On Müntz Theorem for countable compact sets

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Abstract

In this note we solve the Müntz problem for the space $\mathbf{C}(K)$ whenever $K \subset [0, \infty)$ is a countable compact set which satisfies certain additional assumptions and we propose the general case as an open question.

Every introductory course in Approximation Theory should contain at least one proof of the following classical result, which is a nice generalization of the Weierstrass approximation theorem:

Theorem 1 (Müntz, 1914). *Let $\Lambda = \{\lambda_k\}_{k=0}^{\infty}$ be a strictly increasing sequence of real numbers with $\lambda_0 = 0$ and let us denote by*

$$\Pi(\Lambda) = \mathbf{span}\{x^{\lambda_i}\}_{i=0}^{\infty},$$

the space of polynomials with exponents in Λ. Then the following claims are equivalent:
i) $\Pi(\Lambda)$ is a dense subset of $C[0,1]$;
ii) $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty$.

The result was conjectured by Bernstein in 1912 and proved by Müntz in 1914. In 1916, Szász extended the Müntz Theorem in the sense that he was able to prove it also for certain special sequences of complex numbers $\{\lambda_i\}_{i=0}^{\infty}$ as exponents (see [6]). Furthermore, he simplified the final step of the proof, where it is shown that the result in $\mathbf{L}^2(0,1)$ implies the same result in $\mathbf{C}[0,1]$. Since then, many extensions and generalizations have appeared. In particular, for compact intervals $[a,b] \subset (0,\infty)$ the following result that we call the Full Müntz Theorem for compact intervals away from the origin, is classic (see [2],[1]):

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Theorem 2 (Schwartz, Clarkson, Erdös). *Let us assume that* $0 < a < b < \infty$ *and* $\Lambda = \{\lambda_k\}_{k=0}^{\infty} \subset \mathbb{R}$. *Then the following claims are equivalent:*
i) $\Pi(\Lambda)$ *is dense in* $C[a,b]$*;*
ii) $\sum_{k\in\{j:\lambda_j\neq 0\}} \frac{1}{|\lambda_k|} = \infty$.

On the other hand, in 1996 Borwein and Erdelyi published the following result (see [3, Theor. 2.1]),

Theorem 3 (Full Müntz theorem for $\mathbf{C}(0,1)$). *Let* $\Lambda = \{0\} \cup \{\lambda_k\}_{k=0}^{\infty}$*, where* $\lambda_k > 0$ *for all* k*. Then the following claims are equivalent:*
i) $\Pi(\Lambda)$ *is dense in* $C[0,1]$*;*
ii) $\sum_{k=0}^{\infty} \frac{\lambda_k}{\lambda_k^2+1} = \infty$.

Later, Borwein and Erdelyi (see [4]) extended their results to the spaces $\mathbf{L}^p(A)$ and $\mathbf{C}(A)$ for sets A which are Lebesgue measurable with positive measure $|A| > 0$. It is quite surprising that the problem is still open for countable compact sets and also for uncountable sets with Lebesgue measure equal to zero. In this note we are interested in a full Müntz theorem for countable compact sets. We prove that in many cases the Müntz condition can be weakened in a sensible way. More precisely, we obtain the following result.

Theorem 4. *Let* $K \subset [0,\infty)$ *be a countable compact set and let* $\Lambda = \{\lambda_k\}_{k=0}^{\infty} \subset \mathbb{R}$ *be a fixed sequence of exponents, satisfying* $\lambda_0 = 0$*. Then the following holds:*
i) If $\Lambda \subset [0,\infty)$ *is an infinite bounded sequence and* $K \setminus \{0\}$ *is compact then* $\Pi(\Lambda)$ *is dense in* $\mathbf{C}(K)$*.*
ii) If $\Lambda \subset [0,\infty)$ *and* K *does not contain strictly increasing sequences then* $\Pi(\Lambda)$ *is dense in* $\mathbf{C}(K)$ *if and only if* $\#\Lambda = \infty$*. Moreover, if* $\Lambda \subset (-\infty, 0]$ *and* K *does not contain strictly decreasing sequences then* $\Pi(\Lambda)$ *is dense in* $\mathbf{C}(K)$ *if and only if* $\#\Lambda = \infty$*.*

Proof. The main idea in the proof of this result is to use the Riesz representation theorem. Clearly, the unique measures that exist for countable compact sets are atomic. Thus, if $K = \{0\} \cup \{t_i\}_{i=1}^{\infty}$ then $L \in \mathbf{C}(K)'$ if and only if $L(f) = \alpha_0 f(0) + \sum_{i=1}^{\infty} \alpha_i f(t_i)$ for a certain sequence $\{\alpha_i\}_{i=0}^{\infty}$ such that $\sum_{i=0}^{\infty} |\alpha_i| < \infty$. Thus, as a consequence of the Hahn-Banach Theorem, $\mathbf{span}\{x^{\lambda_k}\}_{k=0}^{\infty}$ is dense in $\mathbf{C}(K)$ if and only if the following holds true: if

$$\sum_{i=0}^{\infty} \alpha_i = 0;\ \sum_{i=1}^{\infty} \alpha_i t_i^{\lambda_k} = 0,\ k = 1, 2, \cdots;\ \text{and}\ \sum_{i=0}^{\infty} |\alpha_i| < \infty$$

then $\alpha_i = 0$ for all $i \geq 0$.

Thus, let us assume that $\sum_{i=0}^{\infty} \alpha_i = 0$, $\sum_{i=1}^{\infty} \alpha_i t_i^{\lambda_k} = 0,\ k = 1, \cdots$; and $\sum_{i=0}^{\infty} |\alpha_i| < \infty$. Then we set $\Gamma = \{t_i : \alpha_i \neq 0\}$ and we take $\gamma = \sup \Gamma$. Clearly, $\gamma \in K$ since K is compact. If $\Gamma = \emptyset$ then $L(f) = \alpha_0 f(0)$ and $L(1) = 0$ implies $\alpha_0 = 0$, which ends the proof. If $\Gamma \neq \emptyset$ then $\gamma > 0$ and there exists $t_s \in K$ such that $\gamma = t_s$. Then we take $t_a \in K$ such that $t_a < t_s$ and we set $z_\lambda = (t_a/t_s)^\lambda$. Clearly, the equation $z_\lambda^{p_j} = (t_j/t_s)^\lambda$ is uniquely solved by $p_j = \frac{\ln(t_j/t_s)}{\ln(t_a/t_s)}$, which is a positive real number for all $j \neq s$. Hence $L(x^{\lambda_k}) = 0,\ k = 0, 1, 2, \cdots$ can be written in the following equivalent way:

$$0=\sum_{i=0}^{\infty}\alpha_i \text{ and } 0=(t_s)^{\lambda_k}\sum_{t_i\in\Gamma}\alpha_i\,(t_i/t_s)^{\lambda_k}\,,\ k=1,2,\cdots$$

Hence $\varphi(z_{\lambda_k})=0$ for all $k\geq 1$, where

$$\varphi(z)=\sum_{t_i\in\Gamma}\alpha_i z^{p_i}$$

We decompose the proof in several steps, according to the boundedness properties of the set of exponents Λ.

Step 1. $\Lambda\subset[0,\infty)$ and $\lim_{k\to\infty}\lambda_k=\infty$ and K does not contain strictly increasing sequences.

Under these conditions it is clear that $t_s\in\Gamma$ and $\lim_{k\to\infty}z_{\lambda_k}=0$. Thus $\varphi(0)=\lim_{k\to\infty}\varphi(z_{\lambda_k})=0$ since $\varphi(z)$ is continuous at the origin. On the other hand, $t_s\in\Gamma$ implies that $\alpha_s\neq 0$. Hence we can use that $\varphi(z)=\sum_{t_i\in\Gamma\setminus\{t_s\}}\alpha_i z^{p_i}+\alpha_s$ (since $p_s=\frac{\ln 1}{\ln(t_a/t_s)}=0$) to claim that $\varphi(0)=\alpha_s\neq 0$, a contradiction.

Step 2. $\Lambda=\{\lambda_k\}_{k=0}^{\infty}\subset[0,\infty)$ is bounded and $K\setminus\{0\}$ is compact.

Clearly, we can assume without loss of generality that Λ is itself a convergent sequence. We note that $\varphi(z)=\sum_{t_i\in\Gamma}\alpha_i z^{p_i}$ is analytic in the open set $\Omega=\{z:|z|<1,|1-z|<1\}$. If $\lim_{k\to\infty}\lambda_k=\lambda^*\neq 0$ then $\lim_{k\to\infty}z_{\lambda_k}=z_{\lambda^*}\in(0,1)\subset\Omega$. Hence $\varphi(z)$ vanishes on a set with accumulation points inside Ω, so that $\varphi(z)$ vanishes identically on Ω and $\alpha_i=0$ for all $i>0$. If $0\notin K$ the proof ends. On the other hand, if $0\in K$ then $0=L(1)=\sum_{t_i\in\Gamma}\alpha_i+\alpha_0=\alpha_0$ and the proof also ends. What happens if $\lim_{k\to\infty}\lambda_k=0$? Then we can use the following trick: the equations

$$0=\sum_{t_i\in\Gamma}\alpha_i t_i^{\lambda_k};\ k=1,\cdots$$

can be rewritten as

$$0=\sum_{t_i\in\Gamma}\beta_i t_i^{\lambda_k^*};\ k=1,\cdots,$$

where $\beta_i=\alpha_i/t_i$ for all i and $\lambda_k^*=\lambda_k+1$ for all k (take into account that $\sum_{t_i\in\Gamma}|\beta_i|<\infty$ since $K\setminus\{0\}$ is compact). Thus $\lim_{k\to\infty}\lambda_k^*=1$ and we conclude that $\alpha_j/t_j=0$ for all j. The proof follows.

Step 3. $\Lambda\subset\mathbb{R}$ and $K\setminus\{0\}$ is compact.

Clearly, if Λ is an infinite set then it contains either infinitely many positive elements or infinitely many negative elements. Thus, we may assume that either $\Lambda\subset[0,\infty)$ or $\Lambda\subset(-\infty,0]$. The first case has been already studied in Steps 1 and 2. Thus, let us assume that $\Lambda\subset(-\infty,0]$ and $L(f)=\alpha_0 f(0)+\sum_{j=1}^{\infty}\alpha_j f(t_j)$, $L\in\mathbf{C}(K)'$. Then the equations $L(x^{\lambda_k})=0$, $k=0,1,\cdots$ can be rewritten as

$$\sum_{i=0}^{\infty}\alpha_i=0 \text{ and } \sum_{i=1}^{\infty}\alpha_i\left(\frac{1}{t_i}\right)^{-\lambda_k}=0,\ k=1,2,\cdots.$$

This means that the functional defined by

$$S(f)=\alpha_0 f(0)+\sum_{j=1}^{\infty}\alpha_j f(\frac{1}{t_j}),$$

which belongs to $\mathbf{C}(E)'$, where $E = \{0\} \cup \{\frac{1}{t_j}\}_{j=1}^{\infty}$, which is a countable compact subset of $[0,\infty)$ since $K \setminus \{0\}$ is compact; satisfies $S(x^{-\lambda_k}) = 0$ for all $k \geq 0$. Moreover, if K does not contain decreasing sequences then E does not contain increasing sequences. Now, we use the results proved in Steps 1 and 2 to conclude that $\alpha_i = 0$ for all i. This ends the proof. ∎

Remark. Another proof of step 2: Taking into consideration that $t_i^{\lambda_k} = \exp(\lambda_k \log t_i)$ for all i, we have that the relations

$$\sum_{t_i \in \Gamma} \alpha_i t_i^{\lambda_k} = 0; k = 1, 2, \cdots$$

are equivalent to the relations

$$\Psi(\lambda_k) = 0; k = 1, 2, \cdots;$$

where

$$\Psi(z) = \sum_{t_i \in \Gamma} \alpha_i \exp((\log t_i) z)$$

is an entire function of exponential type. This means, in particular, that Λ cannot be an infinite bounded sequence (otherwise, Ψ should vanish everywhere). This ends the proof. ∎

Remark. Clearly, if Λ is bounded then $|\lambda_k|^{-1} \geq 1/\sup \Lambda$ for all $\lambda_k \neq 0$. Hence $\sum_{k=1}^{\infty} |\lambda_k|^{-1} = \infty$ and case i) of Theorem 3 follows from the theorem by Clarckson, Erdös and Schwartz (Theorem 1) whenever $0 \notin K$. Now, this proof uses a very difficult result in order to prove a simpler one. This is the reason we give our own elementary proof of this fact.

Remark. There are many countable compact sets with the property that they do not have (strictly) increasing sequences. An interesting example is given by:

$$K = \{0\} \cup \{1/n\}_{n=1}^{\infty} \cup \{1/n + 1/m\}_{n,m=1}^{\infty}$$

Obviously, this compact set has infinitely many accumulation points but has no increasing sequence! These cases are covered by Theorem 3 above.

Open question. We have already shown that in order to give a full Müntz theorem for the general case (i.e., for arbitrary countable compact sets $K \subset [0,\infty)$), it is a good idea to study the zero sets of the Müntz type series:

$$\varphi(z) = \sum_{i=1}^{\infty} \alpha_j z^{p_j}; \text{ where } \{p_j\}_{j=1}^{\infty} \text{ decreases to zero and } \sum_{j=1}^{\infty} |\alpha_j| < \infty.$$

and, in the case that $K \setminus \{0\} = \{t_i\}_{i=1}^{\infty}$ is compact, the zero sets of the integral functions of exponential type given by

$$\Psi(z) = \sum_{t_i \in \Gamma} \alpha_i \exp((\log t_i) z) \text{ where } \sum_{j=1}^{\infty} |\alpha_j| < \infty.$$

Is it possible to find a series $\varphi(z)$ with a sequence of infinitely many zeros $\{z_k\}_{k=0}^{\infty}$ which converges to zero? What about a function $\Psi(z)$ with infinitely many zeros?. These questions seem to be still open and not easy.

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