inequality by (3.1)

$$\begin{aligned}
&\frac{4^{n+1}-1}{3\cdot 8^n}\Big\|Df(2^n x_1,\cdots,2^n x_l,2^n x_{l+1})\Big\| \\
&\qquad\qquad \le \frac{4^{n+1}-1}{3\cdot 8^n}\phi(2^n x_1,\cdots,2^n x_l,2^n x_{l+1}), \\
&\frac{4^{n}-1}{6\cdot 8^n}\Big\|Df(2^{n+1} x_1,\cdots,2^{n+1} x_l,2^{n+1} x_{l+1})\Big\| \\
&\qquad\qquad \le \frac{4^{n}-1}{6\cdot 8^n}\phi(2^{n+1} x_1,\cdots,2^{n+1} x_l,2^{n+1} x_{l+1})
\end{aligned}$$

for all $x_1,\cdots,x_l,x_{l+1}\in X$. Thus it follows from above two relations that

$$\begin{aligned}
&Dg_n(x_1,\cdots,x_l,x_{l+1}) \\
&= \Big[\frac{4^{n+1}-1}{3\cdot 8^n}\Big(Df(2^n x_1,\cdots,2^n x_l,2^n x_{l+1})\Big) \\
&\qquad\qquad -\frac{4^{n}-1}{6\cdot 8^n}\Big(Df(2^{n+1} x_1,\cdots,2^{n+1} x_l,2^{n+1} x_{l+1})\Big)\Big] \\
&\le \Big[\frac{4^{n+1}-1}{3\cdot 8^n}\phi(2^n x_1,\cdots,2^n x_l,2^n x_{l+1})+\frac{4^{n}-1}{6\cdot 8^n}\phi(2^{n+1} x_1,\cdots,2^{n+1} x_l,2^{n+1} x_{l+1})\Big],
\end{aligned}$$

which yields by letting $n\to\infty$ that $DF(x_1,\cdots,x_l,x_{l+1})=0$ for all $x_1,\cdots,x_l,x_{l+1}\in X$, that is, F is a solution of (1.4). By Theorem 2.2 there exist a cubic mapping T and an additive mapping A such that $F(x)=T(x)+A(x)$ for all $x\in X$. The proof is complete. ∎

Corollary 3.2. *Suppose that a mapping $f: X\to Y$ satisfies*

$$\|Df(x_1,\cdots,x_l,x_{l+1})\|\le\varepsilon$$

for all $x_1,\cdots,x_l,x_{l+1}\in X$ and some $\varepsilon\ge 0$. Then there exist a cubic mapping $T: X\to Y$ and an additive mapping $A: X\to Y$ which satisfy the equation (1.4) and the inequality

$$\|f(x)-f(0)-A(x)-T(x)\|\le\frac{4\varepsilon(2l-3)}{7(l-1)}$$

for all $x\in X$.

We now investigate the generalized Hyers-Ulam stability problem for the equation (1.4) under the approximately cubic condition. Thus we investigate situations that there exists a true cubic mapping near an approximately cubic mapping.

Theorem 3.3. *Suppose that a mapping $f: X\to Y$ satisfies*

$$\|Df(x_1,\cdots,x_l,x_{l+1})\| \le \phi(x_1,\cdots,x_l,x_{l+1}), \tag{3.11}$$

$$\|f(2x)+8f(-x)-9f(0)\| \le \psi(x) \tag{3.12}$$

for all $x, x_1, \cdots, x_l, x_{l+1} \in X$. If the approximate remainders $\phi : X^{l+1} \to \mathbb{R}^+$, $\psi : X \to \mathbb{R}^+$ are mappings such that the series

$$\sum_{i=1}^{\infty} \frac{\phi(3^i x_1, \cdots, 3^i x_l, 3^i x_{l+1})}{27^i} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{\psi(3^i x)}{27^i}$$

converge for all $x, x_1, \cdots, x_l, x_{l+1} \in X$, then there exists a unique cubic mapping $T : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\begin{aligned} &\|f(x) - f(0) - T(x)\| \\ &\le \sum_{i=1}^{\infty} \Big[\frac{1}{2}\Big(\frac{1}{27^i} - \frac{(-1)^{i-1}}{37^i}\Big)\Phi_1(3^{i-1}x) \\ &\qquad\qquad + \frac{1}{2}\Big(\frac{1}{27^i} + \frac{(-1)^{i-1}}{37^i}\Big)\Phi_1(-3^{i-1}x)\Big] \end{aligned} \tag{3.13}$$

for all $x \in X$, where

$$\Phi_1(x) := \hat{\phi}(x, x, x) + \frac{l-2}{l-1}\hat{\phi}(0, 0, x) + 4\psi(x).$$

The mapping T is given by

$$T(x) = \lim_{n \to \infty} \frac{f(3^n x)}{27^n} \tag{3.14}$$

for all $x \in X$. If, moreover, for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T(rx) = r^3 T(x)$ for all $r \in \mathbb{R}$.

Proof. We use the same notation as in Theorem 2.2. If we put $g(x) := f(x) - f(0)$ in (3.11), then $g : X \to Y$ satisfies the functional inequality (3.4) and

$$\|g(2x) + 8g(-x)\| \le \psi(x) \tag{3.15}$$

for all $x \in X$. Combining (3.6) and (3.7) to eliminate the term $g(x) + g(-x)$, and then applying (3.15) to the resulting inequality, we obtain

$$\begin{aligned} &\|g(3x) + 5g(x) + 32g(-x)\| \\ &\le \hat{\phi}(x, x, x) + \frac{l-2}{l-1}\hat{\phi}(0, 0, x) + 4\psi(x) := \Phi_1(x). \end{aligned} \tag{3.16}$$

By substituting $-x$ for x in (3.16), we have

$$\|f(-3x) + 5f(-x) + 32f(x)\| \le \Phi_1(-x). \tag{3.17}$$

We use induction on n to prove our next relation:

$$\begin{aligned} &\Big\|g(x) + \frac{1}{2}\Big(\frac{(-1)^{n-1}}{37^n} - \frac{1}{27^n}\Big)g(3^n x) + \frac{1}{2}\Big(\frac{(-1)^{n-1}}{37^n} + \frac{1}{27^n}\Big)g(-3^n x)\Big\| \\ &\le \sum_{i=1}^{n} \Big[\frac{1}{2}\Big(\frac{1}{27^i} - \frac{(-1)^{i-1}}{37^i}\Big)\Phi_1(3^{i-1}x) + \frac{1}{2}\Big(\frac{1}{27^i} + \frac{(-1)^{i-1}}{37^i}\Big)\Phi_1(-3^{i-1}x)\Big] \end{aligned} \tag{3.18}$$

for all $x \in X$. By (3.16) and (3.17), we get

$$\begin{aligned}&\Big\|g(x) - \frac{5}{999}g(3x) + \frac{32}{999}g(-3x)\Big\| \\ &\le \frac{5}{999}\| - g(3x) - 5g(x) - 32g(-x)\| \\ &+\frac{32}{999}\|g(-3x) + 5g(-x) + 32g(x)\| \\ &\le \frac{5}{999}\Phi_1(x) + \frac{32}{999}\Phi_1(-x),\end{aligned} \tag{3.19}$$

which proves the validity of the inequality (3.18) for $n = 1$. By using (3.16), (3.17), and the following relation:

$$\begin{aligned}&g(x) + \frac{1}{2}\Big(\frac{(-1)^n}{37^{n+1}} - \frac{1}{27^{n+1}}\Big)g(3^{n+1}x) + \frac{1}{2}\Big(\frac{(-1)^n}{37^{n+1}} + \frac{1}{27^{n+1}}\Big)g(-3^{n+1}x) \\ &= g(x) + \frac{1}{2}\Big(\frac{(-1)^{n-1}}{37^n} - \frac{1}{27^n}\Big)g(3^n x) + \frac{1}{2}\Big(\frac{(-1)^{n-1}}{37^n} + \frac{1}{27^n}\Big)g(-3^n x) \\ &+\frac{1}{2}\Big(\frac{(-1)^n}{37^{n+1}} - \frac{1}{27^{n+1}}\Big)\Big[g(3^{n+1}x) + 5g(3^n x) + 32g(-3^n x)\Big] \\ &+\frac{1}{2}\Big(\frac{(-1)^n}{37^{n+1}} + \frac{1}{27^{n+1}}\Big)\Big[g(-3^{n+1}x) + 5g(-3^n x) + 32g(3^n x)\Big],\end{aligned} \tag{3.20}$$

we can easily verify the relation (3.18) for $n + 1$.

It follows from (3.18) and (3.7) that

$$\begin{aligned}&\Big\|g(x) - \frac{g(3^n x)}{27^n}\Big\| \\ &\le \sum_{i=1}^{n}\Big[\frac{1}{2}\Big(\frac{1}{27^i} - \frac{(-1)^{i-1}}{37^i}\Big)\Phi_1(3^{i-1}x) + \frac{1}{2}\Big(\frac{1}{27^i} + \frac{(-1)^{i-1}}{37^i}\Big)\Phi_1(-3^{i-1}x)\Big] \\ &+\frac{1}{2(l-1)}\Big(\frac{1}{27^n} + \frac{(-1)^{n-1}}{37^n}\Big)\hat{\phi}(0, 0, 3^n x)\end{aligned} \tag{3.21}$$

for all $x \in X$.

In order to prove convergence of the sequence $\{\frac{g(3^n x)}{27^n}\}$, we show that it is a Cauchy sequence in Y. By (3.21), we obtain that for positive integers n, m with $n > m$, the following inequality

$$\begin{aligned}&\Big\|\frac{g(3^m x)}{27^m} - \frac{g(3^{m+n}x)}{27^{m+n}}\Big\| \\ &= \frac{1}{27^m}\Big\|g(3^m x) - \frac{g(3^n(3^m x))}{27^n}\Big\| \\ &\le \frac{1}{27^m}\sum_{i=1}^{n}\Big[\frac{1}{2}\Big(\frac{1}{27^i} - \frac{(-1)^{i-1}}{37^i}\Big)\Phi_1(3^{m+i-1}x) + \frac{1}{2}\Big(\frac{1}{27^i} + \frac{(-1)^{i-1}}{37^i}\Big)\Phi_1(-3^{m+i-1}x)\Big] \\ &+\frac{1}{27^m 2(l-1)}\Big(\frac{1}{27^n} + \frac{(-1)^{n-1}}{37^n}\Big)\hat{\phi}(0, 0, 3^{n+m}x)\end{aligned} \tag{3.22}$$

holds for all $x \in X$. Since $\sum_{i=1}^{\infty} \frac{\phi(3^i x_1, \cdots, 3^i x_l, 3^i x_{l+1})}{27^i} < \infty$ by assumption, the right hand side of the inequality (3.22) tends to 0 as m tends to infinity, and thus the sequence

$\{\frac{g(3^n x)}{27^n}\}$ is Cauchy in Y. Therefore, we may define

$$T(x) := \lim_{n\to\infty} \frac{g(3^n x)}{3^{3n}} = \lim_{n\to\infty} \frac{f(3^n x)}{3^{3n}}$$

for all $x \in X$. By letting $n \to \infty$ in (3.21), we arrive at the formula (3.13).

Replace $(x_1, \cdots, x_l, x_{l+1})$ by $(3^n x_1, \cdots, 3^n x_l, 3^n x_{l+1})$ in (3.11) and divide by 27^n, then it follows that

$$27^{-n}\|Df(3^n x_1, \cdots, 3^n x_l, 3^n x_{l+1})\| \le 27^{-n}\phi(3^n x_1, \cdots, 3^n x_l, 3^n x_{l+1}).$$

Taking the limit as $n \to \infty$, we find that T satisfies (1.4). Obviously, it follows from (3.12) and (3.7) that $T(x) + T(-x) = 0$, and $T(2x) + 8T(-x) = 0$. Therefore T is a cubic mapping defined by $T(x) = B(x, x, x)$ by Theorem 2.2.

To prove the uniqueness of the cubic mapping T subject to the equation (1.4) and the inequality (3.13), let us assume that there exists a cubic mapping $S : X \to Y$ which satisfies (1.4) and the inequality (3.13). Obviously, we have $S(3^n x) = 27^n S(x)$ and $T(3^n x) = 27^n T(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence it follows from (3.13) that

$$\begin{aligned}
&\|S(x) - T(x)\| = 27^{-n}\|S(3^n x) - T(3^n x)\| \\
&\le 27^{-n}(\|S(3^n x) - f(3^n x) + f(0)\| + \|f(3^n x) - f(0) - T(3^n x)\|) \\
&\le \frac{2}{27^n}\sum_{i=1}^{\infty}\Big[\frac{1}{2}\Big(\frac{1}{27^i} - \frac{(-1)^{i-1}}{37^i}\Big)\Phi_1(3^{n+i-1}x) \\
&\qquad + \frac{1}{2}\Big(\frac{1}{27^i} + \frac{(-1)^{i-1}}{37^i}\Big)\Phi_1(-3^{n+i-1}x)\Big]
\end{aligned}$$

for all $x \in X$. By letting $n \to \infty$ in the preceding inequality, we immediately find the uniqueness of T.

The proof of the last assertion in the theorem goes through in the same way as that of [19]. This completes the proof of the theorem. ■

From the main Theorem 3.3, we obtain the following Hyers-Ulam-Rassias stability of the equation (1.4) under the approximately cubic condition.

Corollary 3.4. *Let X and Y be a real normed space and a Banach space, respectively, and let $\delta, \varepsilon \ge 0$, $0 < p < 3$ be real numbers. Suppose that a mapping $f : X \to Y$ satisfies*

$$\begin{aligned}
&\|Df(x_1, \cdots, x_l, x_{l+1})\| \le \varepsilon(\|x_1\|^p + \cdots + \|x_l\|^p + \|x_{l+1}\|^p), \qquad (3.23)\\
&\|f(2x) + 8f(-x) - 9f(0)\| \le \delta
\end{aligned}$$

for all $x \in X$ and $(x_1, \cdots, x_l, x_{l+1}) \in X^{l+1}$. Then there exists a unique cubic mapping $T : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\begin{aligned}
&\|f(x) - f(0) - T(x)\| \qquad (3.24)\\
&\le \frac{3\varepsilon\|x\|^p}{27 - 3^p} + \frac{(l-2)\varepsilon\|x\|^p}{(l-1)(27 - 3^p)} + \frac{2\delta}{13}
\end{aligned}$$

for all $x \in X$. The mapping T is given by

$$T(x) = \lim_{n\to\infty} \frac{f(3^n x)}{27^n}$$

for all $x \in X$. Furthermore, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T(rx) = r^3 T(x)$ for all $r \in \mathbb{R}$.

Proof. The conclusion follows from Theorem 3.3. ∎

By Theorem 3.3, we obtain the following Hyers-Ulam stability of the equation (1.4) under the approximately cubic condition.

Corollary 3.5. *Let X and Y be a real normed space and a Banach space, respectively, and let $\delta, \varepsilon \geq 0$ be real numbers. Suppose that a mapping $f : X \to Y$ satisfies*

$$\begin{aligned} &\|Df(x_1, \cdots, x_l, x_{l+1})\| \leq \varepsilon, \qquad (3.25)\\ &\|f(2x) + 8f(-x) - 9f(0)\| \leq \delta \end{aligned}$$

for all $x \in X$ and $(x_1, \cdots, x_l, x_{l+1}) \in X^{l+1}$. Then there exists a unique cubic mapping $T : X \to Y$ defined by $T(x) = \lim_{n\to\infty} \frac{f(3^n x)}{27^n}$ which satisfies the equation (1.4) and the inequality

$$\|f(x) - f(0) - T(x)\| \leq \frac{\varepsilon}{26} + \frac{(l-2)\varepsilon}{(l-1)26} + \frac{2\delta}{13} \tag{3.26}$$

for all $x \in X$. If moreover, for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T(rx) = r^3 T(x)$ for all $r \in \mathbb{R}$.

In the next part, we investigate the generalized Hyers-Ulam stability problem for the equation (1.4) under the approximately additive condition.

Theorem 3.6. *Suppose that a mapping $f : X \to Y$ satisfies*

$$\|Df(x_1, \cdots, x_l, x_{l+1})\| \leq \phi(x_1, \cdots, x_l, x_{l+1}), \tag{3.27}$$
$$\|f(2x) + 2f(-x) - 3f(0)\| \leq \psi(x) \tag{3.28}$$

for all $x, x_1, \cdots, x_l, x_{l+1} \in X$ and for some $\delta \geq 0$. If the approximate remainder $\phi : X^{l+1} \to \mathbb{R}^+$, $\psi : X \to \mathbb{R}^+$ are mappings such that the series

$$\sum_{i=1}^{\infty} \frac{\phi(3^i x_1, \cdots, 3^i x_l, 3^i x_{l+1})}{3^i} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{\psi(3^i x)}{3^i}$$

converge for all $x, x_1, \cdots, x_l, x_{l+1} \in X$, then there exists a unique additive mapping $A : X \to Y$ which satisfies the equation (1.4) and the inequality

$$\begin{aligned} &\|f(x) - f(0) - A(x)\| \qquad (3.29)\\ &\leq \sum_{i=1}^{\infty} \Big[\frac{1}{2}\Big(\frac{1}{3^i} - \frac{(-1)^{i-1}}{13^i}\Big)\Phi_1(3^{i-1}x)\\ &\qquad + \frac{1}{2}\Big(\frac{1}{3^i} + \frac{(-1)^{i-1}}{13^i}\Big)\Phi_1(-3^{i-1}x)\Big] \end{aligned}$$

for all $x \in X$. The mapping A is given by

$$A(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^n} \tag{3.30}$$

for all $x \in X$. Moreover, if for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $A(rx) = rA(x)$ for all $r \in \mathbb{R}$.

Proof. The proof of this theorem goes through in the same way as that of Theorem 3.3. ∎

4 Applications

Let B be a unital Banach algebra with norm $|\cdot|$, and let ${}_B\mathbb{B}_1$ and ${}_B\mathbb{B}_2$ be left Banach B-modules with norms $\|\cdot\|$ and $||\cdot||$, respectively. A cubic mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is called B-cubic if

$$T(ax) = a^3T(x), \quad \forall a \in B, \forall x \in {}_B\mathbb{B}_1.$$

In the last part of this paper, we prove the generalized Hyers-Ulam stability problem for the equation (1.4) in Banach modules over a unital Banach algebra.

Theorem 4.1. *Let ϕ, ψ be defined mappings as in Theorem 3.3. Suppose that a mapping $f : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies*

$$\begin{aligned}
&\Big\|f\Big((\sum_{i=1}^{l}\alpha x_i)+\alpha x_{l+1}\Big)+f\Big((\sum_{i=1}^{l}\alpha x_i)-\alpha x_{l+1}\Big)+2\sum_{i=1}^{l}f(\alpha x_i)-2\alpha^3f(\sum_{i=1}^{l}x_i)\\
&\qquad -\alpha^3\sum_{i=1}^{l}[f(x_i+x_{l+1})+f(x_i-x_{l+1})]\Big\| \le \phi(x_1,\cdots,x_l,x_{l+1}),\\
&||f(2\alpha x)+8\alpha^3 f(-x)-9f(0)|| \le \psi(x)
\end{aligned}$$

for all $\alpha \in B(|\alpha| = 1)$, all $x, x_1, \cdots, x_l, x_{l+1} \in {}_B\mathbb{B}_1$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, then there exists a unique B-cubic mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$, defined by (3.14), which satisfies the equation (1.4) and the inequality (3.13).

Proof. By Theorem 3.3, it follows from the inequality of the statement for $\alpha = 1$ that there exists a unique cubic mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$, defined by $T(x) = \lim_{n\to\infty}\frac{f(3^n x)}{27^n}$, which satisfies satisfies the equation (1.4) and the inequality (3.13). Under the assumption that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_B\mathbb{B}_1$, by the same reasoning as the proof of [19], the cubic mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ satisfies

$$T(tx) = t^3T(x), \quad \forall x \in {}_B\mathbb{B}_1, \forall t \in \mathbb{R}.$$

That is, T is $\mathbb{R}$-cubic. Since T satisfies the equation

$$\begin{aligned}
&T\Big((\sum_{i=1}^{l}\alpha x_i)+\alpha x_{l+1}\Big)+T\Big((\sum_{i=1}^{l}\alpha x_i)-\alpha x_{l+1}\Big)+2\sum_{i=1}^{l}T(\alpha x_i)\\
&\qquad -2\alpha^3T(\sum_{i=1}^{l}x_i)-\alpha^3\sum_{i=1}^{l}[T(x_i+x_{l+1})+T(x_i-x_{l+1})]=0,
\end{aligned}$$

putting $(x, 0, \cdots, 0, 0)$ instead of $(x_1, \cdots, x_l, x_{l+1})$ in the last equation we obtain that for each fixed $\alpha \in B(|\alpha| = 1)$, $T(\alpha x) = \alpha^3T(x)$ for all $x \in {}_B\mathbb{B}_1$. The last relation is also true for $\alpha = 0$. Since T is $\mathbb{R}$-cubic and $T(\alpha x) = \alpha^3T(x)$ for each element $\alpha \in B(|\alpha| = 1)$, for each element $a \in B(a \neq 0)$ a is written by $a = |a| \cdot \frac{a}{|a|}$ and thus

$$\begin{aligned}
T(ax) = T(|a|\cdot\frac{a}{|a|}x) &= |a|^3\cdot T(\frac{a}{|a|}x) = |a|^3\cdot\frac{a^3}{|a|^3}\cdot T(x)\\
&= a^3T(x), \quad \forall a \in B(a \neq 0), \forall x \in {}_B\mathbb{B}_1.
\end{aligned}$$

So the unique $\mathbb{R}$-cubic mapping $T : {}_B\mathbb{B}_1 \to {}_B\mathbb{B}_2$ is also B-cubic, as desired. This completes the proof of the theorem. ∎

Since $\mathbb{C}$ is a unital Banach algebra, the Banach spaces E_1 and E_2 are considered as Banach modules over $\mathbb{C}$. Thus we have the following corollary.

Corollary 4.2. *Let ϕ, ψ be defined mappings as in Theorem 3.6 and let E_1 and E_2 be Banach spaces over the complex field $\mathbb{C}$. Suppose that a mapping $f : E_1 \to E_2$ satisfies*

$$\Big\|f\Big((\sum_{i=1}^{l} \alpha x_i) + \alpha x_{l+1}\Big) + f\Big((\sum_{i=1}^{l} \alpha x_i) - \alpha x_{l+1}\Big) + 2\sum_{i=1}^{l} f(\alpha x_i) - 2\alpha^3 f(\sum_{i=1}^{l} x_i)$$
$$-\alpha^3 \sum_{i=1}^{l} [f(x_i + x_{l+1}) + f(x_i - x_{l+1})]\Big\| \le \phi(x_1, \cdots, x_l, x_{l+1}),$$
$$||f(2\alpha x) + 8\alpha^3 f(-x) - 9f(0)|| \le \psi(x)$$

for all $\alpha \in \mathbb{C}(|\alpha| = 1)$, all $x, x_1, \cdots, x_l, x_{l+1} \in E_1$ and for some $\delta \ge 0$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then there exists a unique $\mathbb{C}$-cubic mapping $T : E_1 \to E_2$ which satisfies the equation (1.4) and the inequality (3.13).

Similarly, we obtain the alternative results of Theorem 4.1 and Corollary 4.2 for the approximately additive condition.

References

[1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge Univ. Press, 1989.

[2] J. Baker, *The stability of the cosine equation*, Proc. Amer. Math. Soc. 80(1980), 411-416.

[3] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg, 62(1992), 59-64.

[4] S. Czerwik, *The stability of the quadratic functional equation*, In 'Stability of Mappings of Hyers-Ulam Type' (edited by Th. M. Rassias and J. Tabor), Hadronic Press, Florida, 1994, pp 81-91.

[5] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias Stability of approximately additive mappings*, J. Math. Anal. Appl. 184(1994), 431-436.

[6] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. 27(1941), 222-224.

[7] D. H. Hyers, G. Isac and Th. M. Rassias, *"Stability of Functional Equations in Several Variables"*, Birkhäuser, Basel, 1998.

[8] D. H. Hyers, G. Isac and Th. M. Rassias, *On the asymptoticity aspect of Hyers-Ulam stability of mappings*, Proc. Amer. Math. Soc. 126(1998), 425-430.

[9] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. 44(1992), 125-153.

[10] B. E. Johnson, *Approximately multiplicative maps between Banach algebras*, J. London Math. Soc., 37(1988), 294-316.

[11] K. W. Jun and Y. H. Lee, *On the Hyers-Ulam-Rassias stability of a pexiderized quadratic inequality*, Math. Ineq. Appl. 4(1)(2001), 93-118.

[12] K. W. Jun and H. M. Kim, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl. 274(2002), 867-878.

[13] K. W. Jun and H. M. Kim, *On the Hyers-Ulam-Rassias stability of a general cubic functional equation*, Math. Ineq. Appl. 6(1)(2003), 289-302.

[14] K. W. Jun and H. M. Kim, *On the stability of Appolonius' equation*, Bull. Belg. Math. Soc. Simon Stevin, 11(2004), 615-624.

[15] S. -M. Jung, *On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl. 222(1998), 126-137.

[16] Pl. Kannappan, *Quadratic functional equation and inner product spaces*, Results Math. 27(1995), 368-372.

[17] C.-G. Park, *On the stability of quadratic mapping in Banach modules*, J. Math. Anal. Appl. 276(2002), 135-144.

[18] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72(1978), 297-300.

[19] Th. M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. 251(2000), 264-284.

[20] Th. M. Rassias, *On the stability of the quadratic functional equations and its applications*, Studia Univ. "Babes Bolyai", Mathematica, Vol. XLIII (1998), 89-124.

[21] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publ., New York, 1960.

Kil-Woung Jun
Department of Mathematics
Chungnam National University
220 Yuseong-Gu, Daejeon, 305-764
Republic of Korea
email : kwjun@math.cnu.ac.kr

Hark-Mahn Kim
Department of Mathematics Education
Cheongju University
NaeDok-Dong, SangDang-Gu, Cheongju, 360-764
Republic of Korea
email: harkkim@cju.ac.kr