Structure of commutative cancellative subarchimedean semigroups

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Abstract

A commutative semigroup S is subarchimedean if there is an element $z \in S$ such that for every $a \in S$ there exist a positive integer n and $x \in S$ such that $z^n = ax$. Such a semigroup is archimedean if this holds for all $z \in S$. A commutative cancellative idempotent-free archimedean semigroup is an $\mathfrak{N}$-semigroup. We study the structure of semigroups in the title as related to $\mathfrak{N}$-semigroups.

1 Introduction

The best structure theorem for a class of commutative cancellative semigroups is that due to Tamura [6] for idempotent-free archimedean semigroups, called $\mathfrak{N}$-semigroups. It amounts to a representation in the style of an abelian Schreier extension of the group of integers by an abelian group but replacing the former by the additive semigroup of nonnegative integers. As the archimedean property is weakened to subarchimedean and idempotent-freeness is dropped altogether, the same type of representation remains possible. Even though some steps were taken in this direction, see [1], the situation still had to be clarified. This includes the role of $\mathfrak{N}$-semigroups and their (left) translations as well as the role of the groups.

Some terminology and notation comprises Section 2. Section 3 contains a brief discussion of the semigroup of left translations of an $\mathfrak{N}$-semigroup. The main structure theorems can be found in Section 4 and the last Section 5 includes some complementary results.

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2 Notation and terminology

We generally follow the book [5]. *Throughout S denotes a commutative cancellative semigroup.* We shall thus omit these qualifiers, in particular all groups will be abelian. The identity of a monoid S will usually be denoted by e. We shall often use the following semigroups (under addition): $\mathbb{P}$ - positive integers, $\mathbb{N}$ -nonnegative integers. If ρ is an equivalence relation on a set X and $x \in X$, then $x\rho$ denotes the ρ-class of x.

On S the relation $\mathcal{N}$ defined by

$$a \,\mathcal{N}\, b \iff a^m = bx \text{ and } b^n = ay \text{ for some } m, n \in \mathbb{P} \text{ and } x, y \in S\,,$$

is the least semilattice congruence on S. Its classes are the *archimedean components* of S, *components* for short. The semigroup S is *archimedean* if it has only one component; *subarchimedean* if the semilattice $S/\mathcal{N}$ has a least element, the corresponding component is the *pivot* of S, denoted by Piv$S$. If it exists, the pivot of S is the unique component of S which is an ideal of S and

$$\text{Piv}S = \{a \in S \mid \text{for every } b \in S,\ a^n = bx \text{ for some } n \in \mathbb{P} \text{ and } x \in S\}.$$

An $\mathfrak{N}$-semigroup was defined in Section 1.

A transformation λ on S is a *left translation* of S if $(\lambda a)b = \lambda(ab)\quad (a, b \in S)$. For $a \in S$, define a transformation λ_a by $\lambda_a x = ax \quad (x \in S)$. Denote the set of all left translations of S by $\Lambda(S)$. The mapping π_S defined by

$$\pi_S : a \mapsto \lambda_a \qquad (a \in S)$$

is the *canonical homomorphism* of S into $\Lambda(S)$. Its image in $\Lambda(S)$ is denoted by $\Gamma(S)$. By commutativity and cancellation, we have that $\Lambda(S) \cong \Omega(S)$, the translational hull of S. For a complete discussion of this subject, see ([5], Section III.1).

3 Left translations of an $\mathfrak{N}$-semigroup

We consider here the semigroup of all left translations of an $\mathfrak{N}$-semigroup relative to the properties of its pivot. Here are some simple properties of $\Lambda(S)$ and $\Gamma(S)$.

Lemma 3.1. *Let S be a commutative cancellative semigroup.*

(i) $\Lambda(S)$ is a commutative cancellative semigroup.
(ii) For any $a \in S$ and $\lambda \in \Lambda(S)$, we have $\lambda\lambda_a = \lambda_{\lambda a}$.
(iii) $\Gamma(S)$ is an ideal of $\Lambda(S)$.
(iv) π_S is an isomorphism of S onto $\Gamma(S)$.

Proof. This requires straightforward verification which is omitted. Part (ii) is proved in ([5], Lemma III.1.6, part i)). Note that parts (i) and (ii) imply part (iii). ∎

We now explore briefly the mutual relationship of $\Gamma(S)$ and $\Lambda(S)$ for an $\mathfrak{N}$-semigroup S in the context of archimedean components. Let

$$\Psi(S) = \{\lambda \in \Lambda(S) \mid \lambda^n \in \Gamma(S) \text{ for some } n \in \mathbb{P}\}.$$

Lemma 3.2. *Let S be an $\mathfrak{N}$-semigroup. Then* $\mathrm{Piv}\Lambda(S) = \Psi(S)$.

Proof. Let $\lambda \in \mathrm{Piv}\Lambda(S)$. Then for any $a \in S$, there exist $n \in \mathbb{P}$ and $\lambda' \in \Lambda(S)$ such that $\lambda^n = \lambda_a \lambda'$. Since $\Gamma(S)$ is an ideal of $\Lambda(S)$, we get $\lambda^n \in \Gamma(S)$. Conversely, let $\lambda \in \Psi(S)$ so that $\lambda^n = \lambda_a$ for some $a \in S$ and $n \in \mathbb{P}$, and let $\lambda' \in \Lambda(S)$. Then $\lambda'\lambda_a \in \Gamma(S)$ and since $\Gamma(S)$ is archimedean by Lemma 3.1(iv), $\lambda_a^m = \lambda'\lambda_a\lambda_b$ for some $b \in S$. It follows that $\lambda^{nm} = \lambda'\lambda_{ab}$ which proves that $\lambda \in \mathrm{Piv}\Lambda(S)$. ∎

An $\mathfrak{N}$-semigroup S is said to be *steady* if for any $a, b \in S$, $aS \subseteq bS$ and $a^2 \in b^2S$ imply that $a \in bS$.

Proposition 3.3. *The following conditions on a $\mathfrak{N}$-semigroup S are equivalent.*
(i) S is steady.
(ii) $\Gamma(S) = \Psi(S)$.
(iii) $\Gamma(S)$ *is an archimedean component of* $\Lambda(S)$.
(*iv*) $\mathrm{Piv}\Lambda(S) = \Gamma(S)$.
In such a case,
(α) $\Lambda(S)$ *is under inclusion a maximal commutative cancellative semigroup having* $\Gamma(S)$ *as pivot,*
(β) if T is a commutative cancellative semigroup having S as a pivot, then the canonical homomorphism π_S extends uniquely to an embedding of T into $\Lambda(S)$.

Proof. (*i*) *is equivalent to* (*ii*). This is a consequence of ([3], Theorem 6).
(*ii*) *implies* (*iii*). This follows directly from Lemma 3.2.
(*iii*) *implies* (*iv*). It suffices to observe that by Lemma 3.1(iii), $\Gamma(S)$ is an ideal of $\Lambda(S)$.
(*iv*) *implies* (*ii*). This is immediate from Lemma 3.2.
Under the hypothesis that part (iv) holds, both (α) and (β) follow easily by applying ([5], Corollary III.5.15, Theorems III.5.9 and III.1.12, and Corollary III.5.5). ∎

Note that the identity mapping on S is the identity element of $\Lambda(S)$. Since $\Lambda(S)$ contains an ideal, namely $\Gamma(S)$, which is an $\mathfrak{N}$-semigroup, $\Lambda(S)$ is a commutative cancellative subarchimedean nongroup monoid even without the hypothesis that S be steady.

It is of some interest to introduce the following concept. A commutative cancellative semigroup S with pivot P is *p-maximal* (for pivot maximal) if for any commutative cancellative oversemigroup T of S with $\mathrm{Piv}T = P$, we have $T = S$. We shall now see that these semigroups are easily characterized.

Lemma 3.4. *Let S be a (commutative cancellative) semigroup with pivot P. Then S is p-maximal if and only if the function*

$$\chi : s \mapsto \lambda_s|_P \qquad (s \in S)$$

maps S onto $\Lambda(P)$.

Proof. For any semigroup T having P as its pivot, by ([5], Corollary III.5.18), T is commutative cancellative if and only if T is a dense extension of P. The assertion now follows by ([5], Corollary III.5.5). ∎

This leads to the following embedding.

Corollary 3.5. *Every commutative cancellative subarchimedean semigroup S with pivot P is embeddable into a p-maximal commutative cancellative semigroup with pivot P, in particular into $\Lambda(P)$.*

Proof. This follows easily from Lemma 3.4. ∎

If the pivot P of S is not a group, then it is an $\mathfrak{N}$-semigroup. Hence we have the following consequence.

Corollary 3.6. *For an $\mathfrak{N}$-semigroup P, any two p-maximal commutative semigroups with pivot P are P-isomorphic.*

4 Subarchimedean semigroups

Our structure theorems are based on a construction related to abelian Schreier extensions of abelian groups as adapted to semigroups by T. Tamura to describe $\mathfrak{N}$-semigroups. We offer here a somewhat modified version of this construction.

Let G be an abelian group and $I : G \times G \longrightarrow \mathbb{N}$ be a function which satisfies the axioms:

(A) $\quad I(a,b) + I(ab,c) = I(a,bc) + I(b,c) \qquad (a,b,c \in G),$
(C) $\quad I(a,b) = I(b,a) \qquad (a,b \in G),$
(N_i) $\quad I(e,e) = i \qquad (i = 0,1).$

On $\mathbb{N} \times G$ define a multiplication by

(M) $\quad (m,a)(n,b) = (m+n+I(a,b), ab).$

We denote the resulting groupoid by $\mathbb{N}_i(G,I)$ where $i = 0,1$.

The content of the following lemma was observed many times.

Lemma 4.1. *If S is idempotent-free, then $xy \neq x$ for all $x, y \in S$.*

In the statement of the next lemma, we exhibit the basic construction due to Tamura [6], see also [1], and will use it in the proof of Proposition 4.4 to suitably modify each step. For $z \in S$, let (z) denote the monogenic subsemigroup of S generated by the element z.

Lemma 4.2. *Let S be a subarchimedean idempotent-free semigroup and let $z \in \mathrm{Piv}S$. Define a relation ρ on S by*

$$a \;\rho\; b \;\Leftrightarrow\; z^m a = z^n b \text{ for some } m, n \in \mathbb{P}.$$

(i) The relation ρ is a group congruence on S with unit class equal to (z).

(ii) Every element of S admits a unique representation of the form $z^n p$ for some $n \in \mathbb{N}$ and $p \in S \setminus zS$.

(iii) For every $x \in S$, there exists a unique element $p_{x\rho}$ such that $x \;\rho\; p_{x\rho}$ and $p_{x\rho} \in S \setminus zS$.

(iv) For any $a, b \in G = S/\rho$ define $I(a,b)$ as the unique non-negative integer satisfying $p_a p_b = z^{I(a,b)} p_{ab}$. Then the function $I : G \times G \to \mathbb{N}$ satisfies (A), (C) and (N_1).

(v) The mapping χ defined by

$$\chi : (m, a) \mapsto z^m p_a \qquad ((m,a) \in \mathbb{N}_1(G,I))$$

is an isomorphism of $\mathbb{N}_1(G,I)$ onto S.

Proof. (i) Simple verification shows that ρ is a congruence. Letting $e = z\rho$, we get for any $x \in S$, $(x\rho)e = (xz)\rho = x\rho$ and e is the identity of S/ρ. Further, $z^n = xu$ for some $n \in \mathbb{P}$ and $u \in S$ which yields $e = (x\rho)(u\rho)$. Therefore S/ρ is an abelian group. If $x \,\rho\, z$, then $z^m x = z^n$ for some $m, n \in \mathbb{P}$; Lemma 4.1 implies that $m < n$ so that $x = z^{n-m}$. Hence $z\rho = (z)$.

(ii) Let $x \in S$. Then $z^k = xy$ for some $k \in \mathbb{P}$ and $y \in S$. Lemma 4.1 implies that $x \notin z^k S$. Hence there is a greatest $n \in \mathbb{N}$ for which $x = z^n p$ for some $p \in S \setminus zS$.

Assume that $z^m p = z^n q$ where $p, q \in S \setminus zS$. If $m < n$, then we get $p = z^{n-m} q \in zS$ contradicting the choice of p. Similarly $n < m$ is impossible. Thus $m = n$ whence also $p = q$, proving uniqueness.

(iii) By part (ii), for any $x \in S$, we have $x = z^n p$ for some $p \in S \setminus zS$. Hence $x\rho = p\rho$. If $q \in S \setminus zS$ is such that $p \;\rho\; q$, then $z^m p = z^k q$. Again by part (ii) we conclude that $m = n$ whence $p = q$.

(iv) For any $a, b \in G = S/\rho$, the existence and uniqueness of $I(a,b)$ are guaranteed by parts (ii) and (iii). If also $c \in G$, we get

$$\begin{aligned} (p_a p_b) p_c &= z^{I(a,b)} p_{ab} p_c = z^{I(a,b)} z^{I(ab,c)} p_{abc} = z^{I(a,b)+I(ab,c)} p_{abc}\,, \\ p_a (p_b p_c) &= p_a z^{I(b,c)} p_{bc} = z^{I(b,c)} z^{I(a,bc)} p_{abc} = z^{I(a,bc)+I(b,c)} p_{abc}, \end{aligned}$$

which by part (ii) implies condition (A). Condition (C) follows from commutativity. By Lemma 4.1, we have $z \notin zS$ so that $z = p_e$. Hence

$$z^2 = p_e p_e = z^{I(e,e)} p_{e^2} = z^{I(e,e)} z$$

and by cancellation, we get $I(e,e) = 1$. Therefore (N_1) holds as well.

(v) It follows from part (ii) that χ is bijective. For any $(m,a), (n,b) \in \mathbb{N}_1(G,I)$, we obtain

$$\begin{aligned} (m,a)\chi(n,b)\chi &= (z^m p_a)(z^n p_b) = z^m z^n \Big(z^{I(a,b)} p_{ab} \Big) = z^{m+n+I(a,b)} p_{ab} \\ &= (m + n + I(a,b), ab)\chi = \Big((m,a)(n,b) \Big) \chi\,. \end{aligned}$$

Therefore χ is an isomorphism. ∎

We are now able to prove the first statement needed in the general structure theorem cf. [6] and ([1], Section 3).

Proposition 4.3. *The groupoid $\mathbb{N}_1(G,I)$ is a commutative cancellative subarchimedean idempotent-free semigroup. Conversely, every semigroup with these properties is isomorphic to some $\mathbb{N}_1(G,I)$.*

Proof. Direct part. Associativity follows from axiom (A), commutativity from the commutativity of G and axiom (C), and cancellation from cancellation in G and $\mathbb{N}$ by straightforward verification. For any $(m,a) \in S$,

$$(0,e)^{m+I(a,a^{-1})+1} = (m + I(a,a^{-1}), e) = (m,a)(0,a^{-1})$$

and $(0,e)$ is a pivot element. Hence S is subarchimedean. If $(m,a)^2 = (m,a)$, then $2m + I(a,a) = m$ and $a^2 = a$ whence $m+1 = 0$ by (N_1) which is impossible. Hence S is idempotent-free.

Converse. This follows directly by Lemma 4.2 ∎

Next we characterize the groupoid $\mathbb{N}_0(G,I)$, cf. ([1], Section 3).

Proposition 4.4. *The groupoid* $\mathbb{N}_0(G,I)$ *is a commutative cancellative subarchimedean nongroup monoid. Conversely, every semigroup with these properties is isomorphic to some* $\mathbb{N}_0(G,I)$.

Proof. Direct part. Associativity follows from axiom (A), commutativity from the commutativity of G and axiom (C), and cancellation from cancellation in G and $\mathbb{N}$ by straightforward verification. For any $(m,a) \in S$,

$$(1,e)^{m+I(a,a^{-1})} = (m + I(a,a^{-1}), e) = (m,a)(0,a^{-1})$$

and $(1,e)$ is a pivot element. Thus S is subarchimedean. Clearly $(0,e)^2 = (0,e)$ and hence $(0,e)$ is the identity of $\mathbb{N}_0(G,I)$ and $(1,e)$ has no inverse.

Converse. We follow the steps of the proof of Lemma 4.2 with necessary changes. Let e be the idempotent of S.

(i) $zz = z^2e$ implies that $e \in z\rho$ so that $(z) \cup \{e\} \subseteq z\rho$. Let $x \,\rho\, z$. Then $z^m x = z^n$ for some $m,n \in \mathbb{P}$. If $m = n$, then $x = e$; if $m < n$, then $x = z^{m-n}$; if $m > n$, then $z^{m-n}x = e$ which is impossible since $z \in \mathrm{Piv}S$ and $e \notin \mathrm{Piv}S$ since S is not a group. In any case, we get $x \in (z) \cup \{e\}$. Therefore $z\rho = (z) \cup \{e\}$.

(ii) Let $x \in S$. Then $z^k = xy$ for some $y \in S$. If $x \in z^{k+1}S$, then z would be invertible which is impossible because z is in $\mathrm{Piv}S$ and S is not a group, hence $x = z^n p$ for a greatest $n \in \mathbb{N}$ and some $p \in S \setminus zS$.

(iii) The argument is the same as before.

(iv) In particular $p_e p_e = z^{I(e,e)} p_e$ where $p_e = e$ so that $e = z^{I(e,e)}$ and $I(e,e) = 0$, where e denotes the identity of both S and G.

(v) This is the same as before. ∎

We are finally ready for the structure theorem in which, exceptionally, S stands for a not necessarily commutative cancellative semigroup .

Theorem 4.5. *Let S be an arbitrary semigroup which is not a group. Then the following conditions are equivalent.*

(i) S is commutative cancellative and subarchimedean.

(ii) S is isomorphic to some $\mathbb{N}_0(G,I)$ *or* $\mathbb{N}_1(G,I)$.

(iii) There exists an $\mathfrak{N}$*-semigroup T and a semigroup S' such that* $\Gamma(T) \subseteq S' \subseteq \Lambda(T)$ *and* $S \cong S'$.

Proof. The equivalence of parts (*i*) and (*ii*) is a direct consequence of Propositions 4.3 and 4.4.

(i) implies (iii) Let $T = \text{Piv}S$. Clearly T is an $\mathfrak{N}$-semigroup. Since T is an ideal of S, the map

$$\chi: \ s \mapsto \lambda_s|_T \qquad (s \in S)$$

is a homomorphism of S into $\Lambda(T)$ whose restriction to T is the canonical homomorphism π_T. Cancellation in S immediately implies that χ is injective. Hence $S' = \chi S$ has the required property.

(iii) implies (i). As we observed in Lemma 3.1(i), $\Lambda(T)$ is a commutative cancellative semigroup, and in particular so is S' and thus also S. By Lemma 3.1(iv), the mapping π_T is an isomorphism, and since T is archimedean, we obtain that $\Gamma(T)$ is archimedean, and is thus contained in a component, say C, of $\Lambda(T)$. Since $\Gamma(T)$ is an ideal of $\Lambda(T)$, $\Lambda(T)$ cannot have a lower component than C which proves that C is the pivot of $\Lambda(T)$. Hence $C \cap S'$ must be the pivot of S'. Therefore S' is also subarchimedean and hence so is S. ∎

Part (iii) of the above theorem has the general form of a part of the Jacobson structure theorem for primitive rings with a nonzero socle, see ([2], p.75, Structure Theorem, part (3)). It is interesting to note that in this comparison $\Gamma(S)$ corresponds to the (nonzero) socle.

We now explore the natural relationship of the two classes of subarchimedean semigroups.

Lemma 4.6. *Let T be a subarchimedean semigroup and an ideal of S. Then S is subarchimedean.*

Proof. Straightforward. ∎

Proposition 4.7. *Let S be a subarchimedean nongroup. Then $\Lambda(S)$ is a subarchimedean commutative cancellative nongroup monoid.*

Proof. Lemma 3.1 provides the following information: by part (i), $\Lambda(S)$ is commutative and cancellative, by part (ii), $\Gamma(S)$ is an ideal of $\Lambda(S)$ and by part (iv), $S \cong \Gamma(S)$. Then $\Lambda(S)$ has an ideal which is subarchimedean. By Lemma 4.6, $\Lambda(S)$ is subarchimedean. The identity mapping ι_S is the identity element of $\Lambda(S)$ and is not contained in $\Gamma(S)$. Therefore $\Lambda(S)$ is a nongroup monoid. ∎

As a converse to the proposition, we have a more elaborate result.

If $I : G \times G \to \mathbb{N}$ is a function, let $(I + 1)(a, b) = I(a, b) + 1$ for all $a, b \in G$.

Theorem 4.8. *Let $S = \mathbb{N}_0(G, I)$ and set*

$$T = \{(m, a) \in S \mid m > 0\}.$$

(i) T is an $\mathfrak{N}$-semigroup and an ideal of S.

(ii) The mapping

$$\chi : (m, a) \mapsto (m-1, a) \qquad \big((m, a) \in T\big)$$

is an isomorphism of T onto $\mathbb{N}_1(G, I+1)$

(iii) The mappings

$$\begin{array}{ll} \varphi : x \mapsto \lambda_x|_T & (x \in S)\,, \\ \psi : \lambda \mapsto (m, a) & (\lambda \in \Lambda(T))\,, \end{array}$$

where $\lambda(1, e) = (m+1, a)$ and e is the identity of G, are mutually inverse isomorphisms between S and $\Lambda(T)$.

Proof. (i) Obviously T is an ideal of S and is idempotent-free. For any $(m, a), (n, b) \in T$ we have $(m, a)^p = (n, b)(k, c)$ for some $p \in \mathbb{P}$ and $(k, c) \in S$ since $T \subseteq \mathrm{Piv}S$. Hence $(m, a)^{p+1} = (n, b)(k', c')$, where $(k', c') = (k, c)(m, a)$ belongs to T, and thus T is archimedean.

(ii) Indeed, for $(m, a), (n, b) \in T$, we get

$$\begin{aligned} \chi(m, a)\chi(n, b) &= (m-1, a)(n-1, b) = (m+n+I(a, b)-1, ab) \\ &= \chi(m+n+I(a, b), ab) = \chi\big((m, a)(n, b)\big) \end{aligned}$$

and χ is a homomorphism. Clearly χ is a bijection between the sets $\mathbb{P} \times G$ and $\mathbb{N} \times G$.

(iii) Clearly φ maps S into $\Lambda(S)$, and for $x, y \in S$, we have

$$(\varphi x)(\varphi y) = (\lambda_x|_T)(\lambda_y|_T) = (\lambda_x\lambda_y)|_T = \lambda_{xy}|_T = \varphi(xy)$$

and φ is a homomorphism. Obviously ψ maps $\Lambda(T)$ into S.

For $(m, a) \in S$, we obtain

$$\lambda_{(m,a)}|_T(1, e) = (m, a)(1, e) = (m+1, a)$$

and thus $\psi\varphi(m, a) = (m, a)$ so that $\psi\varphi = \iota_S$, the identity mapping on S. For $\lambda \in \Lambda(T)$, we first have $\lambda(1, e) = (m+1, a)$ for some $m \in \mathbb{N}$ and $a \in G$. Hence $(m, a) \in S$. For $(n, b) \in T$, we get

$$\begin{array}{rclcl} [\lambda(n, b)](1, e) & = & [\lambda(1, e)](n, b) & = & (m+1, a)(n, b) \\ & = & (m+1+n+I(a, b), ab) & = & (m+n+I(a, b), ab)(1, e) \end{array}$$

so that

$$\lambda(n, b) = (m+n+I(a, b), ab) = (m, a)(n, b)$$

and thus $\lambda = \lambda_{(m,a)}|_T$. Therefore

$$\varphi\psi\lambda = \phi(m, a) = \lambda_{(m,a)}|_T = \lambda$$

which proves that $\varphi\psi = \iota_{\Lambda(T)}$, the identity mapping on $\Lambda(T)$. ∎

Proposition 4.7 indicates that if S is idempotent-free and subarchimedean, then $\Lambda(S)$ is a subarchimedean nongroup monoid (this proposition is trivial if S is a monoid). Theorem 4.8 implies that if S is a subarchimedean monoid, then an ideal T of S has the properties: T is idempotent-free, subarchimedean and $\Lambda(T) \cong S$. This establishes a close relationship between these two classes of subarchimedean semigroups: idempotent-free and nongroup monoids. The latter are, up to isomorphism, precisely $\Lambda(T)$ for T belonging to the former.

5 Supplements

We shall now characterize the semigroups in Theorem 4.5 in terms of the Malcev product. To this end, we need the following concepts.

Let ρ be a congruence on S and $\mathcal{B}$ a class of semigroups. If $S/\rho \in \mathcal{B}$, we say that ρ is a *$\mathcal{B}$-congruence*. If every idempotent ρ-class belongs to $\mathcal{B}$, we say that ρ is *over* $\mathcal{B}$.

We introduce the following notation:

$\mathcal{N}_0$ (respectively, $\mathcal{N}_1$)- infinite cyclic monoids (respectively, semigroups), $\mathcal{N}_2 = \mathcal{N}_0 \cup \mathcal{N}_1$;

$\mathcal{P}_0$ (respectively, $\mathcal{P}_1$)- commutative cancellative subarchimedean nongroup monoids (respectively, idempotent-free semigroups), $\mathcal{P}_2 = \mathcal{P}_0 \cup \mathcal{P}_1$;

$\mathcal{A}$- abelian groups, $\mathcal{C}$- commutative cancellative semigroups.

The Malcev product of $\mathcal{N}_i$ and $\mathcal{A}$ in the class $\mathcal{C}$ is defined by

$$(\mathcal{N}_i \circ \mathcal{A})_{\mathcal{C}} = \{S \in \mathcal{C} \mid S \text{ has a group congruence over } \mathcal{N}_i\}, \ i = 0, 1, 2.$$

Proposition 5.1. *For $i = 0, 1, 2$, we have $(\mathcal{N}_i \circ \mathcal{A})_{\mathcal{C}} = \mathcal{P}_i$.*

Proof. We treat only the case $i = 1$; the proof for $i = 0$ follows along the same lines with obvious modifications. The case $i = 2$ follows immediately from cases $i = 0, 1$.

Let $S \in (\mathcal{N}_1 \circ \mathcal{A})_{\mathcal{C}}$ and ρ be a congruence on S over $\mathcal{N}_1$ and $S/\rho \in \mathcal{A}$. Let $\varphi : S \to S/\rho$ be the natural epimorphism, and set $A = \varphi^{-1}e$ where e is the identity of $G = S/\rho$. Then A is the only ρ-class which is a subsemigroup of S and, by hypothesis, A is an infinite cyclic semigroup. Therefore S is idempotent-free.

Let a be the generator of A and $b \in S$. Then $\varphi a = e$ and φb has an inverse, say φc. Hence $\varphi a = (\varphi b)(\varphi c)$ so that $a \ \rho \ bc$. Then $bc = a^n$ for some $n \in \mathbb{P}$. Therefore $a \in \mathrm{Piv}S$ and thus S is subarchimedean.

Consequently $S \in \mathcal{P}_1$. We have proved that $(\mathcal{N}_1 \circ \mathcal{A})_{\mathcal{C}} \subseteq \mathcal{P}_1$.

For the opposite inclusion, let $S \in \mathcal{P}_1$. By Theorem 4.5, $S \cong \mathbb{N}_1(G, I)$ for some $G \in \mathcal{A}$ and suitable function $I : G \times G \to \mathbb{N}$. The projection $\pi : (m, a) \mapsto a$ is a homomorphism of $\mathbb{N}_1(G, I)$ onto G. The mapping $(m, e) \mapsto m+1$ is an isomorphism of $\pi^{-1}e$ onto $\mathbb{P}$ where e is the identity of G. Therefore the congruence induced on $\mathbb{N}_1(G, I)$ by π has all the requisite properties which implies that $\mathbb{N}_1(G, I) \in (\mathcal{N}_1 \circ \mathcal{A})_{\mathcal{C}}$ and thus by isomorphism, also $S \in (\mathcal{N}_1 \circ \mathcal{A})_{\mathcal{C}}$. ∎

Proposition 5.2. *The class of commutative (sub)archimedean semigroups is closed under finite direct products.*

Proof. Let $S_1, \cdots, S_n$ be commutative (sub)archimedean semigroups and $S = \prod_{i=1}^{n} S_i$. Let $(a_i), (b_i) \in S$ (where $a_i \in \mathrm{Piv}S_i$ for $i = 1, \cdots, n$). Then $a_i^{m_i} = b_i c_i$ for some $m_i \in \mathbb{P}$ and $c_i \in S_i$, $i = 1, \cdots, n$. Letting $m = m_1 \cdots m_n$, we get

$$(a_i)^m = (a_i^m) = \Big((a_i^{m_i})^{\frac{m}{m_i}}\Big) = \Big(b_i^{\frac{m}{m_i}}\Big)\Big(c_i^{\frac{m}{m_i}}\Big) = (b_i)\Big(b_i^{\frac{m}{m_i}-1} c_i^{\frac{m}{m_i}}\Big)$$

and S is (sub)archimedean. ∎

The classes $\mathcal{P}_i$ have very restricted closure properties. It follows from Lemma 5.2 that for $i = 0, 1, 2$, $\mathcal{P}_i$ is closed under finite direct products. If I is an ideal of a commutative subarchimedean semigroup T, then a simple argument shows that $\mathrm{Piv}S \cap I = \mathrm{Piv}I$. As a consequence, we have that $\mathcal{P}_1$ and $\mathcal{P}_2$ are closed for taking ideals; obviously $\mathcal{P}_0$ is not.

Proposition 5.3.
(i) For $i = 0$ or $i = 1$ and $S = \mathbb{N}_i(G, I)$, the pivot of S equals

$$\mathrm{Piv}S = \{(m, a) \in S \mid \textit{either } m > 0 \textit{ or } \; I(a, a^n) > 0 \;\; \textit{for some } n \in \mathbb{P}\} .$$

(ii) For $S = \mathbb{N}_0(G, I)$, the identity element is $(0, e)$ and the group of units

$$U(S) = \{(0, a) \in S \mid I(a, a^{-1}) = 0\} .$$

Proof. (i) We treat the case $i = 0$; the case $i = 1$ requires a similar, but not identical, argument.

Assume that $I(a, a^n) = 0$ for all $n \in \mathbb{P}$. Then $(0, a)^n = (0, a^n)$ for all $n \in \mathbb{P}$. Hence an equality of the form $(0, a)^n = (1, e)(m, x)$ is not possible in S and hence $(0, a) \notin \mathrm{Piv}S$.

For the opposite inclusion, if $m > 0$, then the equality $(m, a) = (1, e)(m-1, a)$ in conjunction with $(1, e) \in \mathrm{Piv}S$, proved in the direct part of Proposition 4.4, implies that $(m, a) \in \mathrm{Piv}S$. We consider next the element $(0, a)$ where $I(a, a^n) > 0$ for some $n \in \mathbb{P}$. Then

$$(0, a)^{n+1} = (\sum_{j=1}^{n} I(a, a^j), a^{n+1})$$

where $\sum_{j=1}^{n} I(a, a^j) > 0$ so that $(0, a)^{n+1} \in \mathrm{Piv}S$ by the first part of this paragraph. But then $(0, a) \in \mathrm{Piv}S$.

(ii) The simple verification is omitted. ■

This clarifies somewhat the "size" of $\mathrm{Piv}S$ and $U(S)$ within S.

References

[1] H. B. Hamilton, T.E. Nordahl and T. Tamura, Commutative cancellative semigroups without idempotents, Pacific J. Math. **61** (1975), 441-456.

[2] N. Jacobson, Structure of rings, Amer. Math. Soc. Coll. Publ. Vol. **37** (1964), Providence (2^{nd} edition).

[3] M. Petrich, Normal bands of commutative cancellative semigroups, Duke Math. J. **40** (1973), 17-32.

[4] M. Petrich, L'enveloppe de translations d'un demi-treillis de groupes, Canad. J. Math. **25** (1973), 164-177.

[5] M. Petrich, Introduction to semigroups, Merrill, Columbus, 1973.

[6] T. Tamura, Commutative nonpotent archimedean semigroups with cancellative law I, J. Gakugei, Tokushima Univ. **8** (1957), 5-11.

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