# Pappus-Guldin theorems for weighted motions

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## 1 Introduction

Pappus of Alexandria (fl. c. 300-c. 350), who is regarded as one of the last great mathematicians of the Hellenistic Age, formulated, in the introduction to book VII of his mathematical collections, a rule to determine the volume (resp. area) of a domain in $\mathbb{R}^3$ (resp. surface) generated by rotation of a plane domain $D_0$ around an axis in its plane: multiply the area (resp. perimeter) of $D_0$ by the length of the circle generated by the center of mass of $D_0$ (resp. the center of mass of the boundary of $D_0$). This rule was rediscovered in 1641 by P. Guldin and was also proved by several mathematicians of the 17th century, such as Kepler and Cavalieri. For that reason, the rule is mainly known as the Pappus-Guldin theorem. In [2] and [16] one can find two reasons which explain why the mathematicians of the early 17th century did not know the Pappus rule. In any case, both articles of the History of Mathematics conclude that P. Guldin did not plagiarize Pappus.

In 1969 A. W. Goodman and G. Goodman [8] developed a generalization of the Pappus-Guldin theorem for domains $D$ generated by moving a plane region $D_0$ around an arbitrary space curve $c$. When the center of mass of $D_0$ is on $c$, they obtained a Pappus type formula

$$Volume(D) = Area(D_0) \times Length(c). \tag{1.1}$$

Formulas for the area of the surface $C$ generated by moving a plane curve $C_0$ around $c$ are also considered in [8] but the authors obtain a generalization of the Pappus-Guldin theorem only for plane curves $c$ and 'natural motions' of a plane curve $C_0$ along $c$. One year later, L. E. Pursell [15] and H. Flanders [7] supplement the results

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in [8] by showing that there exists a unique spin function such that the area of the surface $C$ generated as $C_0$ moves with this spin (and with the center of mass on $c$) obeys a Pappus type formula

$$Area(C) = Length(C_0) \times Length(c). \tag{1.2}$$

L. E. Pursell obtains these results using elementary, classical methods of differential geometry and shows that the 'natural motion' in [8] means a motion with $C_0$ fixed to a Frenet frame. On the other hand, H. Flanders obtain the results in [15] more efficiently using moving frames and differential forms.

The above results on curves in $\mathbb{R}^3$ where generalized, by A. Gray, and the second author, to curves in $n$-dimensional spaces of constant sectional curvature in [12], where, also, was stressed the importance of the momenta of $D_0$ in the formula for $Volume(D)$ and the influence of the motion on the vector normal to $C$, giving, with this last remark, a first explanation of the difference between the behaviors of $Volume(D)$ and $Volume(C)$. A further detailed study of $Volume(C)$ was done in [3], where Domingo-Juan and ourselves introduced a limited use of motions as curves in the Lie algebra of the orthogonal group in order to obtain a better comprehension of the relation between the motion and $Volume(C)$.

For motions along submanifolds new phenomena appear, many of which have been studied in [4] and [5].

As is revealed, for instance in [15] and [3], there is an obvious connection between the Pappus-Guldin formula and a different line of research that was initiated by H. Hotelling ([14]) around 1939. Motivated by a problem of statistical inference, he computed the first terms of the asymptotic expansion of the volume of a tube around a curve in a Euclidean or Spherical $n$-dimensional space. In the same year and journal ([18]), H. Weyl published a formula for the volume of a tube $P_r$ and of the corresponding tubular hypersurface $\partial P_r$ around a $q$-dimensional submanifold $P$ in a Euclidean or Spherical space. The tube of radius $r$ around $P$ is defined as the set of points at distance from $P$ lower or equal to $r$, and the corresponding tubular hypersurface $\partial P_r$ is the set of points at distance from $P$ equal to $r$. Apart from the interest (at least in statistical inference) of having a precise formula for $Volume(P_r)$ and $Volume(\partial P_r)$, a remarkable and striking fact of these formulae is that both volumes depend only on the radius $r$ and the intrinsic geometry of $P$. This formula was used later in the first proof of the generalized Gauss-Bonnet Theorem by Allendoerfer and Fenchel ([1, 6]). A lot of work related to Weyl's formulae has been done after, and it is possible to find many references in the book [10] and the survey [17].

Other approaches for a better understanding of Weyl's formula are: the study of subtubes by A. Gorin ([9]) and the consideration of tubes of non constant radius by the first author ([13]). The importance of this last work in relation with Weyl's formulae is that, although these tubes also have spherical section, the different behavior between $Volume(D)$ and $Area(\partial D)$ appears again, showing that 'having a spherical section' is not a sufficient condition to have a Pappus or Weyl's type formula.

In this paper we go back to $\mathbb{R}^3$, and we will try to understand the light that, on Weyl's formula, can shed the union of the approaches "motions along curves" and "tubes of non-constant radius" (which will lead to the notion of weighted motion), and also the systematic consideration, from the beginning, of motions along a curve as curves in a Lie group or its corresponding Lie algebra. Then, our results will be on volumes of domains and areas of surfaces obtained by weighted motions, and they will be given by formulae where the expression of the motion as a curve in a Lie group/algebra will appear in a fairly explicit form.

Given a domain $D_0$ or a plane curve $C_0$ in the plane $P_0$ orthogonal to a curve $c$ in $\mathbb{R}^3$ at $c(0)$, a weighted motion of $D_0$ (or $C_0$) along $c$ from $c(0)$ to $c(t)$, with weight function $g(t)$, gives a domain $D_t$ homothetic to $D_0$ (or a curve $C_t$ homothetic to $C_0$). So the area (resp. the length) of the section of the body obtained by such a motion will be that of $D_0$ (resp. $C_0$) multiplied by the factor $g(t)^2$ (resp. $g(t)$), then, we would expect a Pappus or Weyl's type formula (like (1.1) and (1.2)) for the volume of the domain (resp. area of the surface) generated by a weighted motion of $D_0$ (resp. $C_0$) along $c$ of the form

$$V = Area(D_0) \int_0^L g^2(t)\mathrm{d}t, \quad \left(\text{resp. } A = Length(C_0) \int_0^L g(t)\mathrm{d}t,\right) \tag{1.3}$$

where $L$ is the length of the arc-length parametrized curve $c(t)$.

Then we will study under which conditions imposed on $D_0$ or $C_0$ and the weighted motion, formulae (1.3) hold.

On the other hand, the consideration of motions as curves in a Lie group or in its corresponding Lie algebra will allow to distinguish better between the influence of the curve $c$ and of the motion on the volume of a domain and the area of a surface obtained by a motion. In particular, this will allow us to clarify some remarks made in [3].

## 2 Weighted motions

In this section we will give the definition of weighted motion, which mixes the notions of "motions along a curve" ([8]) and "tubes of non-constant radius" ([13]).

Let $c : I = [0, L] \longrightarrow \mathbb{R}^3$ be a $C^\infty$ curve parametrized by arc-length $t$. Let $g : I = [0, L] \longrightarrow \mathbb{R}^+$ be a positive and differentiable function with $g(0) = 1$. We shall denote by $P_t$ the plane through $c(t)$ orthogonal to $c(t)$.

**Definition 1.** *A weighted motion of weight $g(t)$ along $c$ (or $g(t)-$weighted motion for short) associated to a positively oriented smooth orthonormal frame $\{E_1(t) = c'(t), E_2(t), E_3(t)\}$ along $c(t)$ is a map $\phi : [0, L] \times P_0 \longrightarrow \mathbb{R}^3$ defined by*

$$\phi\left(t, \left(c(0) + \sum_{i=2}^{3} x^i E_i(0)\right)\right) = c(t) + g(t) \sum_{i=2}^{3} x^i E_i(t). \tag{2.1}$$

*Let us denote by $\overline{P}_t$ the vectorial plane defined by $\overline{P}_t = P_t - c(t)$. A motion $\phi$ defines a family $\Phi := \{\varphi_t : \overline{P}_0 \longrightarrow \overline{P}_t\}_{t\in[0,L]}$ of conformal isomorphisms with conformal factor $g(t)$ given by*

$$\varphi_t(x - c(0)) = \phi(x, t) - c(t), \tag{2.2}$$

*and a family of conformal maps*

$$\phi_t : P_0 \longrightarrow P_t \text{ defined by } \phi_t(x) := \phi(t, x) = c(t) + \varphi_t(x - c(0)) \tag{2.3}$$

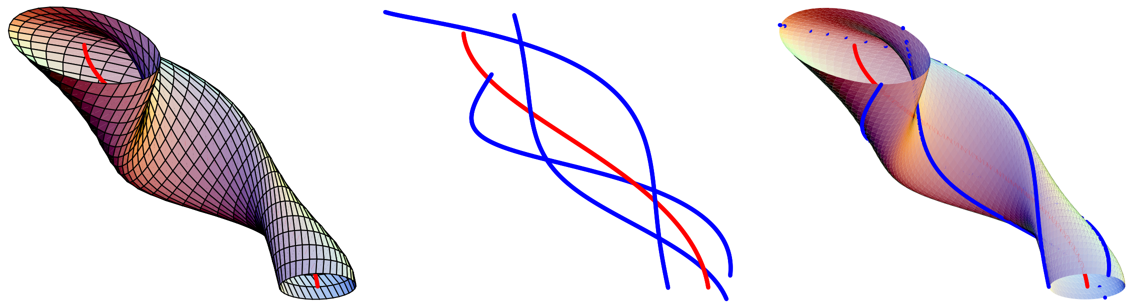

Fig. 1: $1 + \sin(t/(2\sqrt{2}))$-weighted motion of an ellipse and three points along a helix, for $t \in [0, \sqrt{2}\,\pi]$

A particular case of weighted motions are those called 'motions' (and we shall use the same name in this paper) in [8], [15], [12] and [3]. They are 1-weighted motions. Then, in a natural way, *every $g(t)$-weighted motion $\phi$, has an associated motion $\overline{\phi}$* defined by

$$\overline{\varphi}_t = \frac{1}{g(t)}\varphi_t \quad \text{and} \quad \overline{\phi}(t,x) = c(t) + \overline{\varphi}_t(x - c(0)).$$

This associated motion will play an important role in the formulae for the volume and the area.

*From now on, we shall suppose that all the curves that we consider have a Frenet frame* $\{f_1(t) = c'(t), f_2(t), f_3(t)\}$. Then, every curve $c(t)$ has two special kinds of frames: Frenet frames and parallel frames. Parallel frames are defined as follows. The *normal derivative* of a vector field $X(t)$ along $c(t)$ in the direction of $c'(t)$ is defined as the component of the usual derivative $X'(t)$ orthogonal to $c(t)$, that is

$$\frac{DX(t)}{dt} = X'(t) - \langle X'(t), c'(t)\rangle\, c'(t). \tag{2.4}$$

We say that *an orthonormal frame $\{e_1(t) := c'(t), e_2(t), e_3(t)\}$ is parallel along* $c(t)$ if $\frac{De_2(t)}{dt} = 0 = \frac{De_3(t)}{dt}$.

As special cases, we will consider *Frenet motions $\phi^F$ and parallel motions $\phi^P$, which are the 1-weighted motions associated to Frenet frames and parallel frames, respectively.* Frenet motions are called motions in a natural manner in [8] and parallel motions are called motions without spin in [15].

Since we are going to look at the motions as curves in a Lie group or its associated Lie algebra, we shall recall that the group of conformal maps of $\mathbb{R}^2 \equiv \overline{P}_0$ is $]0,\infty[\times SO(2)$ with the inner law $(a,A)(b,B) = (ab, AB)$, with the action on $\overline{P}_0$ given by $(a,A)v = a\ Av$, and its Lie algebra is $\mathbb{R} \times \mathfrak{o}(2)$ with the inner laws $(\alpha,\mathcal{A}) + (\beta,\mathcal{B}) = (\alpha+\beta, \mathcal{A}+\mathcal{B})$ and $[(\alpha,\mathcal{A}),(\beta,\mathcal{B})] = (0, \mathcal{AB} - \mathcal{BA})$, and the action on $\overline{P}_0$ given by $(\alpha,\mathcal{A})v = \alpha v + \mathcal{A}v$. Moreover $SO(2) \cong S^1$ with the isomorphism given by $A_\theta \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto e^{i\theta}$, and $\mathfrak{o}(2) \cong \mathbb{R}$ with the isomorphism given by $\mathcal{A}_\theta \equiv \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \mapsto \theta$.

Using these isomorphisms, the group of conformal maps of $\mathbb{R}^2$ can be identified with the group $]0,\infty[\times S^1$ with the inner law $(a,e^{i\theta})(b,e^{i\varphi}) = (ab, e^{i(\theta+\varphi)})$, and its Lie algebra can be identified with the commutative Lie algebra $\mathbb{R}\oplus\mathbb{R}$ with the inner law $(\alpha,\theta)+(\beta,\varphi) = (\alpha+\beta,\theta+\varphi)$. The exponential map between the Lie algebra and the group is given by

$$\exp(\alpha,\theta) = (e^{\alpha}, e^{i\theta}), \tag{2.5}$$

which is a covering map.

Now, if we choose an auxiliary model weighted motion $\phi^M$ with weight $g^M$, given any $g(t)$-weighted motion $\phi$, for each $t\in I$, we consider the maps

$$A^M(t) := (\varphi_t^M)^{-1}\circ\varphi_t : \overline{P}_0 \longrightarrow \overline{P}_0, \quad t\in I, \tag{2.6}$$

which are conformal isomorphisms with conformal factor $g(t)/g^M(t)$.

Therefore, once $\phi^M$ is fixed, *we can identify a weighted motion $\phi$ along $c(t)$ with a curve $A^M : I \longrightarrow ]0,\infty[\times S^1$ such that $A^M(0) = (1,1)$.*

Moreover, since $\exp : \mathbb{R}\oplus\mathbb{R} \longrightarrow ]0,\infty[\times S^1$ is a covering map, there is a unique lifting $\ln A^M : I \to \mathbb{R}\oplus\mathbb{R}$ of $A^M$ satisfying $\ln A^M(0) = (0,0)$, and *a weighted motion can be considered as a curve $\ln A^M : I \to \mathbb{R}\oplus\mathbb{R}$ satisfying $\ln A^M(0) = (0,0)$.*

When we restrict our attention to motions, the above representation has a simpler form. If $\phi$ is any motion (then $g(t) = 1$) and the model $\phi^M$ that we choose is also a motion, then the maps $A^M(t)$ are isometries of $\overline{P}_0$, and a motion can be considered as a curve $A^M : I \longrightarrow S^1$ satisfying $A^M(0) = 1$, and it has a unique lifting to a curve $\ln A^M : I \longrightarrow \mathbb{R}$ satisfying $\ln A^M(0) = 0$, which can also be considered as a representation of the motion as a curve in the Lie algebra $\mathbb{R}\cong\mathfrak{o}(2)$.

If $\phi$ is a generic $g(t)$-weighted motion and we choose as model a motion $\phi^M$ (that is, $\phi^M$ is a 1-weighted motion), we have the curves $A^M$ and $\ln A^M$, in $\mathbb{R}\times S^1$ and $\mathbb{R}\oplus\mathbb{R}$ respectively, representing the weighted motion $\phi$. Moreover, there are the associated motion $\overline{\phi}$ of $\phi$ and the curves $R^M$ and $\ln R^M$, in $S^1$ and $\mathbb{R}$ respectively, representing the motion $\overline{\phi}$. In this situation we have the following relation between the curves representing the weighted motion and its associated motion:

$$A^M(t) = (g(t), R^M(t)) \quad \text{and} \quad \ln A^M = (\ln g(t), \ln R^M(t)). \tag{2.7}$$

In the next sections, we shall use, as models $\phi^M$, the Frenet motion $\phi^F$ (which will appear in a natural way when we consider volumes of domains) and the parallel motion $\phi^P$ (which will appear in the formulae for the area of a surface), then the associated curve $A^M$ will be denoted by $A^F$ and $A^P$ respectively, and $R^M$ will be denoted by $R^F$ and $R^P$ respectively.

Let us remark that the action of $]0,\infty[\times S^1$ (resp. $\mathbb{R}\oplus\mathbb{R}$) on $\overline{P}_0$ defines an action of the same group (resp. the same algebra) on $P_0$ by $(a,A)(c(0)+v) = c(0)+(a,A)v$ (resp. $(\alpha,\mathcal{A})(c(0)+v) = c(0)+(\alpha,\mathcal{A})v$). Then every curve $A^F$, $A^P$, $R^F$, $R^P$ can also be considered as acting on $P_0$.

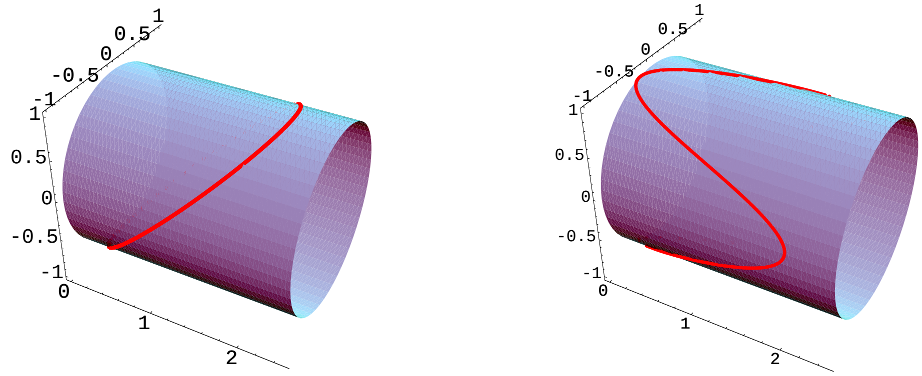


Fig 2: The curves $A^F(t)$ and $A^P(t)$ of the motion in Fig.1.

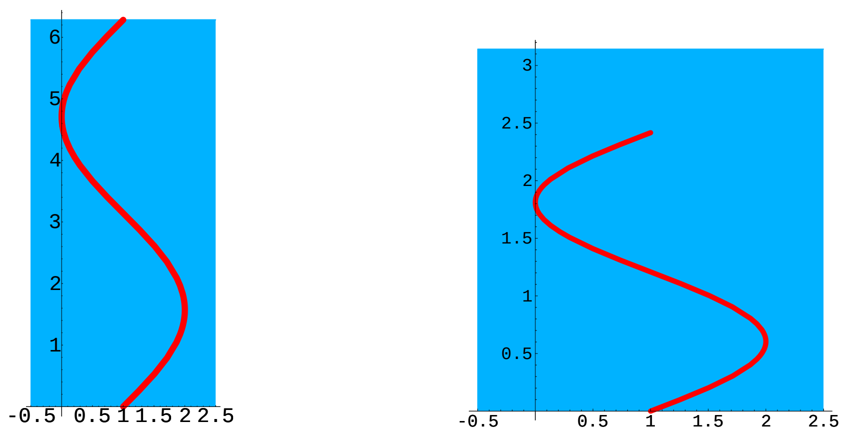


Fig 3: The curves $\ln A^F(t)$ and $\ln A^P(t)$ of the motion in Fig.1.

We finish this section recalling the definitions of moment and center of mass.

Let $\Gamma$ be an oriented line in $\mathbb{R}^2$, $o \in \Gamma$ and $\xi$ the unit vector normal to $\Gamma$ in $o$ which defines the orientation of $\Gamma$. Given a set (domain or curve) $B$ of $\mathbb{R}^2$, we define the moment $M_\Gamma(B)$ of $B$ with respect to $\Gamma$ by the integral

$$M_\Gamma(B) = \int_B \langle \xi, x - o \rangle \, \mathrm{d}x, \tag{2.8}$$

where $\mathrm{d}x$ is the area or line element of $B$. It can be checked by using elementary trigonometry that it does not depend on the choice of $o$ in $\Gamma$.

A point $o \in \mathbb{R}^2$ is the *center of mass* of $B$ if and only if $M_\Gamma(B) = 0$ for every line $\Gamma$ through $o$.

## 3 Volume of domains obtained by weighted motions

Let $D_0$ be a domain in $P_0$, $D_t = \phi(\{t\} \times D_0)$, and $D = \phi([0, L] \times D_0)$ (*the domain obtained by the* $g(t)-$*weighted motion* $\phi$ *of* $D_0$ *along* $c$). We suppose that $c(t)$, $D_0$ and $g(t)$ are such that $D$ has no selfintersections.

**Theorem 1.** *Let $\Gamma$ be the line of $P_0$ through $c(0)$ orthogonal to $f_2(0)$ and oriented by $f_2(0)$; then*

$$Volume(D) = Area(D_0)\int_0^L g^2(t)\mathrm{d}t - \int_0^L g^3(t)\kappa(t)M_{R^F(t)^{-1}\Gamma}(D_0)\mathrm{d}t \qquad (3.1)$$

*where $\kappa(t)$ is the curvature of $c(t)$.*

*Proof.* Let $x = \sum_{i=2}^{3} x^i E_i(0)$, $x(t) = \sum_{i=2}^{3} x^i E_i(t) = \overline{\phi}(t,x) - c(t) = \overline{\varphi}_t(x)$ (then $|x(t)| = |x|$), $N(t) = \frac{x(t)}{|x|}$. Let $\mathrm{d}x$ be the area element of $D_0$. By the rule of change of variable in multiple integrals, we have

$$Volume(D) = \int_0^L \int_{D_0} |\mathrm{detJac}(\phi)|\ \mathrm{d}t\ \mathrm{d}x. \qquad (3.2)$$

Further, it holds

$$|\mathrm{detJac}(\phi)| = \left|\left\langle \frac{\partial\phi}{\partial t}\ ,\ \frac{\partial\phi}{\partial x^2} \wedge \frac{\partial\phi}{\partial x^3}\right\rangle\right|, \qquad (3.3)$$

$$\frac{\partial\phi}{\partial t}(t,x) = c'(t) + g'(t)x(t) + g(t)|x|N'(t), \qquad (3.4)$$

$$\frac{\partial\phi}{\partial x^2} \wedge \frac{\partial\phi}{\partial x^3}(t,x) = g^2(t)E_2 \wedge E_3 = g^2(t)c'(t) \qquad (3.5)$$

$$\langle N'(t), c'(t)\rangle = -\langle N(t), c''(t)\rangle = -\kappa(t)\langle N, f_2(t)\rangle \qquad (3.6)$$

First we substitute (3.4), (3.5) and (3.6) in (3.3), then the result of this substitution in (3.2), and we obtain

$$Volume(D) = \int_0^L g^2(t)\ Area(D_0)\mathrm{d}t - \int_0^L \int_{D_0} |x|g^3(t)\kappa(t)\langle N(t), f_2(t)\rangle\, \mathrm{d}x\mathrm{d}t. \quad (3.7)$$

But

$$\langle N(t), f_2(t)\rangle = \left\langle \varphi_t^F \circ R^F(t)N(0), \varphi_t^F f_2(0)\right\rangle = \left\langle N(0), R^F(t)^{-1}f_2(0)\right\rangle, \qquad (3.8)$$

and

$$\int_{D_0} |x|\,\langle N(t), f_2(t)\rangle\, \mathrm{d}x = \int_{D_0} \left\langle x, R^F(t)^{-1}f_2(0)\right\rangle\ \mathrm{d}x = M_{R^F(t)^{-1}\Gamma}(D_0). \qquad (3.9)$$

Then (3.1) follows from (3.7) and (3.9).

Formula (3.1) for $Volume(D)$ has two summands. The extrinsic geometry of $c(t)$ is present only in the second one (through $\kappa(t)$). As a consequence,

**Corollary 1.** *$Volume(D)$ does not depend on $\kappa(t)$ if and only if one of the following conditions hold:*

*a) The motion $\overline{\phi}$ associated to $\phi$ is a Frenet motion and $M_\Gamma(D_0) = 0$,*

*b) $c(0)$ is the center of mass of $D_0$.*

*Proof.* $Volume(D)$ does not depend on $\kappa(t)$ if and only if

$$\int_0^L g^3(t)\kappa(t)M_{R^F(t)^{-1}\Gamma}(D_0)\mathrm{d}t$$

is constant for every function $\kappa(t)$, and this condition holds if and only if $M_{R^F(t)^{-1}\Gamma}(D_0) = 0$ for every $t$. For this, there are two possibilities:

a) $R^F(t)^{-1}\Gamma = \Gamma$ for every $t$ (then $R^F(t) = Id$), and $M_\Gamma(D_0) = 0$, which is the first condition in the corollary, or

b) $\Gamma_t := R^F(t)^{-1}\Gamma \neq \Gamma$ at some $t$. If $\xi_t$ is the oriented unit vector orthogonal to $\Gamma_t$ used to define $M_{\Gamma_t}(D_0)$ (see (2.8)), then $\{f_2(0), \xi_t\}$ is a basis of $P_0$. Since the integral expression 2.8 is linear in $\xi$, it follows that the vanishing of the moments respect to $\Gamma_t$ and $\Gamma$ is equivalent to the vanishing of the moment with respect to any line through $c(0)$, that is, to $c(0)$ being the center of mass of $D_0$.

The motion is present in both terms in (3.1), in the first only through its weight $g(t)$, whereas in the second, both the weight and the rotation part $R^F(t)$ are present, but the contribution of the last is only through the moment $M_{R^F(t)^{-1}\Gamma}(D_0)$. As a consequence:

**Corollary 2.** *$Volume(D)$ does not depend on the motion $\overline{\phi}$ associated to $\phi$ if and only if $c(0)$ is the center of mass of $D_0$ or $\kappa(t) = 0$.*

*Proof.* From (3.1), it is clear that $Volume(D)$ does not depend on $\overline{\phi}$ if and only if $\kappa(t) = 0$ or $M_{\mathcal{L}}(D_0)$ is constant as a function of the unit vector $\xi$ orthogonal to the line $\mathcal{L} \in P_0$ through $c(0)$. Since $M_{\mathcal{L}}(D_0)$ is linear in $\xi$ (as we remarked before), it is constant if and only if it is zero.

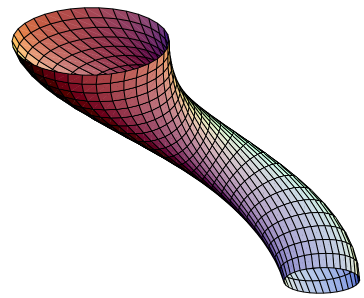

Fig. 4: A consequence of Corollary 2 is that the domain enclosed by this surface has the same volume as that of Figure 1

Arguments like those used in the above corollaries give the following answer to the question arisen in the introduction:

**Corollary 3.** *Given $D_0$, the formula (1.3) holds for a straight line ($\kappa(t) = 0$) and it holds for every curve (or, fixed a curve with $\kappa(t) \neq 0$, for every weight $g$) if and only if $c(0)$ is the center of mass of $D_0$.*

## 4 Area of surfaces obtained by weighted motions

Let $C_0$ be a plane curve in $P_0$. For any weighted motion $\phi$, we write $C_t = \phi_t(C_0)$, and $C = \phi([0, L] \times C_0) = \cup_{t\in[0,L]} C_t$ will be called *the surface obtained by the $g(t)-$weighted motion $\phi$ of $C_0$ along $c$.*

$c(0) + u(s)$ will be a parametrization of the curve $C_0$ by its arclength. Then $u : [0, \ell] \longrightarrow \overline{P}_0$ is a curve in $\overline{P}_0$ with image $C_0 - c(0)$, $|\dot{u}(s)| = 1$ and $u(s) = \sum_{i=2}^3 u^i(s)E_i(0)$.

We shall write

$$\phi(t, s) := \phi(t, c(0) + u(s)) = c(t) + g(t)u_t(s), \tag{4.1}$$

where $u_t(s) = \overline{\varphi}_t(u(s)) = \sum_{i=2}^3 u^i(s)E_i(t)$. As a consequence, $|u_t(s)| = |u(s)|$. Moreover, $N(t)$ will denote the unit vector in the direction of $u_t(s)$, that is, $N(t) =$

$\frac{u_t(s)}{|u|}$. Of course, $N(t)$ is not well defined at the points $s$ where $u(s)=0$, but this may happen for only a finite number of $s$ at most (only one if $C_0$ has no self-intersections), and it has no influence in our computations.

For every $t \in [0, L]$, we shall denote by $J$ the isometry of $\overline{P}_t$ satisfying that, if $e \in \overline{P}_t$ is a unit vector, then $\{c'(t), e, Je\}$ is a positively oriented orthonormal basis of $\mathbb{R}^3$.

**Theorem 2.** *With the notation as above, it holds*

$$Area(C) = \int_0^L \int_0^\ell \Big[\Big\langle g(t)(\ln A^P)'(t)(u(s)), J\dot{u}(s)\Big\rangle^2 \tag{4.2}$$

$$+\Big(1 - g(t)\ \Big\langle u(s), R^F(t)^{-1} f_2(0)\Big\rangle\, \kappa(t)\Big)^2\Big]^{1/2} g(t)\mathrm{d}s\ \mathrm{d}t$$

*Proof.* Using the parametrization $\phi(s,t)$ of the surface $C$ given by (4.1), we have

$$Area(C) = \int_0^L \int_0^\ell |\frac{\partial\phi}{\partial t} \wedge \frac{\partial\phi}{\partial s}|\ \mathrm{d}s\ \mathrm{d}t \tag{4.3}$$

where

$$\frac{\partial\phi}{\partial t} = c'(t) + |u|\ (g(t)N(t))', \quad \frac{\partial\phi}{\partial s} = g(t)\dot{u}_t(s), \tag{4.4}$$

$$(g(t)N(t))' \wedge \dot{u}_t(s) = \langle (gN)'(t), c'(t)\rangle\ \ c'(t) \wedge \dot{u}_t(s) + \frac{D(gN)}{dt} \wedge \dot{u}_t(s), \tag{4.5}$$

$$\langle (gN)'(t), c'(t)\rangle = -\langle (gN)(t), c''(t)\rangle = -\kappa(t)\langle (gN)(t), f_2(t)\rangle. \tag{4.6}$$

From the definition of $J$ and the cross vector product,

$$c'(t) \wedge \dot{u}_t(s) = J\dot{u}_t(s), \tag{4.7}$$

$$\frac{D(gN)}{dt} \wedge \dot{u}_t(s) = \left\langle \frac{D(gN)}{dt}, J\dot{u}_t(s)\right\rangle J\dot{u}_t(s) \wedge \dot{u}_t(s) = -\left\langle \frac{D(gN)}{dt}, J\dot{u}_t(s)\right\rangle c'(t). \tag{4.8}$$

From (4.5), (4.6), (4,7), and (4.8),

$$\frac{\partial\phi}{\partial t} \wedge \frac{\partial\phi}{\partial s}(s,t) = \Big(g(t) - \kappa(t)|u|g^2(t)\langle N(t), f_2(t)\rangle\Big) J\dot{u}_t(s)$$
$$- g(t)\left\langle \frac{D(gN)}{dt}, J\dot{u}_t(s)\right\rangle c'(t) \tag{4.9}$$

but, with the notation $A^P(t)N(0) = \sum_{i=2}^3 A_i^j(t)N^i E_j(0)$,

$$\begin{aligned}\frac{D(gN)}{\mathrm{dt}} &= \frac{D}{\mathrm{dt}}(\varphi_t(N(0))) = \frac{D}{\mathrm{dt}}(\varphi_t^P \circ (\varphi_t^P)^{-1} \circ \varphi_t(N(0)))\\ &= \frac{D}{\mathrm{dt}}(\varphi_t^P \circ A^P(t)(N(0))) = \sum_{i=2}^3 \frac{D}{\mathrm{dt}}(A_i^j(t)N^i\varphi_t^P(E_j(0)))\\ &= \sum_{i=2}^3 A_i^{j\prime}(t)N^i \circ \varphi_t^P(E_j(0)) = \varphi_t^P\left(\sum_{i=2}^3 A_i^{j\prime}(t)N^i E_j(0)\right)\\ &= \varphi_t^P \circ A^{P\prime}(t)(N(0)).\end{aligned} \tag{4.10}$$

Then,

$$\left\langle \frac{D(gN)}{\mathrm{d}t}(s,t), J\dot{u}_t(s) \right\rangle = g^2(t) \left\langle \varphi_t^{-1} \frac{D(gN)}{\mathrm{d}t}, \varphi_t^{-1} J\dot{u}_t(s) \right\rangle$$
$$= g^2(t) \left\langle A^{P^{-1}} \circ A^{P\prime}(t)(u), \frac{1}{g(t)} J\dot{u}_t(0) \right\rangle$$
$$= g(t) \left\langle (\ln A^P)'(t)(N(0)), J\dot{u}_t(0) \right\rangle. \tag{4.11}$$

Finally, by substitution of (4.9) and (4.11) in (4.3) we obtain (4.2).

**Remark** For motions ($g(t) = 1$), we assured in [3] that, in general, $Area(C)$ depends on the torsion $\tau$ of the curve $c$, and that this dependence was encoded in a part of the formula carrying the normal covariant derivative. As a proof of this statement, we gave some examples showing that a Frenet motion of the same curve $C_0$ along two curves with the same curvature and different torsion give two surfaces with different area. However, formula (4.2) shows that $Area(C)$ depends on the motion, but not on $\tau$. This is one of the advantages of formula (4.2) to express the area. But, then, what about the examples in [3]? The reason for the dependence of the area of these two surfaces on $\tau$ is that Frenet motion is a motion defined using a Frenet frame, and this Frenet frame has encoded information on $\tau$. So, the dependence on $\tau$ of $Area(C)$ is due to the motion, not to the curve. In fact, although Frenet motions on curves with different torsion have the same name, the motions as curves $A^P$ in $S^1$ are different.

With more detail, formula (4.2) shows that, for a $g(t)$-motion $\phi(t)$, two associated curves $A^P(t)$ and $A^F(t)$ in $]0,\infty[\times S^1$ are relevant for $Area(C)$, and it is the interplay of these two curves which makes the torsion appear. In fact, if these two associated curves have the form $A^F(t) = \left(g(t), e^{i\theta_F(t)}\right)$, and $A^P(t) = \left(g(t), e^{i\theta_P(t)}\right)$, then

$$A^F(t) \circ (A^P)^{-1}(t) = \left(1, e^{i(\theta_F - \theta_P)(t)}\right) \quad \text{and}$$
$$A^F(t) \circ (A^P)^{-1}(t) = (\varphi_t^F)^{-1} \circ \varphi_t \circ \varphi_t^{-1} \circ \varphi_t^P = (\varphi_t^F)^{-1} \circ \varphi_t^P$$

which is the curve $R_F^P(t)$ in $S^1$ defined by a parallel motion when we take the Frenet motion as the model. Then

$$\theta_P(t) - \theta_F(t) = \theta_P^F(t),$$

where $\theta_P^F(t)$ is the angle of the rotation $R_F^P(t)$. But, if $\{c'(t), E_2(t), E_3(t)\}$ is a parallel frame along $c(t)$, with $E_i(0) = f_i(0)$,

$$f_2(t) = \cos\theta_P^F(t)\ E_2(t) - \sin\theta_P^F(t) E_3(t),$$
$$f_3(t) = \sin\theta_P^F(t)\ E_2(t) + \cos\theta_P^F(t) E_3(t),$$

and, taking normal derivatives in both equalities and applying Frenet equations, we obtain

$$\tau(t) = -(\theta_P^F)'(t) = {\theta_F}'(t) - {\theta_P}'(t),$$

which shows how the torsion of $c(t)$ is determined by the two curves $A^F(t)$ and $A^P(t)$ defined by the motion $\phi$.

If we compare the expression (4.2) with (3.1), we see that $Area(C)$ has a part similar to $Volume(D)$ which depends on the motion through its representation as a curve in $]0,\infty[\times S^1$ (using $\phi^F$ as the model motion); and a particular part (4.9) which depends on the derivatives of the motion considered as a curve in the Lie algebra $\mathbb{R}\oplus\mathbb{R}$ (using $\phi^P$ as a model motion).

**Corollary 4**. *If $C_0$ is a piecewise $C^1$-curve, $Area(C)$ does not depend on the derivative of the motion $\phi$ considered as a curve $\ln A^P(t)$ in the Lie algebra $\mathbb{R}\oplus\mathbb{R}$ if and only if one of the following conditions hold:*

*a) $C_0$ is a logarithmic spiral with polar equation $r(\varphi) = b\ e^{a\varphi}$ and $\ln A^P(t)$ is a straight line in $\mathbb{R}\oplus\mathbb{R}$ with slope $a$. When $a = 0$, $C_0$ is a circle and $g(t) = 1$ (that is, $\phi$ is a motion).*

*b) $C_0$ is a segment of a straight line through $c(0)$ and the motion $\overline{\phi}$ associated to $\phi$ is parallel.*

*c) $C_0$ is any curve and $\phi$ is the parallel motion.*

*Proof.* From (4.2), $Area(C)$ does not depend on $(\ln A^P)'(t)$ if and only if $\Big\langle(\ln A^P)'(t)(u(s)), J\dot{u}(s)\Big\rangle = 0$ for every $t \in I$ and every $s \in [0, L]$. If $\ln A^P(t) = (\ln g(t), \theta(t))$, using the identification between $\mathbb{R}\oplus\mathbb{R}$ and $\mathbb{R}\times\mathfrak{o}(2)$ indicated in Section 2, we have

$$\begin{aligned}(\ln A^P)'(t)(u) &= \frac{g'(t)}{g(t)}u + \begin{pmatrix}0 & -\theta'(t)\\ \theta'(t) & 0\end{pmatrix}\begin{pmatrix}u_2\\ u_3\end{pmatrix}\\ &= \frac{g'(t)}{g(t)}u + \theta'(t)\begin{pmatrix}-u_3\\ u_2\end{pmatrix} = \frac{g'(t)}{g(t)}u + \theta'(t)\ Ju, \end{aligned} \tag{4.12}$$

Now, let us suppose that $C_0$ is of class $C^1$. Let us consider the case when there is a $s_0 \in [0,\ell]$ such that $\dot{u}(s_0) \neq \frac{u(s_0)}{|u(s_0)|}$. Since $C_0$ is $C^1$, the set $\{s \in [0,\ell]; \dot{u}(s) \neq \frac{u(s)}{|u(s)|}\}$ is open and contains a maximal open subinterval $\mathcal{J}$ of $[0,\ell]$ containing $s_0$. On this interval $\mathcal{J}$ we can write $u(s) = r(s)(\cos\beta(s), \sin\beta(s))$, where $\beta(s)$ is a $C^1$ function which gives, modulo $2\pi$, the angle between $u(s)$ and $f_2(0)$. Then, for every $s \in \mathcal{J}$,

$$\dot{u}(s) = \dot{r}(s)\frac{u(s)}{|u(s)|} + r(s)\dot{\beta}(s)J\frac{u(s)}{|u(s)|}, \tag{4.13}$$

$$1 = |\dot{u}(s)| = \dot{r}(s)^2 + r(s)^2\dot{\beta}(s)^2,$$

$$J\dot{u}(s) = -r(s)\dot{\beta}(s)u(s) + \dot{r}(s)Ju(s),$$

$$\langle u(s), J\dot{u}(s)\rangle = -r(s)\ \dot{\beta}(s) = -\sqrt{1-\dot{r}(s)^2} \tag{4.14}$$

and

$$\langle Ju(s), J\dot{u}(s)\rangle = \dot{r}(s). \tag{4.15}$$

then

$$\begin{aligned}\Big\langle(\ln A^P)'(t)(u), J\dot{u}\Big\rangle &= \left\langle\frac{g'(t)}{g(t)}u + \theta'(t)\ Ju, J\dot{u}\right\rangle\\ &= -\frac{g'(t)}{g(t)}\sqrt{1-\dot{r}(s)^2} + \theta'(t)\dot{r}(s) = 0.\end{aligned} \tag{4.16}$$

Since the variables $s$ and $t$ appear separated, the equation (4.16) holds if and only if $\theta(t) = 0$ and $g(t) = 1$ (parallel motion) or

$$\frac{g'(t)}{g(t)} \frac{1}{\theta'(t)} = a = \frac{\dot{r}(s)}{\sqrt{1 - \dot{r}(s)^2}}, \tag{4.17}$$

where $a$ is some constant. If $\alpha = a/\sqrt{1+a^2}$, the general solution of the right equation in (4.17) is $r(s) = \alpha\ s + \delta$, where $\delta$ is an arbitrary constant. This is a logarithmic spiral with polar equation $r(\beta) = b\ e^{a\beta}$ (we have implicitly used the relation $1 = \dot{r}(s)^2 + r^2\dot{\beta})$. The left equation in (4.17) is satisfied if and only if $(\ln g)'(t) = a\theta'(t)$.

Since $u$ is $C^1$, $\beta$ and $\dot{\beta}$ are also well defined at the boundaries $s_0, s_1$ of the interval $\mathcal{J}$, and (4.13) is still true on them, and also $r = b\ e^{a\beta}$, from which we have $\dot{r} = a\ b\ e^{a\beta}\dot{\beta}$. Then, using (4.11), for $i = 0, 1$, $\dot{u}(s_i) = \frac{u(s_i)}{|u(s_i)|}$ if and only if $\dot{\beta}(s_i) = 0$ if and only if $\dot{r}(s_i) = 0$ if and only if $\dot{u}(s_i) = 0$, which is in contradiction with the fact that we have chosen $s$ as the arc-length parameter. Then $\mathcal{J}$ is closed and open in $[0, \ell]$, so $\mathcal{J} = [0, \ell]$. Then we have proved that if there is some $s_0 \in [0, \ell]$ with $\dot{u}(s_0) \neq \frac{u(s_0)}{|u(s_0)|}$, then the conditions a) in this corollary are satisfied.

Now, let us suppose that $\dot{u}(s) = \frac{u(s)}{|u(s)|}$ for every $s \in [0, \ell]$, then $u(s) - r(s)\dot{u}(s) = 0$, which is equivalent to saying that $u(s) = (s + k)\ u_0$, with $u_0$ a constant unit vector and $k \in \mathbb{R}$. From this, $J\frac{u(s)}{|u(s)|} = J\dot{u}(s)$ and $\left\langle (\ln A^P)'(t)(u), J\dot{u} \right\rangle = \theta'(t)$, then $\left\langle (\ln A^P)'(t)(u), J\dot{u} \right\rangle = 0$ if and only if $\theta(t) = 0$, that is, $\overline{\phi}$ is a parallel motion. This finishes the proof of the case $C^1$.

If $u$ is only piecewise $C^1$, for each $C^1$ piece of the curve $u$ we must be in one of the cases a), b) or c). If we are not in the case c), we must be in cases a), or b), or we must have some pieces in the case a) and others in the case b). But in the last situation, both conditions on the motion b) and a) must be satisfied. Then $\theta = 0$ from the conditions of case b), and the proof of case a) shows that $\theta = 0$ also implies that $\phi$ is a motion (then, a parallel motion).

Let us remark that the statement "*Area*$(C)$ does not depend on the derivative of the motion considered as a curve in the Lie algebra $\mathbb{R} \oplus \mathbb{R}$ (using $\phi^P$ as the model motion)" is equivalent to saying that

$$Area(C) = Length(C_0) \int_0^L g(t)\mathrm{d}t - \int_0^L g^2(t)\ \kappa(t))M_{R^F(t)^{-1}\Gamma}(C_0)\mathrm{d}t. \tag{4.18}$$

This will help to answer the question in the introduction "When is 1.3 valid for *Area*$(C_0)$?". Of course (1.3) is valid when $c(t)$ is a straight line ($\kappa(t) = 0$) and one of the conditions of Corollary 4 holds. Moreover, we have:

**Corollary 5.** *The formula* (1.3) *holds for Area*$(C)$ *on any curve* $c$ *with* $\kappa(t) \neq 0$ *if and only if*

i) $\phi$ *is a parallel motion and* $c(0)$ *is the center of mass of* $C_0$,

*or*

ii) *the following conditions are satisfied:*

*(a)* $C_0$ *is a logarithmic spiral* $r = b\ e^{a\beta}$, *where* $\beta$ *denotes the angle with the axis* $f_2(0)$ *and* $\beta \in ]\beta_1, \beta_2[$ *satisfying* $\int_{\beta_1}^{\beta_2} e^{2a\beta} \cos\beta\ \mathrm{d}\beta = 0$,

*(b) the associated motion* $\overline{\phi}$ *is Frenet, and*

*(c)* $g(t) = \exp(a\int_0^t \tau(t)\mathrm{d}t)$.

*or*

iii) $C_0$ *is a circle, and* $\phi$ *is a motion* ($g(t) = 1$)

*or*

iv) $C_0$ *is a segment of a straight line with its middle point in* $c(0)$ *and the motion* $\overline{\phi}$ *associated to* $\phi$ *is parallel.*

*Proof.* The remark made before tells us that (1.3) holds if and only if the conditions of Corollary 4 are satisfied and the second summand in (4.18) vanishes. But, arguing as in the proof of Corollary 1, we have that this summand vanishes if and only if one of the two conditions hold:

d) The motion $\overline{\phi}$ associated to $\phi$ is a Frenet motion (which is equivalent to saying that

$$\theta(t) = \int_0^t \tau(t)\mathrm{d}t \;) \tag{4.19}$$

and $M_\Gamma(C_0) = 0$,

e) $c(0)$ is the center of mass of $C_0$.

The union of one of these conditions with the conditions of Corollary 4 gives the following possibilities:

d) and 4.a): $C_0$ is a logarithmic spiral and it can be parametrized by $u(\beta) = b\ e^{a\beta}(\cos\beta, \sin\beta)$. Then

$$0 = M_\Gamma(C_0) = b^2\sqrt{1+a^2}\int_{\beta_1}^{\beta_2} e^{2a\beta}\cos\beta \mathrm{d}\beta \tag{4.20}$$

and the condition 4a) on the motion is given by the equation

$$(\ln g)'(t) = a\theta'(t) = a\ \tau(t) \tag{4.21}$$

and all this is just the set of conditions ii).

d) and 4b) or d) and 4c) are only compatible if $c(t)$ is a plane curve.

e) and 4a) together imply that $C_0$ is a logarithmic spiral given by $u(\beta) = b\ e^{a\beta}(\cos\beta, \sin\beta)$ with the center of mass at $c(0)$, which implies that both (4.18) and

$$0 = M_{\Gamma^\perp}(C_0) = b^2\sqrt{1+a^2}\int_{\beta_1}^{\beta_2} e^{2a\beta}\sin\beta\ \mathrm{d}\beta \tag{4.22}$$

are satisfied, which is equivalent to $a = 0$ and $\beta_2$ being congruent with $\beta_1$ modulo $2\pi$, that is, to $C_0$ be a circle. This also implies that the slope of the motion $\ln A^P(t)$ is zero, so $g(t) = 1$. This gives case iii).

e) and 4b) is case iv).

e) and 4c) is case i).

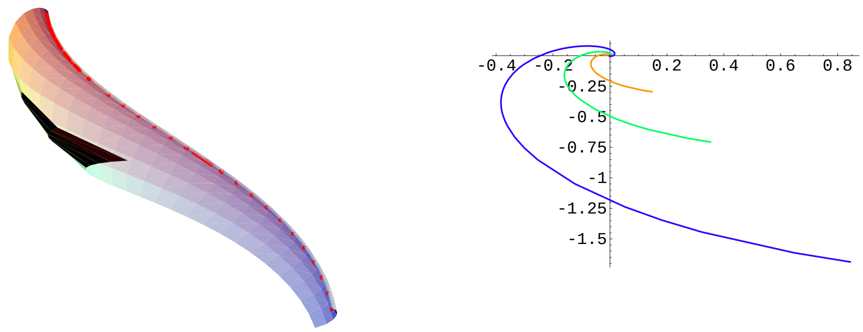


Fig. 5: An example of surface satisfying conditions of Corollary 5 ii) obtained for a motion along the helix of Fig. 1, and the normal sections of this surface at the first, middle and end point of the helix

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