On A_p^* -algebras of the first kind

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Abstract

We introduce a new class of algebras which extends classical A^*-algebras to the p-normed case with generalized involution. We give results concerning the symmetry and the C^*-algebra structure in such algebras.

Introduction

An A^*-algebra E is an involutive Banach algebra which possesses, in addition to its given complete norm, a second algebra norm, called the auxiliary norm, satisfying the C^*-property. The completion $\mathcal{U}$ of E with respect to auxiliary norm is then a C^*-algebra and E is said to be of the first kind if it is a two-sided ideal of $\mathcal{U}$. For a detailed account of the basic properties of A^*-algebras and A^*-algebras of the first kind, we refer the reader to [9] and [11]. In this paper, we extend the class of A^*-algebras to the p-normed case with generalized involution. Thus, we obtain a new class of algebras which will be called A_p^*-algebras. Given an A_p^*-algebra $\left(E, \|.\|_p\right)$, $0 < p \leq 1$, with a generalized involution $x \longmapsto x^*$ and an auxiliary q-norm $|.|_q$, $0 < q \leq 1$, we prove that $|.|_q^{\frac{1}{q}}$ is a norm and hence the completion of $\left(E, |.|_q^{\frac{1}{q}}\right)$ is a C^*-algebra. We also show that an A_p^*-algebra of the first kind E is hermitian. As a consequence, we obtain in this case the uniqueness of the auxiliary norm. If moreover E has a bounded left or right approximate identity, then E is a C^*-algebra.

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1 Preliminaries

A generalized involution on a complex algebra E is a vector involution $x \longmapsto x^*$ [5] which is either an algebra involution (i.e., $(xy)^* = y^*x^*$, for every $x, y \in E$) or an involutive anti-morphism (i.e., $(xy)^* = x^*y^*$, for every $x, y \in E$). We define an A_p^*-algebra as being a complex p-Banach algebra $\left(E, \|.\|_p\right)$, $0 < p \leq 1$, endowed with a generalized involution $x \longmapsto x^*$ on which there is defined a second algebra q-norm $|.|_q$, $0 < q \leq 1$, called auxiliary q-norm, with the C^*-property, that is $|xx^*|_q = |x|_q^2$, for all $x \in E$. If $p = 1$ and $x \longmapsto x^*$ is an algebra involution, we obtain the classical A^*-algebras ([11]). As in the Banach case ([9], [1]), we say that an A_p^*-algebra E is of the first kind if it is a two-sided ideal of its completion with respect to the auxiliary q-norm. Let $\left(E, \|.\|_p\right)$, $0 < p \leq 1$, be a complex p-Banach algebra endowed with a generalized involution $x \longmapsto x^*$. An element a of E is said to be hermitian (resp., normal) if $a = a^*$ (resp., $a^*a = aa^*$). We designate by $H(E)$ (resp., $N(E)$) the set of hermitian (resp., normal) elements of E. The algebra E is said to be hermitian if the spectrum of every hermitian element is real. We denote the Ptàk function, on E, by P_E that is, for every $x \in E$, $P_E(x) = \rho(xx^*)^{\frac{1}{2}}$, where ρ is the spectral radius i.e., $\rho(x) = \sup\{|\lambda| : \lambda \in Spx\}$.

Taking in account the fact that, in any p-Banach algebra $\left(E, \|.\|_p\right)$, we have $\rho(x)^p = \lim_n \|x^n\|_p^{\frac{1}{n}}$, for all $x \in E$, we prove, as in ([2], p.115-117), the following result.

Proposition 1.1. Let $\left(E, \|.\|_p\right)$, $0 < p \leq 1$, be a p-Banach algebra with a generalized involution $x \longmapsto x^*$. Then E is hermitian if, and only if, $\rho_E(a) \leq cP_E(a)$, for some $c > 0$ and every $a \in N(E)$. In this case, if $x \longmapsto x^*$ is an algebra involution, we obtain that the Ptàk function P_E is an algebra semi-norm such that $P_E(xx^*) = P_E(x)^2$, for $x \in E$. Moreover $Rad(E) = \{x \in E : P_E(x) = 0\}$, where $Rad(E)$ is the Jacobson radical of E.

Using Theorem 3.10 of [13], we prove that Theorem 4.8 of [8] extends to the p-Banach case as follows.

Proposition 1.2. A real semi-simple p-Banach algebra, $0 < p \leq 1$, in which every square is quasi-invertible, is necessarily commutative.

2 A_p^*-algebras of the first kind

The following result shows that any A_p^*-algebra possesses an auxiliary norm which satisfies the C^*-property.

Theorem 2.1.Let $\left(E, \|.\|_p\right)$, $0 < p \leq 1$, be an A_p^*-algebra and $|.|_q$, $0 < q \leq 1$, its auxiliary q-norm. Then $|.|_q^{\frac{1}{q}}$ is a norm and the completion of $\left(E, |.|_q^{\frac{1}{q}}\right)$ is a C^*-algebra.

Proof. Since $x \longmapsto x^*$ is continuous for $|.|_q$, it follows that the equality $|xx^*|_q = |x|_q^2$ extends to the completion $\overset{\wedge}{E_q}$ of $\left(E, |.|_q\right)$. In particular, we obtain $|h|_q = \rho_{\overset{\wedge}{E_q}}(h)^q$, for every $h \in H(\overset{\wedge}{E_q})$. Whence

$$\rho_{\overset{\wedge}{E_q}}(a)^2 \leq |a|_q^{\frac{2}{q}} = |aa^*|_q^{\frac{1}{q}} = P_{\overset{\wedge}{E_q}}(a), \text{ for every } a \in N(\overset{\wedge}{E_q}).$$

By Proposition 1.1, the algebra $\overset{\wedge}{E_q}$ is hermitian. Consider first an algebra involution $x \longmapsto x^*$. In this case, the Ptàk function is an algebra semi-norm. But

$$P_E(x) = \rho(xx^*)^{\frac{1}{2}} = |xx^*|_q^{\frac{1}{2q}} = |x|_q^{\frac{1}{q}}, \text{ for every } x \in E.$$

Whence $|.|_q^{\frac{1}{q}}$ is a norm in E and hence $\left(\overset{\wedge}{E_q}, |.|_q^{\frac{1}{q}}\right)$ is a C^*-algebra. Suppose now that $x \longmapsto x^*$ is an involutive anti-morphism. We will show that the algebra E is commutative. It is sufficient to prove that the real algebra $H(E)$ is commutative. Since $|h|_q = P_{\overset{\wedge}{E_q}}(h)^q$, for every $h \in H(\overset{\wedge}{E_q})$, it follows that $H(\overset{\wedge}{E_q})$ is semi-simple. Moreover, every square in $H(\overset{\wedge}{E_q})$ is quasi-invertible for $\overset{\wedge}{E_q}$ is hermitian. Thus, by Proposition 1.2, the algebra $H(\overset{\wedge}{E_q})$ is commutative. Whence the commutativity of $H(E)$.

As a consequence, we obtain the following result.

Corollary 2.2. Let $\left(E, \|.\|_p\right)$, $0 < p \leq 1$, be an A_p^*-algebra and $|.|_q$, $0 < q \leq 1$, its auxiliary q-norm. Then

1) E is semi-simple.

2) The involution is continuous for $\|.\|_p$.

3) $|a|_q^{\frac{1}{q}} \leq c\, \|a\|_p^{\frac{1}{p}}$, for some $c > 0$ and every $a \in E$.

Proof.

1) We have $|h|_q = \left|h^{2^n}\right|_q^{\frac{1}{2^n}}$, for every $h \in H(E)$ and $n = 1, 2, \ldots$. Since, by Theorem 2.1, $|.|_q^{\frac{1}{q}}$ is a norm, it follows from Theorem 7 ([5], p. 22), that

$$|h|_q^{\frac{1}{q}} = \lim_n \left(\left|h^{2^n}\right|_q^{\frac{1}{q}}\right)^{\frac{1}{2^n}} \leq \rho(h), \text{ for every } h \in H(E).$$

Then $|a|_q^2 = |aa^*|_q \leq \rho(aa^*)^q$, for every $a \in E$. Whence $|a|_q^{\frac{1}{q}} \leq P_E(a)$, for every $a \in E$. It follows from Proposition 1.1 and Theorem 2.1 that the algebra E is semi-simple.

2) We have $|a|_q^2 \leq \|aa^*\|_p^{\frac{q}{p}} \leq \|a\|_p^{\frac{q}{p}} \|a^*\|_p^{\frac{q}{p}}$, for every $a \in E$. A simple application of the closed graph Theorem shows that $x \longmapsto x^*$ is continuous for $\|.\|_p$.

3) Let $M > 0$ such that $\|a^*\|_p \leq M\, \|a\|_p$, for every $a \in E$. Then

$$|a|_q^{\frac{1}{q}} \leq \rho(aa^*)^{\frac{1}{2}} \leq \|aa^*\|_p^{\frac{1}{2p}} \leq M^{\frac{1}{2p}}\, \|a\|_p^{\frac{1}{p}}, \text{ for every } a \in E.$$

According to Theorem 2.1, we use in the sequel the notation $\left(E, \|.\|_p, |.|\right)$ to declare an A_p^*-algebra $\left(E, \|.\|_p\right)$ with an algebra involution $x \longmapsto x^*$ and an auxiliary norm $|.|$ satisfying $|xx^*| = |x|^2$ for all $x \in E$. The completion $\hat{E}$ of E with respect to the norm $|.|$ is then a C^*-algebra. By Corollary 2.2(3), $\|.\|_p$ is finer than $|.|$. Since every A_p^*-algebra $\left(E, \|.\|_p, |.|\right)$ is an F-space (Fréchet space) for the metric $d(x,y) = \|x-y\|_p$, using the closed graph theorem and Theorem 2.17 of [12], we can prove that an A_p^*-algebra of the first kind satisfies

$$\|ax\|_p^{\frac{1}{p}} \leq c\,\|a\|_p^{\frac{1}{p}}\,|x| \text{ and } \|xa\|_p^{\frac{1}{p}} \leq c\,\|a\|_p^{\frac{1}{p}}\,|x|,$$

for some $c > 0$ and every $a \in E$, $x \in \hat{E}$. Conversely, if E is an A_p^*-algebra and there is a constant $c > 0$ such that $\|ab\|_p^{\frac{1}{p}} \leq c\,\|a\|_p^{\frac{1}{p}}\,|b|$, for all $a, b \in E$, then E is an A_p^*-algebra of the first kind.

The following example shows that an A_p^*-algebra $\left(E, \|.\|_p\right)$of the first kind is not necessarily an A^*-algebra for a norm equivalent to $\|.\|_p$.

Example 2.3. For $0 < p < 1$, consider

$$E = \left\{(x_n)_n \subset C : \sum_{n=1}^{\infty} |x_n|^p < +\infty\right\},$$

equipped with the pointwise operations and the p-norm given by $\|x\|_p = \sum\limits_{n=1}^{\infty} |x_n|^p$, where $x = (x_n)_n \in E$. Then $\left(E, \|.\|_p\right)$ is a p-Banach (not Banach) algebra. Endowed with the algebra involution $((x_n)_n)^* = (\overline{x_n})_n$, E is an A_p^*-algebra with auxiliary norm $|.|$ defined by $|x| = \sup\limits_n |x_n|$. Moreover, it is easily seen that $\|xy\|_p^{\frac{1}{p}} \leq \|x\|_p^{\frac{1}{p}}\,|y|$, for every $x, y \in E$. Hence, the A_p^*-algebra E is of the first kind.

If the algebra admits a bounded left or right approximate identity, the situation is different as the following result shows.

Theorem 2.4. Let $\left(E, \|.\|_p, |.|\right)$, $0 < p \leq 1$, be an A_p^*-algebra of the first kind. If E has a bounded left or right approximate identity $(e_i)_{i \in I}$ with respect to $\|.\|_p$, then $(E, |.|)$ is a C^*-algebra.
Proof. We will show that $\|.\|_p$ and $|.|$ are equivalent. By Theorem 2.1 and Corollary 2.2, it remains to show that $\|a\|_p^{\frac{1}{p}} \leq c\,|a|$, for some $c > 0$ and every $a \in E$. Since E is an A_p^*-algebra of the first kind, we have $\|ba\|_p^{\frac{1}{p}} \leq c\,\|b\|_p^{\frac{1}{p}}\,|a|$, for some $c > 0$ and every $a, b \in E$. In particular $\|e_i a\|_p^{\frac{1}{p}} \leq c\,\|e_i\|_p^{\frac{1}{p}}\,|a|$, for every $a \in E$, and so $\|a\|_p^{\frac{1}{p}} \leq c'\,|a|$, for some $c' > 0$ and every $a \in E$.

Remark 2.5. Theorem 2.4 shows that the unitization of an A_p^* -algebra of the first kind is not in general of the same type. But we can always adjoin an identity element so as to preserve the A_p^*-algebra structure. In fact, Let $\left(E, \|.\|_p, |.|\right)$

be an A_p^*-algebra. By lemma 4.1.13 of [11], there exists a normed algebra B and an isometric $*$-isomorphism of E into B such that B has an identity element and its norm satisfies the C^*-property. The algebra B consists of all operators of the form $L_x + \alpha I$, $x \in E$, $\alpha \in C$, where I the identity operator of $L(E)$ and L_x the operator defined, in E, by $L_x(a) = xa$. If E does not have an identity element, define $\|L_x + \alpha I\|_p = \|x\|_p + |\alpha|^p$. Then $\left(B, \|.\|_p\right)$ is a p-Banach algebra and hence an A_p^*-algebra.

By Proposition 1.1, a semi-simple p-Banach algebra endowed with a hermitian algebra involution is an A_p^*-algebra with auxiliary norm the Ptàk function. In [7], Gelfand and Naïmark give an example of an A^*-algebra which is not hermitian. However, the following result shows that an A_p^*-algebra of the first kind is hermitian.

Theorem 2.6. Let $\left(E, \|.\|_p, |.|\right)$, $0 < p \leq 1$, be an A_p^*-algebra of the first kind. Then E is hermitian.
Proof. There exists $c > 0$ such that $\|ab\|_p^{\frac{1}{p}} \leq c\,\|a\|_p^{\frac{1}{p}}\,|b|$, for all $a, b \in E$. In particular, for every $a \in E$ and $n = 1, 2, ...$, we have $\|a^{n+1}\|_p^{\frac{1}{np}} \leq c^{\frac{1}{n}}\,\|a\|_p^{\frac{1}{np}}\,|a^n|^{\frac{1}{n}}$. Tending n to infinity, we obtain $\rho_E(a) \leq \rho_{\mathcal{U}}(a)$, where $\mathcal{U}$ is the completion of $(E, |.|)$. On the other hand $Sp_{\mathcal{U}}(a) \subset Sp_E(a)$, $a \in E$, and hence $\rho_E(a) = \rho_{\mathcal{U}}(a)$ for every $a \in E$. Moreover, the algebra $(\mathcal{U}, |.|)$ is hermitian for it is a C^*-algebra. It follows, by Proposition 1.1, that $\rho_{\mathcal{U}} \leq P_{\mathcal{U}}$ in $\mathcal{U}$. But $P_E = P_{\mathcal{U}}$ in E. Thus $\rho_E \leq P_E$ in E. This implies, by Proposition 1.1, that the algebra E is hermitian.

As a consequence, we obtain the uniqueness of the auxiliary norm in A_p^*-algebras of the first kind.

Corollary 2.7. An A_p^*-algebra E of the first kind has a unique auxiliary norm. This norm is exactly the Ptàk function.
Proof. If $|.|$ is an auxiliary norm in E and $\mathcal{U}$ the completion of E with respect to $|.|$, then $|.| = P_{\mathcal{U}}$ in $\mathcal{U}$ for $(\mathcal{U}, |.|)$ is a C^*-algebra. But $P_E = P_{\mathcal{U}}$ in E, by Theorem 2.6. Whence $|.| = P_E$ in E.

Remark 2.8. Using Theorem 2.6, we can deduce Corollary 2.7 from a result of Bhatt-Inoue-Kürsten [3; Lemma 4.5(1)] according to which every spectral C^*-seminorm is unique and coincides with Ptàk function. In fact, Theorem 2.6 shows that every A_p^*-algebra E of the first kind is a C^*-spectral algebra, in the sense that there is a C^*-semi-norm $|.|$ (in this case Ptàk function) with $\rho_E(x) \leq |x|$, for every x in E ([4]).

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