Approximation and Leray-Schauder Type Results for Multimaps in the S-KKM class

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Abstract

The paper presents new approximation and Leray-Schauder type results for multimaps in the class S-KKM.

1 Introduction

In 1969, Ky Fan [6] established the following result:

Let C be a nonempty, compact, convex subset of a normed space E. Then for any continuous mapping f from C to E, there exists an $x_0 \in C$ with

$$||x_0 - f(x_0)|| = \inf_{y \in C} ||f(x_0) - y||.$$

Since then, various analogues of this result have been obtained for other sets C and other types of maps; see, for instance, [2, 9, 10, 12, 18, 19]. Recently, Lin and Park [11] obtained a Fan type approximation result for α-condensing $\mathfrak{A}_c^\kappa$ maps defined on a closed ball in a Banach space. Their results have been extended to other classes of maps by O'Regan and Shahzad [13, 14]. More recently, Shahzad [20] obtained some Fan type approximation results for a Φ-condensing closed s-KKM multimap F with an additional assumption that the composition $f \circ F$ is closed whenever f is continuous. The aim of this paper is to establish Fan type approximation result for s-KKM multimaps in the general setting. As an application, we also obtain the Leray-Schauder type result.

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2 Preliminaries

Let E be a Hausdorff locally convex space. For a nonempty set $Y \subseteq E$, 2^Y denotes the family of nonempty subsets of Y. If L is a lattice with a minimal element 0, a mapping $\Phi : 2^E \to L$ is called a generalized measure of noncompactness provided that the following conditions hold:

(a). $\Phi(A) = 0$ if and only if $\overline{A}$ is compact.

(b). $\Phi(\overline{co}(A)) = \Phi(A)$; here $\overline{co}(A)$ denotes the closed convex hull of A.

(c). $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$.

It follows that if $A \subseteq B$, then $\Phi(A) \leq \Phi(B)$. The Kuratowskii measure and Hausdorff measure of noncompactness are examples of the generalized measure of noncompactness (see [17]).

Let C be a nonempty subset of a Hausdorff locally convex space E and $F : C \to 2^E$. Then F is called Φ-condensing provided that $\Phi(A) = 0$ for any $A \subseteq C$ with $\Phi(F(A)) \geq \Phi(A)$.

Suppose X and Y are Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathfrak{A}$ of maps [15, 16] is defined by the following properties:

(i). $\mathfrak{A}$ contains the class $\mathbb{C}$ of single valued continuous functions;

(ii). each $F \in \mathfrak{A}_c$ is upper semicontinuous and compact valued; and

(iii). for any polytope P, $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathfrak{A}$.

Definition 2.1. $F \in \mathfrak{A}_c^\kappa(X, Y)$ if for any compact subset K of X, there is a $G \in \mathfrak{A}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Definition 2.2. Let X be a convex subset of a Hausdorff topological vector space and Y a topological space. If $S, T : X \to 2^Y$ are two set-valued maps such that $T(co(A)) \subseteq S(A)$ for each finite subset A of X, then we say that S is a generalized KKM map w.r.t. T. The map $T : X \to 2^Y$ is said to have the KKM property if for any generalized KKM w.r.t. T map S, the family

$$\{\overline{S(x)} : x \in X\}$$

has the finite intersection property. We let

$$\mathrm{KKM}(X, Y) = \{T : X \to 2^Y : T \,\text{has the KKM property}\}.$$

Remark 2.1. If X is a convex space, then $\mathfrak{A}_c^\kappa(X, Y) \subset \mathrm{KKM}(X, Y)$ (see [5]).

Definition 2.3. Let X be a nonempty set, Y a nonempty convex subset of a Hausdorff topological vector space and Z a topological space. If $S : X \to 2^Y$, $T : Y \to 2^Z$, $F : X \to 2^Z$ are three set-valued maps such that $T(co(S(A))) \subseteq F(A)$ for each nonempty finite subset A of X, then F is called a generalized S-KKM map w.r.t. T. If the map $T : X \to 2^Z$ is such that for any generalized S-KKM w.r.t. T map F, the family

$$\{\overline{F(x)} : x \in X\}$$

has the finite intersection property, then F is said to have the S-KKM property. The class

$$\text{S-KKM}(X, Y, Z) = \{T : Y \to 2^Z : T \text{ has the S-KKM property}\}.$$

Remark 2.2. Note that S-KKM(X, Y, Z) = KKM(X, Z) whenever $X = Y$ and S is the identity mapping $\mathbf{1}_X$. Also KKM(Y, Z) is a proper subset of S-KKM(X, Y, Z) for any $S : X \to 2^Y$ (see [3, 4] for examples).

Remark 2.3. Let X be a convex space, Y a convex subset of a Hausdorff locally convex space, and Z a normal space. Suppose $s : Y \to Y$ is surjective, $F \in$ s-KKM(Y, Y, Z) is closed, and $f \in \mathbb{C}(X, Y)$. Then $F \circ f \in \mathbf{1}_X -KKM(X, X, Z)$ (see [4]).

Let C be a subset of a Hausdorff topological space X. We let $\overline{C}$ (respectively, $\partial(C)$, $int(C)$) denote the closure (respectively, boundary, interior) of C.

Let C be a subset of a Hausdorff topological vector space E and $x \in X$. Then the inward set $I_C(x)$ is defined by

$$I_C(x) = \{x + r(y - x) : \ y \in C, r \geq 0\}.$$

Let C be a convex subset of Hausdorff locally convex space E with $0 \in int(C)$. The Minkowski functional p of C, defined by

$$p(x) = \inf\{r > 0 : x \in rC\},$$

has the following properties:
(i). p is continuous on E;
(ii). $p(x + y) \leq p(x) + p(y)$, $x, y \in E$;
(iii). $p(\lambda x) = \lambda p(x)$, $\lambda \geq 0$, $x \in E$;
(iv). $0 \leq p(x) < 1$ if $x \in int(C)$;
(v). $p(x) > 1$, if $x \notin \overline{C}$;
(vi). $p(x) = 1$, if $x \in \partial C$.

For $x \in E$, let

$$d_p(x, C) = \inf\{p(x - y) : y \in C\}.$$

The following result [1] will be needed in the sequel.

Lemma 2.1. *Let Ω be a closed, convex subset of a Hausdorff locally convex topological vector space E with $x_0 \in \Omega$. Suppose $s : \Omega \to \Omega$ is surjective and $F \in s - KKM(\Omega, \Omega, \Omega)$ is closed with the following property holding:*

$$A \subseteq \Omega, \ A = \overline{co}\,(\{x_0\} \cup F(A)) \ \textit{implies} \ A \ \textit{is compact}. \tag{2.1}$$

Then F has a fixed point in Ω.

3 Main Results

Theorem 3.1. *Let C be a closed, convex subset of a Hausdorff locally convex space E with $0 \in C$ and U a convex open neighborhood of 0. Suppose C is a normal space, $s : \overline{U} \cap C \to \overline{U} \cap C$ is surjective and $F \in s\text{-}KKM(\overline{U} \cap C, \overline{U} \cap C, C)$ is a closed map satisfying the following condition:*

(3.1) $A \subseteq C,\ A \subseteq \overline{co}\,(\{0\} \cup F(co\,(\{0\} \cup A)))$ *implies* $\overline{A}$ *is compact.*

Then there exist $x_0 \in \overline{U} \cap C$ and $y_0 \in F(x_0)$ with

$$p(y_0 - x_0) \;=\; d_p(y_0, \overline{U} \cap C) \;=\; d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C)\,;$$

here p is the Minkowski functional of U. More precisely, either (i). F has a fixed point $x_0 \in \overline{U} \cap C$, or (ii). there exist $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with

$$0 < p(y_0 - x_0) = d_p(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C);$$

here $\partial_C(U)$ denotes the boundary of U relative to C.

Proof: Let $r : E \to \overline{U}$ be defined by

$$r(x) = \frac{x}{\max\{1, p(x)\}} \text{ for } x \in E.$$

Since $0 \in U = int(U)$, it follows that r is continuous. Let f be the restriction of r to C. Since C is convex and $0 \in C$, $f(C) \subseteq \overline{U} \cap C$. Furthermore $f \in \mathbb{C}(C, \overline{U} \cap C)$. By Remark 2.3, $F \circ f \in \mathbf{1}_C - \mathrm{KKM}(C, C, C)$. Let $G = F \circ f$. Then G is closed. Next we claim

(3.2) if $A \subseteq C$ and $A \subseteq \overline{co}\,(\{0\} \cup G(A))$, then $\overline{A}$ is compact.

To see this notice if $A \subseteq C$ and $A \subseteq \overline{co}\,(\{0\} \cup F\,f(A))$ then since $f(A) \subseteq co\,(\{0\} \cup A)$ we have

$$A \subseteq \overline{co}\,(\{0\} \cup F(co\,(\{0\} \cup A))).$$

Now (3.1) implies $\overline{A}$ is compact, so (3.2) holds. Now Lemma 2.1 guarantees that there exists $z_0 \in C$ with $z_0 \in (F \circ f)(z_0)$. If we let $x_0 = f(z_0) \in \overline{U} \cap C$ then $x_0 \in (f \circ F)(x_0)$. Thus $x_0 = f(y_0)$ for some $y_0 \in F(x_0)$. We now consider two cases: (i) $y_0 \in \overline{U} \cap C$ or (ii) $y_0 \in C \setminus \overline{U}$.
Suppose $y_0 \in \overline{U} \cap C$. Then $x_0 = f(y_0) = y_0$. As a result

$$p(y_0 - x_0) = 0 = d_p(y_0, \overline{U} \cap C)$$

and x_0 is a fixed point of F. On the other hand, if $y_0 \in C \setminus \overline{U}$, then

$$x_0 = f(y_0) = \frac{y_0}{p(y_0)}\,.$$

Now, for any $x \in \overline{U} \cap C$,

$$\begin{aligned} p(y_0 - x_0) &= p\left(y_0 - \frac{y_0}{p(y_0)}\right) = \left(\frac{p(y_0) - 1}{p(y_0)}\right) p(y_0) \\ &= p(y_0) - 1 \le p(y_0) - p(x) = p((y_0 - x) + x) - p(x) \\ &\le p(y_0 - x). \end{aligned}$$

Thus

$$p(y_0 - x_0) = \inf\{p(y_0 - z) : z \in \overline{U} \cap C\} = d_p(y_0, \overline{U} \cap C)\,.$$

Also $p(y_0 - x_0) > 0$ since $p(y_0 - x_0) = p(y_0) - 1$.
Let $z \in I_{\overline{U}}(x_0) \cap C \setminus (\overline{U} \cap C)$. Then there exists $y \in \overline{U}$ and $c \geq 1$ with $z = x_0 + c(y - x_0)$. Assume that

$$p(y_0 - z) < p(y_0 - x_0).$$

Since C is convex, $\frac{1}{c}z + (1 - \frac{1}{c})x_0 \in C$. Since $\frac{1}{c}z + (1 - \frac{1}{c})x_0 = y \in \overline{U}$, we have

$$\begin{aligned} p(y_0 - y) &= p[\frac{1}{c}(y_0 - z) + (1 - \frac{1}{c})(y_0 - x_0)] \\ &\leq \frac{1}{c}p(y_0 - z) + (1 - \frac{1}{c})p(y_0 - x_0) \\ &< p(y_0 - x_0). \end{aligned}$$

This contradicts the choice of y_0. Therefore,

$$p(y_0 - x_0) \leq p(y_0 - z) \text{ for all } z \in I_{\overline{U}}(x_0) \cap C\,.$$

The continuity of p gives that

$$p(y_0 - x_0) \leq p(y_0 - z) \text{ for all } z \in \overline{I_{\overline{U}}(x_0)} \cap C\,.$$

Consequently

$$0 < p(y_0 - x_0) = d_p(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C)\,.$$

If $x_0 \in U$, then $\overline{I_{\overline{U}}(x_0)} = E$. This implies that $d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C) = 0$. Hence $x_0 \in \partial_C(U)$. ■

Remark 3.1. Every Φ-condensing mapping F on C satisfies (3.1). To see this, let $A \subseteq C$ and $A \subseteq \overline{co}\,(\{0\} \cup F(co\,(\{0\} \cup A)))$. Then $\Phi(co\,(\{0\} \cup A)) = \Phi(A) \leq \Phi(F(co(\{0\} \cup A)))$. Since F is Φ-condensing, $\overline{co}\,(\{0\} \cup A)$ is compact. Consequently, $\overline{A}$ is compact.

Corollary 3.2. *Let E be a normed space. Suppose $s : B_R \to B_R$ is surjective and $F \in s\text{-}KKM(B_R, B_R, E)$ is a closed map satisfying*

(3.3) $\qquad A \subseteq B_R,\ A \subseteq \overline{co}\,(\{0\} \cup F(co\,(\{0\} \cup A)))$ *implies* $\overline{A}$ *is compact.*

Then there exist $x_0 \in B_R$ and $y_0 \in F(x_0)$ with

$$||y_0 - x_0|| = d(y_0, B_R) = d(y_0, \overline{I_{B_R}(x_0)})\,.$$

More precisely, either (i). F has a fixed point $x_0 \in B_R$, or (ii). there exist $x_0 \in \partial(B_R)$ and $y_0 \in F(x_0)$ with

$$0 < ||y_0 - x_0|| = d(y_0, B_R) = d(y_0, \overline{I_{B_R}(x_0)}).$$

Proof: Since $p(x) = \frac{||x||}{R}$ is the Minkowski functional on B_R, the result follows from Theorem 3.1. ■

Remark 3.2. Clearly every Φ-condensing mapping F on B_R satisfies (3.3). Thus Corollary 3.2 contains Corollary 3.4 of Shahzad [20]. It also extends Theorem 1 of Lin and Park [11] to the class s-KKM.

As an application of our approximation result, we have the following result.

Theorem 3.3. *Let C be a closed, convex subset of a Hausdorff locally convex space E with $0 \in C$ and U a convex open neighborhood of 0. Suppose C is a normal space, $s : \overline{U} \cap C \to \overline{U} \cap C$ is surjective and $F \in$ s-KKM($\overline{U} \cap C, \overline{U} \cap C, C$) is a closed map satisfying (3.1). If F satisfies any one of the following conditions for any $x \in \partial_C(U) \setminus F(x)$:*
(i). For each $y \in F(x)$, $p(y-z) < p(y-x)$ for some $z \in \overline{I_{\overline{U}}(x)} \cap C$;
(ii). For each $y \in F(x)$, there exists λ with $|\lambda| < 1$ such that $\lambda x + (1-\lambda)y \in \overline{I_{\overline{U}}(x)} \cap C$;
(iii). $F(x) \subseteq \overline{I_{\overline{U}}(x)} \cap C$;
(iv). $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;
(v). For each $y \in F(x)$, $p(y-x) \neq p(y) - 1$;
(vi). For each $y \in F(x)$, there exists $\alpha \in (1, \infty)$ such that $p^\alpha(y) - 1 \leq p^\alpha(y-x)$;
(vii). For each $y \in F(x)$, there exists $\beta \in (0,1)$ such that $p^\beta(y) - 1 \geq p^\beta(y-x)$,
then F has a fixed point.

Proof: Theorem 3.1 guarantees that either
(1). F has a fixed point in $\overline{U} \cap C$
or
(2). there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with $x_0 = f(y_0)$ such that

$$0 < p(y_0) - 1 \;=\; p(y_0 - x_0) \;=\; d_p(y_0, \overline{U} \cap C) \;=\; d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C),$$

where p is the Minkowski functional of U and f is the restriction of the continuous retraction r to C.

Suppose F satisfies condition (i). Assume (2) holds (with x_0 and y_0 as described above) and $x_0 \notin F(x_0)$. Then condition (i) implies that $p(y_0 - z) < p(y_0 - x_0)$ for some $z \in \overline{I_{\overline{U}}(x_0)} \cap C$. This contradicts the choice of x_0. Hence F has a fixed point in $\overline{U} \cap C$.

Suppose F satisfies condition (ii). Assume (2) holds (with x_0 and y_0 as described above) and $x_0 \notin F(x_0)$. Then, by condition (ii), there exists λ with $|\lambda| < 1$ such that $\lambda x_0 + (1-\lambda)y_0 \in \overline{I_{\overline{U}}(x_0)} \cap C$. Therefore

$$\begin{aligned} p(y_0 - x_0) &\leq p(y_0 - (\lambda x_0 + (1-\lambda)y_0)) = p(\lambda(y_0 - x_0)) \\ &= |\lambda| p(y_0 - x_0) < p(y_0 - x_0). \end{aligned}$$

This is impossible. Hence F has a fixed point in $\overline{U} \cap C$.

The proof for condition (iii) is clear.

Suppose F satisfies condition (iv). Assume (2) holds (with x_0 and y_0 as described above) and $x_0 \notin F(x_0)$. Then, by condition (iv), $\lambda x_0 \neq y_0$ for each $\lambda > 1$. But we have $x_0 = f(y_0) = \frac{y_0}{p(y_0)}$ and so $y_0 = \lambda_0 x_0$ with $\lambda_0 = p(y_0) > 1$. Hence F has a fixed point in $\overline{U} \cap C$.

Suppose F satisfies condition (v). Assume (2) holds (with x_0 and y_0 as described above) and $x_0 \notin F(x_0)$. Then, by condition (v), $p(y_0 - x_0) \neq p(y_0) - 1$. But $p(y_0 - x_0) = p(y_0) - 1$. Hence F has a fixed point in $\overline{U} \cap C$.

Suppose F satisfies condition (vi). Assume (2) holds (with x_0 and y_0 as described above) and $x_0 \notin F(x_0)$. Then, by condition (vi), there exists $\alpha \in (1, \infty)$ with $p^\alpha(y_0) - 1 \leq p^\alpha(y_0 - x_0)$. Let $\mu_0 = \frac{1}{p(y_0)}$. Then $\mu_0 \in (0,1)$ and

$$\begin{aligned} \frac{(p(y_0)-1)^\alpha}{p^\alpha(y_0)} &< 1 - \mu_0^\alpha \\ &\leq \frac{p^\alpha(y_0)-1}{p^\alpha(y_0)} \\ &\leq \frac{p^\alpha(y_0 - x_0)}{p^\alpha(y_0)}. \end{aligned}$$

Thus $p(y_0 - x_0) > p(y_0) - 1$. But $p(y_0 - x_0) = p(y_0) - 1$. Hence F has a fixed point in $\overline{U} \cap C$.

Finally suppose F satisfies condition (vii). Then, as above (see the proof of (vi)), it can be verified that F has a fixed point in $\overline{U} \cap C$. ∎

Remark 3.3. We have obtained a Leray-Schauder type result (see Theorem 3.3(iv)) as an application of Theorem 3.1.

Corollary 3.4. *Let E be a normed space. Suppose $s : B_R \to B_R$ is surjective and $F \in s\text{-}KKM(B_R, B_R, E)$ is a closed map satisfying* (3.3). *If F satisfies any one of the following conditions for any $x \in \partial(B_R) \setminus F(x)$:*
(i). For each $y \in F(x)$ $||y - z|| < ||y - x||$ for some $z \in \overline{I_{B_R}(x)}$;
(ii). For each $y \in F(x)$, there exists λ with $|\lambda| < 1$ such that $\lambda x + (1-\lambda)y \in \overline{I_{B_R}(x)}$;
(iii). $F(x) \subseteq \overline{I_{B_R}(x)}$;
(iv). $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;
(v). For each $y \in F(x)$, $||y - x|| \neq ||y|| - R$;
(vi). For each $y \in F(x)$, there exists $\alpha \in (1, \infty)$ such that $||y||^\alpha - R \leq ||y - x||^\alpha$;
(vii). For each $y \in F(x)$, there exists $\beta \in (0,1)$ such that $||y||^\beta - R \geq ||y - x||^\beta$, then F has a fixed point.

Remark 3.4. Corollary 3.4 extends Theorem 2 of Lin and Park [11] and Corollary 3.10 of Shahzad [20].

Remark 3.5. Let C be a nonempty subset of a Hausdorff locally convex space E and $c \geq 1$. A mapping $F : C \to 2^E$ is called pseudocondensing in the sense of Hahn [7] (see also [8]) provided that if A is any subset of C such that $\Phi(A) \leq c\Phi(F(A))$, then A is relatively compact in C; here Φ is the c-measure of noncompactness [7]. We note that every pseudocondensing map satisfies conditions (3.1) and (3.3).

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