Automorphisms of Unitals

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Abstract

It is shown that every automorphism of a classical unital over a certain (not necessarily commutative) field is induced by a semi-similitude of a corresponding hermitian form. In particular, this is true if the form uses an involution of the second kind.

In [16], J. Tits has studied classical unitals, defined by suitable polarities. Using the Borel–Tits Theorem [2], he determines the full group of automorphisms in the special case where the ground field is commutative and infinite. The finite case has been treated by M.E. O'Nan [12]. In [16], it is also claimed that the result is extendable (presumably, within the limitations imposed by the machinery used in [2]: for instance, one would at least require that the ground field has finite dimension over its center). However, no precise statement has been published up to now.

In a recent investigation [13] into non-classical unitals in translation planes obtained via modification of the projective plane over Hamilton's quaternions, complete information about automorphisms of unitals over the quaternions is required, in order to distinguish the unitals in question from the classical one.

We generalize the results of [16], using "elementary methods" in the sense of Dieudonné's review [6]: we treat the groups in question as classical groups rather than as algebraic groups. For many ground fields, we show that a distinguished subgroup of the automorphism group of the unital contains all unitary reflections, and that the set of reflections is invariant in the group of all automorphisms of the unital. See Sections 3 and 5.

The reflections are then used (in Section 6) to reconstruct the ambient projective plane, leading to a determination of the full group of automorphisms under some technical assumptions. For instance, this is possible in the cases where an involution

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of the second kind was used (in particular, if the ground field is commutative), where no interior points exist, or where every interior line contains only exterior points. Prominent examples are given by the unitals over Hamilton's quaternions. The more restrictive assumptions are due to the fact that one has to exclude the existence of reflections that may be confused with Baer involutions, in the sense of 2.2.

Section 7 contains an extension of our results on planar unitals to higher dimensions.

Apart from the obvious geometric relevance of our results, we point out a group-theoretic application: information about $\mathrm{Aut}\,(U,B)$ can be translated into information about automorphisms of a certain simple algebraic group of rank 1, see 4.6 and 4.7. In fact, the classical unitals may be interpreted in terms of buildings of rank one, corresponding to simple algebraic groups of rank one. However, the system of blocks is not encoded in the building alone (because the rank is too small) but in the group (more precisely, in a certain nilpotent normal subgroup of a Borel group). The present paper contributes to J. Tits' program (as outlined in [17]) to characterize algebraic and classical groups of rank one with non-abelian unipotent subgroups, in terms of their action on geometries that generalize the unitals that we discuss here. Even more generally, one considers so-called Moufang sets, see [17].

1 Hermitian Forms, Unitals, and Unitary Groups.

Let K be a (not necessarily commutative) field with $\mathrm{char}\,K \neq 2$, and let $\sigma : x \mapsto \overline{x}$ be an involutory anti-automorphism of K. Note that we exclude the case $\sigma = \mathrm{id}$. The sets of symmetric and of skew-symmetric elements (i.e., of fixed points of σ and $-\sigma$) will be denoted by K^+ and K^-, respectively. For $x = (x_0, x_1, x_2)$ and $y = (y_0, y_1, y_2) \in K^3$, we define $\langle x|y\rangle := x_0\,\overline{y_2} + x_1\,\overline{y_1} + x_2\,\overline{y_0}$, and obtain a hermitian form $\langle\cdot|\cdot\rangle : K^3 \times K^3 \to K$. This form describes a polarity π, with $U := \left\{Kx \mid x \in K^3 \smallsetminus \{0\}, \langle x|x\rangle = 0\right\}$ as the set of *absolute points*. The traces of projective lines meeting U in more than one point will be called *blocks*, and we write B for the set of these blocks. Background information about projective spaces, their automorphisms and polarities may be found in [4] Ch. I or [8] Ch. II.

1.1 Remark. Let β be a non-degenerate σ-hermitian form on a 3-dimensional left vector space over K. If there exists a nontrivial vector v with $\beta(v,v) = 0$ then it is easy to find a basis $b_0 := v, b_1, b_2$ such that the coordinate description for β is of the form $\langle x|y\rangle_d := x_0\,\overline{y_2} + x_1 d\,\overline{y_1} + x_2\,\overline{y_0}$, with $d \in K^+ \smallsetminus \{0\}$.

We may replace the form β by a scalar multiple βs, with $s \in K^+ \smallsetminus \{0\}$. This does not change the unital, but replaces σ by $\sigma\iota_s$, where $\iota_s : h \mapsto s^{-1}hs$ is the inner automorphism. Note that $\sigma\iota_s$ is an involution because $s = \overline{s}$. The scalar d is then replaced by ds, and $d = \overline{d} \neq 0$ implies that we may restrict our attention to the case $d = 1$.

1.2 Remark. A large part of the literature deals with skew-hermitian forms rather than with hermitian ones. Replacing β by βp, with $p \in K^- \smallsetminus \{0\}$, we may pass from a σ-hermitian form to a $\tilde\sigma$-skew-hermitian one, where $\tilde\sigma$ maps x to $p^{-1}\,\overline{x}\,p$. Neither the unitary group nor the unital are changed by this modification, but one has to be careful with explicit assertions about the size or structure of K^-.

Let us briefly recall the (semi-)linearly induced collineations that induce automorphisms of the *unital* (U, B): one has the group (cf. [4] I §9)

$$\Gamma\mathrm{U}\,(\sigma) := \left\{\gamma \in \Gamma\mathrm{L}_3 K \,\middle|\, \exists\, r_\gamma \in K\ \forall\, x, y \in K^3 : \langle x\gamma | y\gamma\rangle = \langle x|y\rangle^{\sigma_\gamma}\, r_\gamma\right\}$$

of *semi-similitudes*, where $\sigma_\gamma \in \mathrm{Aut}\,(K)$ is the field automorphism associated with γ. The conditions $\sigma_\gamma = \mathrm{id}$ and $r_\gamma = 1$ single out the *unitary group* $\mathrm{U}\,(\sigma)$. The groups induced on the projective plane over K will be called $\mathrm{P}\Gamma\mathrm{U}\,(\sigma)$ and $\mathrm{PU}\,(\sigma)$, respectively.

For a point X, there are three possibilities: there may be exactly one absolute line through X (then $X \in U$), more than one absolute line through X (such a point is called an *exterior point*), or no absolute lines through X (such a point is called an *interior point*). Dually, we have the notions of exterior and interior lines. Note that an interior line is a passing line, having empty intersection with U. Exterior lines are also called *secants*. There are many examples of polarities where no interior points exist.

We use affine coordinates for the complement of the image of $\infty := K(0,0,1)$ under the polarity, writing $(x, y) \in K^2$ for $K(1, x, y)$, and $[s, t]$ for the line $\left\{(x, xs + t) \mid x \in K\right\}$. The *vertical line* through (x, y) is $[x] := \left\{(x, h) \mid h \in K\right\}$. The *affine part* of U is $A := U \smallsetminus \{0\} = \left\{(x, p - \frac{x\overline{x}}{2}) \,\middle|\, x \in K, p \in K^-\right\}$.

1.3 Lemma. **a.** *The group* $\mathrm{PU}\,(\sigma)$ *acts two-transitively on* U*, and thus transitively on the set of secants.*

b. *The set* $\Xi := \left\{\xi_{x,p} : (u, v) \mapsto (u + x, v + p - \frac{x\overline{x}}{2} - u\,\overline{x}) \,\middle|\, x \in K, p \in K^-\right\}$ *forms a subgroup of* $\mathrm{PU}\,(\sigma)$ *that acts sharply transitively on the affine part of the unital.*

c. *In affine coordinates, the stabilizer of* $(0, 0)$ *and* ∞ *in* $\mathrm{PU}\,(\sigma)$ *consists of all maps* $(x, y) \mapsto (\overline{a}\,xc, \overline{a}\,ya)$*, with* $a, c \in K \smallsetminus \{0\}$ *and* $c\,\overline{c} = 1$.

Proof: The first assertion follows from Witt's theorem (cf. [4] I §11), the rest is verified by easy computations. Note that assertion a can also be deduced directly from assertion b and its analogue for a sharply transitive subgroup of the stabilizer of $K(1, 0, 0)$. ∎

1.4 Definition. Let T be the (normal) subgroup of $\mathrm{PU}\,(\sigma)$ generated by all conjugates of $\Xi' = \left\{\xi_{0,p} : (u, v) \mapsto (u, v + p) \mid p \in K^-\right\}$.

1.5 Remarks. The elements of Ξ (and their conjugates) are also known as *Eichler transformations*, cf. [7] p. 214f. It is known that T is a simple group, see [4] II § 4, where T appears as $T_3(K, f)/W_3(K, f)$, or [7] 6.3.16, where T is denoted by $\mathrm{PEU}_3(V)$. We will see in 3.2 below that Ξ is contained in T. This yields that T coincides with its commutator subgroup, acts two-transitively (and thus primitively) on U, and is generated by the conjugates of an abelian normal subgroup (namely, Ξ') of a stabilizer. This is the standard situation for Iwasawa's criterion for simplicity [10], cf. [9] II §6, 6.12 or [14] 1.2.

2 Reflections.

Our next aim is to characterize the reflections in $\mathrm{PU}(\sigma)$ by their action on the unital. The following arguments are needed only in the case where K is not commutative. Let Z denote the center of K.

2.1 Lemma. *Assume that $J \in \mathrm{U}(\sigma)$ induces an involution $[J]$ fixing all the absolute points of an exterior line ℓ, but no other absolute point. Then $[J]$ is the reflection at ℓ.*

Proof: If $[J]$ fixes all the points on some line then $[J]$ is the unitary reflection at that line. In the present case, the axis has to be ℓ because every secant contains more than two absolute points. So assume, to the contrary, that $[J]$ does not have an axis. Then the fixed points of $[J]$ form a Baer subplane $\mathcal{B}$ by [1], cf. [8] IV.3.

Using 1.3.a, we may also assume that ℓ is the polar of $K(0,1,0)$, then $[J]$ fixes the (absolute) points $K(1,0,0)$ and $K(0,0,1)$, and the point $K(0,1,0)$. Thus we have

$$J = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \text{ with } a^2 = b^2 = c^2 \in Z,\ a\,\overline{c} = b\,\overline{b} = 1.$$

Since $[J]$ is a Baer involution, there is a third fixed point $K(0,1,x)$ on the line joining $K(0,0,1)$ and $K(0,1,0)$. This means that there exists $x \in K \setminus \{0\}$ such that $bx = xc$. Now a simple computation shows that $K(1, \overline{x}, -\frac{\overline{x}x}{2})$ is an absolute fixed point outside ℓ, contradicting our hypotheses. ■

2.2 Definitions. A reflection $\gamma \in \mathrm{PU}(\sigma)$ is called *confusable* if there is an automorphism α of the unital (U,B) such that $\beta := \alpha^{-1}\gamma\alpha$ is a Baer involution. We call β a confusable Baer involution, in that case.

A unitary reflection is called *admissible* if it belongs to T, and is not confusable. A point or line is called admissible if it is the center or the axis of an admissible reflection.

A reflection is called *exterior* (*interior*) if its axis — and then also its center — is exterior (interior).

We hasten to remark that we do not have any example of a confusable involution. Most of the evidence collected below indicates that, even if they exist, such examples would be hard to construct explicitly. From 2.1 one knows that $\mathrm{Aut}(U,B)$ leaves invariant the set $\mathcal{E}$ of exterior reflections. Thus we have:

2.3 Lemma. **a.** *Every confusable reflection is interior.*

b. *A confusable Baer involution never fixes an absolute point or line.*

c. *The product of two exterior reflections is never a confusable involution.* ■

Let γ be a confusable reflection. Then $\mathrm{C}_{\mathcal{E}}(\gamma)$ is not empty: every line joining the center of γ to an absolute point is a secant because there are no absolute (i.e., tangent) lines through the (interior) center of γ.

2.4 Corollary. *The centralizer* $\mathrm{C}_{\mathcal{E}}(\varphi)$ *of a confusable involution* φ *in the set* $\mathcal{E}$ *of exterior reflections does not contain any two commuting elements.*

Proof: We may assume that φ is an interior reflection. For any two commuting elements $\varepsilon_1, \varepsilon_2$ in $\mathcal{E}$, the product $\varepsilon_1\varepsilon_2$ is a reflection: in fact, the center of ε_j lies on the axis of ε_{3-j}. Using matrix representations, it is easy to see that the line joining the centers is an axis for the product. If the two commute with φ, the center of φ lies on both axes, and φ coincides with the product. ∎

2.5 Remark. It would be very nice if we could distinguish the centralizers of interior reflections and Baer involutions (taken in T) by group-theoretic properties. It seems that the centralizer of a reflection tends to be something like a unitary group in two dimensions (plus a considerable centralizer), while the centralizer of a Baer involution is a unitary group in three dimensions, over a smaller field. However, we have to deal with anisotropic forms on vector spaces of low dimension here, and everything is quite complicated.

3 Reflections in T.

After 2.1, we are able to recognize the exterior reflections from their action on the unital. Our next aim is to locate as many reflections as possible inside T. We will show in Section 4 below that the group T is characterized inside $\mathrm{Aut}\,(U,B)$ by its action on the unital. Section 5 treats the question of confusability.

3.1 Lemma. *For any two points in* U*, there is an element of* T *interchanging them. Moreover, the stabilizer of the block joining the two points contains the unitary reflection at the line that induces the block.*

Proof: We use homogeneous coordinates. Matrices enclosed in square brackets instead of parentheses denote the induced collineations. Since $\mathrm{PU}\,(\sigma)$ acts two-transitively on U, we may assume that the two points in question are $\infty = K(0,0,1)$ and $K(1,0,0)$. The linear transformation $(x_0,x_1,x_2) \mapsto (x_2,x_1,x_0)$ induces an involution $\iota \in \mathrm{PU}\,(\sigma)$ interchanging the two points, but we do not yet know whether ι belongs to T (cf. 3.5 and 3.7 below, where ι occurs as $[J_{-1}]$). However, we have

$$\iota\Xi'\iota \;=\; \left\{ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p & 0 & 1 \end{array}\right] \;\middle|\; p \in K^- \right\} \le \mathrm{T}\,,\ \text{and for each } p \in K^- \smallsetminus \{0\} \text{ the product}$$

$$\psi_p \;:=\; \left[\begin{array}{ccc} 1 & 0 & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p^{-1} & 0 & 1 \end{array}\right] \left[\begin{array}{ccc} 1 & 0 & p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc} 0 & 0 & p \\ 0 & 1 & 0 \\ -p^{-1} & 0 & 0 \end{array}\right]$$

belongs to $\Xi'(\iota\Xi'\iota)\Xi' \le \mathrm{T}$. Now ψ_p interchanges $K(0,0,1)$ and $K(1,0,0)$, while its square ψ_p^2 is the reflection. ∎

Computing the commutator $\xi_{2x,0} = \psi_p^2(\xi_{x,0}^{-1}\psi_p^2\xi_{x,0}) \in \mathrm{T}$, we obtain:

3.2 Corollary. *The group* Ξ *is contained in* T. ∎

Every unitary reflection $[R]$ is determined by its axis because the center has to be the image of the axis under the polarity π, and center and axis form the spaces of fixed points for R and $-R$. Since T acts two-transitively on U, the exterior lines (secants) form an orbit under T, and we have:

3.3 Lemma. *The group* T *contains the set* $\mathcal{E}$ *of exterior reflections, and this set forms a single conjugacy class in* T. ■

The involution $\iota : K(x_0, x_1, x_2) \mapsto K(x_2, x_1, x_0)$ is the unitary reflection with center $K(1, 0, -1)$. It depends on the field K and the anti-involution σ whether the point $K(1, 0, -1)$ is exterior: for instance, it is interior if $K \in \{\mathbb{C}, \mathbb{H}\}$ and σ is the standard anti-involution (with $K^+ = \mathbb{R}$).

Every unitary reflection at a point $K(x, y, z)$ (necessarily outside U) is a conjugate of a reflection at some point $K(1, 0, s)$, because the center $K(x, y, z)$ lies on some secant, which may be moved to $(0, 1, 0)^\perp$ by some element of $\mathrm{U}(\sigma)$; cf. 1.3.a (this is even possible by some element of T, see 1.5). Note that $\overline{s} \neq -s$ follows from $K(1, 0, s) \notin U$. Using the group Ξ', we may even achieve $s \in K^+ \setminus \{0\}$. Now the point $K(1, 0, s)$ is interior if the equation $x\,\overline{x} = 2s$ does not admit any solution $x \in K \setminus \{0\}$.

3.4 Examples. The standard involution $u + iv \mapsto u - iv$ on $K = F(i)$ yields interior points $K(1, 0, s)$ in the cases where $F \in \{\mathbb{Q}, \mathbb{R}\}$ and $s < 0$, but also in several cases where $F = \mathbb{Q}$ and $s > 0$; the smallest example with integer s is $s = 3 \notin \left\{u^2 + v^2 \mid u, v \in \mathbb{Q}\right\}$.

Straightforward computations show:

3.5 Lemma. *Let* $s \in K^+ \setminus \{0\}$. *The unitary reflection at* $K(1, 0, s)$ *is induced by*

$$J_s := \begin{pmatrix} 0 & 0 & s \\ 0 & -1 & 0 \\ s^{-1} & 0 & 0 \end{pmatrix}.$$

In particular, we have $[J_{-1}] = \iota$. *The reflection* $[J_s]$ *is exterior if, and only if, one has* $s \in \left\{\frac{1}{2}x\,\overline{x} \;\middle|\; x \in K^\times\right\}$. ■

3.6 Lemma. *For each* $s \in \left\{-\frac{1}{2}x\,\overline{x} \;\middle|\; x \in K \setminus \{0\}\right\}$, *the reflection* $[J_s]$ *belongs to* T.

Proof: We compute

$$\begin{bmatrix} 1 & x & -\frac{1}{2}x\,\overline{x} \\ 0 & 1 & -\overline{x} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2x^{-1} & 1 & 0 \\ -2(x\,\overline{x})^{-1} & 2\,\overline{x}^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & x & -\frac{1}{2}x\,\overline{x} \\ 0 & 1 & -\overline{x} \\ 0 & 0 & 1 \end{bmatrix} = [J_{-\frac{1}{2}x\overline{x}}],$$

noting that the factors belong to $\Xi \cup \iota\Xi\iota \subseteq \mathrm{T}$, cf 3.2. ■

3.7 Lemma. *Let* $s \in K^+ \setminus \{0\}$, *and assume that there exists* $p \in K^- \setminus \{0\}$ *such that* $sp = ps$. *Then* $[J_s]$ *belongs to* T.

Proof: Our assumption entails $-2sp \in K^-$, and $\psi_{-2sp} \in \mathrm{T}$, cf. 3.1. We compute that $[J_s]$ equals $\psi_{-2ps}\psi_p[J_{\frac{1}{2}}]$, and lies in T. ∎

3.8 Remark. Lemma 3.7 completely settles the case where σ is an involution of the second kind (i.e., where $K^- \cap Z \neq \{0\}$). In particular, this includes the case where K is commutative; it even shows:

$$\forall z \in Z^+ \setminus \{0\} : [J_z] \in \mathrm{T}\,.$$

Now $[J_1 J_2]$, $[J_{-1} J_2]$, $[J_z] \in \mathrm{T}$ and 3.6 may be used to see

$$\forall z \in Z^+ \setminus \{0\}\, \forall x \in K \setminus \{0\} : [J_{zx\overline{x}}] \in \mathrm{T}\,.$$

3.9 Example. In general, not every reflection in T is obtained by one of the constructions presented above. For instance, many elements of K^+ may have trivial centralizer in K^-:

Consider the involution $\sigma = (x \mapsto -k\,\overline{x}\,k)$ on $K := \mathbb{H}_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}k$: here $K^- = \mathbb{Q}k$, and i anti-commutes with every element of K^-. The set

$$\begin{aligned} &\left\{ zxx^\sigma \,\middle|\, z \in \mathbb{Q}^\times, x \in \mathbb{H}_{\mathbb{Q}}^\times \right\} \cup \left\{ s \in \mathbb{H}_{\mathbb{Q}}^+ \,\middle|\, \exists p \in \mathbb{H}_{\mathbb{Q}}^- \setminus \{0\} : ps = sp \right\} \\ = \;& \left\{ q\left((a^2 - b^2 + c^2 - d^2) + 2(ac - bd)i + 2(ab + cd)j\right) \,\middle|\, q \in \mathbb{Q},\, a, b, c, d \in \mathbb{Z} \right\} \end{aligned}$$

forms a rather small part of $\mathbb{H}_{\mathbb{Q}}^+ = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j$.

3.10 Remark. If the dimension $\dim_Z K$ is finite, it is a perfect square m^2 (for instance, see [11] VII §11 Prop. 1), and σ is of one of the following types (see [5] pp. 378f):

Type I: $Z \leq K^+$, and $\dim_Z K^+ = \frac{1}{2}m(m+1)$.

Type II: $Z \leq K^+$, and $\dim_Z K^+ = \frac{1}{2}m(m-1)$.

Type III: Z is not contained in K^+.

Types I and II occur with an involution σ of the first kind, while type III belongs to involutions of the second kind. Note that the passage between hermitian and skew-hermitian forms (as in 1.2) interchanges types I and II: in fact, for $p \in K^- \setminus \{0\}$, the map $x \mapsto xp$ is a Z-linear bijection from K^+ onto the set $\left\{ q \in K \mid p^{-1}\,\overline{q}\,p = -q \right\}$.

Consider a unitary reflection $[J]$. Without loss, we may assume that we are dealing with the reflection at $K(1, 0, s)$, where $s \in K^+ \setminus \{0\}$. If there exists $p \in K^- \setminus \{0\}$ such that $sp = ps$, we have $[J] \in \mathrm{T}$ by 3.7.

Now assume that $sp \neq ps$ holds for each $p \in K^- \setminus \{0\}$. Then surely σ is not of type III, the Z-linear map $\lambda : K^- \to K^+ : p \mapsto sp - ps$ is injective, and $\dim_Z K^- \leq \dim_Z K^+$ follows. This means that σ is of type I. In order to apply results of [5] or [7], one has to switch to a skew-hermitian form, and σ is replaced by an involution of type II, cf. 1.2. Unfortunately, this is the case which is not understood very well.

4 Translations.

In this section, we identify the group T inside $\mathrm{Aut}\,(U,B)$.

4.1 Definition. Let $\mathrm{T}(\infty)$ denote the set of all elements of $\mathrm{Aut}\,(U,B)$ fixing every block through $\infty = K(0,0,1)$ (i.e., every vertical block).

The commutator group Ξ' is contained in $\mathrm{T}(\infty)$. Our first aim is to show that the two groups coincide. We adapt the argument that was given in [16] for the commutative case:

4.2 Lemma. *The group* $\mathrm{T}(\infty)$ *acts sharply transitively on each vertical block. Therefore, it coincides with* Ξ'.

Proof: Since $\Xi' \le \mathrm{T}(\infty)$ acts sharply transitively on each vertical block, it suffices to show that the stabilizer $\mathrm{T}(\infty)_{(0,0)}$ is trivial. Assume, to the contrary, that $\gamma \in \mathrm{T}(\infty)_{(0,0)}$ moves a point $q := (u,v) \in U$. Then q belongs to A, and we may assume $u \neq 0$ without loss of generality, because every point on $[0]$ is the intersection of two blocks that are determined by their points outside $[0]$. According to 1.3, there exists $\alpha \in \mathrm{PU}\,(\sigma)$ such that $q^\alpha = (u^{-1}u, u^{-1}v\,\overline{u^{-1}}\,)$, and $\delta := \alpha^{-1}\gamma\alpha$ still fixes $(0,0)$ and all verticals, but moves the point $b := q^\alpha = (1,s)$, where $s := u^{-1}v\,\overline{u^{-1}}\,$.

The line joining $(0,0)$ and b is $[s,0]$. Let $[s',0]$ be the line joining $(0,0)$ and b^δ; then $s' \neq s$. For $h \in K$, we define $L_h := \left\{x \in K \mid \exists y \in K : (x,y) \in [h,0] \cap U\right\} = \left\{x \in K \,\middle|\, xh + \overline{xh}\; = -x\,\overline{x}\right\}$. Since δ fixes each vertical, we have $L_{s'} = L_s$. Thus each solution x of $xs + \overline{xs}\; = -x\,\overline{x}$ also has to satisfy $x(s'-s) = -\,\overline{x(s'-s)}\,$. Putting $e := (s'-s)$, we find $xe = -\,\overline{e}\,\overline{x}$. Now $b \in A$ implies $1 \in L_s$, yielding $s + \overline{s}\; = -1$ and $e \in K^-$. On the other hand, the elements $-2\,\overline{s}\, \in L_s$ and $-2\,\overline{s'}\, \in L_{s'}$ give $-2\,\overline{s}\,e = -2es$ and $-2\,\overline{s'}\,e = -2es'$, leading to $e^2 = e(s'-s) = (-\,\overline{s} + \overline{s'}\,)e = (s-1-s'+1)e = -e^2$, contradicting $s \neq s'$. ∎

4.3 Corollary. *The full group* $\mathrm{Aut}\,(U,B)$ *of automorphisms of the unital normalizes* T. ∎

4.4 Corollary. *Via conjugation, the group* $\mathrm{Aut}\,(U,B)$ *acts on the set* $\mathcal{E}$ *of reflections at exterior points, and on the set* $\mathcal{R}$ *of admissible reflections.* ∎

We will use this action to reconstruct the ambient projective plane from the action, see Section 6 below.

4.5 Lemma. *The centralizer of* T *in* $\mathrm{Aut}\,(U,B)$ *is trivial.*

Proof: The group $\mathrm{T}(\infty)$ fixes exactly one point of U, namely $K(0,0,1)$. Therefore, the centralizer of T fixes each point in the orbit U of $K(0,0,1)$ under $\mathrm{PU}\,(\sigma)$. ∎

4.6 Theorem. *The group* $\mathrm{Aut}\,(U,B)$ *coincides with the subgroup* Ψ *of* $\mathrm{Aut}\,(\mathrm{T})$ *that leaves invariant the set of conjugates of* Ξ'.

Proof: From 4.3 and 4.5 we know that Aut (U, B) acts faithfully on T. According to the geometric characterization 4.2 of Ξ', the set C of conjugates of Ξ' is invariant under this action, and Aut (U, B) induces a subgroup of Ψ.

The normalizer of Ξ' in Aut (T) contains the stabilizer of ∞ in T. This stabilizer is a maximal subgroup because T acts two-transitively on U, and we infer that the normalizer and the stabilizer coincide. Thus the action of T on U is equivalent to the action of T on C, and the latter extends to the action of Ψ on C. The blocks may be characterized as unions of orbits of conjugates of Ξ' sharing more than a single point, and Ψ acts by automorphisms on (U, B). ∎

4.7 Remark. If K is a commutative field then T is a simple algebraic group over K, and Ξ' is the commutator subgroup of the unipotent radical of a minimal parabolic subgroup. The algebraic group T has rank 1, and all minimal parabolic subgroups are conjugates. Thus the group Ψ defined in 4.6 contains the group of all automorphisms of the algebraic group T.

After 4.2, one might be tempted to conjecture that the group of automorphisms of the unital fixing all points of a block b and all blocks induced by lines through $b^\perp$ acts sharply transitively on these blocks. However, this is not the case, as the following example shows.

4.8 Example. Let $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ be the field of Hamilton's quaternions, and consider the involution $\sigma : x = x_1 + x_i i + x_j j + x_k k \mapsto \widetilde{x} := x_1 + x_i i + x_j j - x_k k$. (We refrain from our suggestive bar notation here.) In order to simplify notation in this example, we use the σ-hermitian form on $\mathbb{H}^3$ given by $\langle (x, y, z) | (u, v, w) \rangle := (x + y)\widetilde{u} + x\widetilde{v} + z\widetilde{w}$.

The point $e := K(1, 0, 0)$ is exterior, its polar $e^\perp$ is spanned by $(1, -1, 0)$ and $(0, 0, 1)$. The block induced by $e^\perp$ is $b = \left\{K(1, -1, w) \mid w \in \mathbb{R} + \mathbb{R}k,\ w\widetilde{w} = 1\right\}$.

The group of collineations induced by unitary transformations that fix all points in the line $e^\perp$ is strictly larger than the group of collineations induced by unitary transformations that fix all points in the block b. In fact, the first of these groups acts freely on each line through e. The latter group, however, is obtained as

$$\left\{ \begin{bmatrix} a & 0 & 0 \\ a-b & b & 0 \\ 0 & 0 & b \end{bmatrix} \;\middle|\; a \in \mathbb{H},\ b \in \mathbb{R} + \mathbb{R}k,\ a\widetilde{a} = 1 = b\widetilde{b} \right\},$$

and the stabilizer of $K(0, 1, 0)$ is given by the condition $a = b$.

5 Admissible Reflections.

5.1 Proposition. *Let ∞^δ be a point in $U \setminus \{\infty\}$, and consider $\xi \in \Xi$ and $\eta \in \delta^{-1}\Xi\delta$.*

- **a.** *If $\xi\eta\xi\delta^{-1}$ fixes ∞ then $\xi\eta\xi$ is induced by a linear map that fixes the vector $(0, 1, 0)$.*
- **b.** *If $\xi\eta\xi$ is an involution with $\infty^{\xi\eta\xi} = \infty^\delta$ then $\xi\eta\xi$ is a reflection.*

Proof: Without loss, we may assume $\infty^\delta = K(1,0,0)$. Then there are $x, y \in K$ and $p, q \in K^-$ such that

$$\xi = \begin{bmatrix} 1 & x & c \\ & 1 & -\overline{x} \\ & & 1 \end{bmatrix}, \eta = \begin{bmatrix} 1 & & \\ -\overline{y} & 1 & \\ d & y & 1 \end{bmatrix}, \xi\eta\xi = \begin{bmatrix} 1 - x\overline{y} + cd & * & * \\ -\overline{y} - \overline{x}d & * & * \\ d & dx+y & * \end{bmatrix},$$

where we have abbreviated $c := p - \frac{1}{2}x\overline{x}$ and $d := q - \frac{1}{2}y\overline{y}$. The entries marked $*$ are complicated, and will not be used before we have simplified the formula.

The assumption that $\infty = K(0,0,1)$ and $K(1,0,0)$ are interchanged yields the conditions $0 = 1 - x\overline{y} + cd = dx + y$ and $0 = \overline{y} + \overline{x}d$ (because the pole $K(0,1,0)$ of the line joining the two points is fixed).

For $x = 0$ we infer $y = 0$ and $cd = -1$. Then $\xi\eta\xi = \psi_c$ is not an involution; in fact, the square $(\xi\eta\xi)^2$ is the reflection at $K(0,1,0)$. However, the collineation $\xi\eta\xi$ is induced by a linear map that fixes the vector $(0,1,0)$.

There remains the case $x \neq 0$, and we infer $d = \overline{d}$. Using the conditions derived above, we compute $\xi\eta\xi = [M]$ with

$$M = \begin{bmatrix} 0 & 0 & c \\ 0 & 1 - \overline{x}y & \overline{x}y\overline{x} - 2\overline{x} \\ d & 0 & 0 \end{bmatrix},$$

and $M \in \mathrm{U}(\sigma)$ implies $\overline{x}y = 2$ and $c\overline{d} = 1$. If $\xi\eta\xi$ is an involution, we find $cd = -1$, and $c \in K^-$. Now $M = J_c$, and $\xi\eta\xi = [J_c]$ is the reflection with axis $K(1,0,c) + K(0,1,0)$. ■

Applying 5.1 to the description for $[J_{-\frac{1}{2}x\overline{x}}]$ obtained in 3.6, we find:

5.2 Corollary. *For every $x \in K^\times$, the reflection $[J_{-\frac{1}{2}x\overline{x}}]$ is admissible.* ■

5.3 Lemma. *Assume that $\rho_1, \dots, \rho_n$ are admissible reflections such that their axes pass through an exterior point p, and that the product $\pi = \rho_1 \cdots \rho_n$ is an involution. Then π is an admissible reflection.*

Proof: It suffices to show that every conjugate π^α with $\alpha \in \mathrm{Aut}\,(U,B)$ is a reflection. We will use the fact that a linear map λ fixing a non-zero vector induces an involution $[\lambda]$ only if $\lambda^2 = \mathrm{id}$. Any non-trivial element of $\mathrm{U}\,(\sigma)$ sharing this property is a reflection.

Let ρ be the reflection at p. Then our assumption on the axes means that ρ commutes with ρ_k, for each k, and every conjugate ρ_k^α commutes with ρ^α. Now each of the reflections ρ_k is described by a linear map fixing a generator of the center p of ρ, and each of the conjugates is described by a linear map fixing a generator v of the center of ρ^α. Consequently, the product π^α is described by a linear map fixing v, and π^α is a reflection. ■

Writing $[J_{zx\overline{x}}] = [J_{-\frac{1}{2}x\overline{x}}][J_{-\frac{1}{2}}][J_z]$, we obtain:

5.4 Theorem. *For each $x \in K^\times$ and each $z \in (Z^+)^\times$, the reflection $[J_{zx\overline{x}}]$ is admissible.* ■

5.5 Theorem. *If σ is an involution of the second kind, then every reflection is admissible.*

Proof: Our assumption means that there exists a non-trivial element $p \in K^- \cap Z$. For each $s \in K^+ \setminus \{0\}$, we find an exterior reflection ε and elements $\xi_j \in \Xi'$ and $\eta_j \in \iota(\Xi')\iota$ such that $[J_s] = (\xi_1\eta_1\xi_1)(\xi_2\eta_2\xi_2)\varepsilon$, cf. 3.1. Explicitly, we may choose

$$\xi_j := \begin{bmatrix} 1 & 0 & -u_j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \eta_j := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u_j^{-1} & 0 & 1 \end{bmatrix},\ \text{with } u_1 := -2ps,\ u_2 := p, \text{ and } \varepsilon := [J_{\frac{1}{2}}].$$

Conversely, a straightforward matrix computation shows that each of the involutions in the set $\mathcal{J} := \left\{(\xi_1\eta_1\xi_1)(\xi_2\eta_2\xi_2)\varepsilon \mid \xi_j \in \Xi',\ \eta_j \in \iota(\Xi')\iota,\ \varepsilon \in \mathcal{E},\ \infty^\varepsilon = K(1,0,0)\right\}$ is a reflection. Since the description of $\mathcal{J}$ is invariant under all automorphisms of (U, B) that fix the set $\{\infty, \infty^\iota\}$, all these reflections are admissible. ■

Our results so far may be summarized, as follows:

5.6 Theorem. **a.** *If the center of some unitary reflection has a representative v with $\langle v|v\rangle \in \left\{zx\overline{x} \mid z \in Z^\times, x \in K^\times\right\} \cup \left\{s \in K^+ \mid \exists p \in K^- \setminus \{0\} : ps = sp\right\}$ then that reflection is admissible.*

b. *Every exterior reflection is admissible, and the product of two commuting exterior reflections is always an admissible reflection.*

c. *If σ is of the second kind then every reflection is admissible.* ■

6 Reconstructing the Projective Plane.

In this section, we reconstruct the projective plane from the action of the group T. However, we will need the assumption that every unitary reflection is admissible (i.e., belongs to T, and all its conjugates under $\mathrm{Aut}\,(U, B)$ are reflections).

Let ρ_c be the unitary reflection with center c. Then $c \mapsto \rho_c$ defines a bijection from the set R of admissible points onto the set $\mathcal{R}$ of admissible reflections. Composing this bijection with the polarity, we also obtain a model of the sets of admissible lines. It remains to describe the incidence relation; we will do this in such a way that it is obvious that the action of $\mathrm{Aut}\,(U, B)$ on $P := U \cup \mathcal{R}$ (cf. 4.4) is an action by collineations. We define the following binary relation $*$ on P:

$$\begin{array}{lcll} \text{For } u, v \in U: & u * v & \iff & u = v. \\ \text{For } u \in U, a \in \mathcal{R}: & u * \rho_a & \iff & u^{\rho_a} = u \iff \rho_a * u \\ \text{For } e, f \in \mathcal{R}: & \rho_e * \rho_f & \iff & \rho_e\rho_f = \rho_f\rho_e \wedge \rho_e \neq \rho_f \end{array}$$

We obtain:

6.1 Theorem. *If all unitary reflections are admissible, then $(\,U \cup \mathcal{R}\,,\, U \cup \mathcal{R}\,,\, *\,)$ is isomorphic to the projective plane over K, and the action of $\mathrm{Aut}\,(U, B)$ on (U, B) extends to an action on the projective plane. Consequently, the groups $\mathrm{Aut}\,(U, B)$ and $\mathrm{P\Gamma U}\,(\sigma)$ coincide.* ■

6.2 Remarks. See 5.6 for criteria that ensure that every reflection is admissible. One could interpret the reconstruction also in the general case (where we do not know whether all reflections are admissible): Then the action of the normalizer of the set of reflections extends to an action on the plane, and we obtain that this normalizer coincides with $\mathrm{P\Gamma U}(\sigma)$.

7 Higher Dimensions.

Let X be a left vector space of finite[1] dimension $d+1 := \dim_K X \geq 3$ over K. We identify X with the space K^{d+1} of row vectors, and write these vectors as $x = (x_0, u, x_d)$, where $x_0, x_d \in K$ and $u \in V := K^{d-1}$. We obtain a polarity of the projective space $\mathrm{PG}(X) \cong \mathrm{PG}_d(K)$, described by a sesquilinear form $\langle x|y\rangle := \langle (x_0, u, x_d)|(y_0, v, y_d)\rangle := x_0\,\overline{y_d} + uH\,\overline{v}\,' + x_d\,\overline{y_0}$, where H is an invertible σ-hermitian matrix: $\overline{H}\,' = H$. Here, for any matrix M with entries from K, the matrix $\overline{M}$ is obtained by applying σ to each entry, and M' is the transpose of M.

The case where $uH\,\overline{v}\,'$ describes a hermitian form of positive Witt index corresponds to the case where U completely contains nontrivial projective subspaces, and these subspaces form a Tits building (more traditionally, a polar space). This case is understood rather well (see [15]), we shall henceforth concentrate on the case where H describes an *anisotropic form*, that is, the case where $uH\,\overline{u}\,' = 0$ implies $u = 0$.

We extend the definitions of ∞, A and $\mathrm{T}(\infty)$ from the discussion of the plane case, writing $\mathbf{0}$ for the zero vector in V. Applying the arguments of the proof of 4.2 to the plane spanned by $\infty = K(0, \mathbf{0}, 1)$, $K(1, \mathbf{0}, 0)$ and a hypothetical point q moved by an element of $\mathrm{T}(\infty)_{(\mathbf{0},0)}$, we obtain:

7.1 Proposition. *The group* $\mathrm{T}(\infty)$ *acts sharply transitively on each vertical block. Therefore, it is contained in* $\mathrm{PU}(\sigma)$*, and the subgroup* $\mathrm{T} \leq \mathrm{PU}(\sigma)$ *generated by all conjugates of* $\mathrm{T}(\infty)$ *forms a normal subgroup of* $\mathrm{Aut}(U,B)$*, again. The centralizer of* T *in* $\mathrm{Aut}(U,B)$ *is trivial, and* $\mathrm{Aut}(U,B)$ *may be interpreted as a group of automorphisms of* T. ∎

We interpret the projective space as a point–hyperplane geometry, and identify non-absolute points and hyperplanes with the corresponding unitary reflections. Proceeding as in the proof of 6.1, we find:

7.2 Theorem. *If all unitary hyperplane reflections are admissible, then every automorphism of* (U,B) *is induced by an automorphism of the projective space. Consequently, the groups* $\mathrm{Aut}(U,B)$ *and* $\mathrm{P\Gamma U}(\sigma)$ *coincide under these circumstances.* ∎

7.3 Remark. Note that it may happen that a unital has quite different embeddings into projective spaces. For instance, the unital in projective 3-space over $\mathbb{C}$ (with respect to the standard involution on $\mathbb{C}$) is isomorphic to a unital in the plane over $\mathbb{H}$ (with respect to the involution σ discussed in 4.8). The classes of involutions that

[1]Finiteness of dimension is implied by our requirement that the projective space admits a polarity.

act as reflections are different, but both consist of admissible involutions in these two cases.

In general, it appears that the passage between embeddings in different projective spaces might help in cases where not all reflections are admissible.

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