Characterizing the half-spin geometries by a class of singular subspaces

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1 Introduction

Although characterizations of Lie incidence geometries go back to Veblen and Young, little was accomplished before the magnificent creation of the Theory of Buildings by Jacques Tits. The reader should bear in mind that characterizing buildings of rank at least three and characterizing Lie incidence geometries are two different things: the former are chamber systems and the latter are point-lines geometries which may appear under different axiomatic guises for the same building. In fact, Tits' monumental work, contains two such characterizations of Lie incidence geometries: one for metasymplectic spaces and one for the non-degenerate polar spaces of rank at least three.

The characterization theorems of Cohen and Cooperstein [4] were particularly spectacular since they characterized at least one geometry derived from each spherical building of rank three or more. The key hypothesis of their work restricts the possible relationships between a point and a symplecton not containing it. Here we examine an analogous theory involving the relation between a point and maximal singular spaces belonging to some class rich enough to contain all lines.

We offer this theorem.

Theorem 1. *Suppose Γ is a parapolar space of symplectic rank at least three possessing a class $\mathcal{M}$ of maximal singular subspaces of finite projective rank with these properties:*

1. *Every line lies in at least one member of $\mathcal{M}$.*
2. *There exists a positive integer d such that if x is a point not in a subspace M of the class $\mathcal{M}$, then $x^{\perp} \cap M$ is either empty, or is a projective space of dimension d.*

3. In case $d = 1$, assume that at least one line lies in at least two members of $\mathcal{M}$.

Then one of the following three conclusions holds:

1. $d = 1$ and Γ is either a rank three polar space or is the Grassmannian whose points are the k-spaces of a (possibly infinite dimensional) vector space V. (Here the vector-space dimension k is a positive integer greater than 1.)

2. $d = 2$ and Γ is either a polar space of rank four, or an appropriate homomorphic image of a classical half-spin geometry, $D_{n,n}(F)$ where F is a field.

3. $d \geq 3$, and Γ is a polar space of polar rank $d + 2$.

The result for $d = 1$ is contained in [8], so in this note we will describe how the breakup into alternative conclusions occurs, and then sketch how these alternatives emerge when $d > 1$. As is usually the case with such theorems, the analysis hinges upon the nature of a point residual.

2 Terminology

This section contains the basic definitions needed to understand Theorem 1 and its discussion. The reader who is already familiar with these notions may proceed directly to the next section.

2.1 The language of point-line geometries

A point-line geometry is a rank two incidence geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ of two sorts of objects: *points* ($\mathcal{P}$), and *lines* ($\mathcal{L}$), which are special subsets of points of cardinality at least two. If all lines have at least three points, Γ is said to "have thick lines". Two distinct points are said to be **collinear** if and only if they are equal or there is a line in $\mathcal{L}$ containing both of them. Associated with each such geometry Γ is a point-collinearity graph $\Delta\Gamma := (\mathcal{P}, \sim)$ whose vertex set is the set of points, two *distinct* points being adjacent if and only if they are collinear. This allows one to employ the language of graph theory to describe sets of points. Thus we say that the point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is **connected** if and only if the graph $\Delta\Gamma := (\mathcal{P}, \sim)$ is connected. Also, following standard notation of graph theory, for any point p, the symbol $p^\perp$ denotes the collection of all points x which are equal to, or are otherwise collinear with p. Similarly for any non-singleton subset X of $\mathcal{P}$, $X^\perp$ is the collection of all points collinear with *every* point of X – that is, $X^\perp = \cap_{x \in X} x^\perp$.

A point-line geometry is said to be a **partial linear space** if and only if any two distinct points are together incident with *at most one* line.

A **subspace** is a subset X of points with the property that any line $L \in \mathcal{L}$ intersects X at zero, exactly one, or all of its points. The point-collinearity graph of a subspace is always an induced subgraph of the point-collinearity graph of the whole geometry. A subspace X is a **singular** if and only if its point-collinearity graph is a clique (i.e. any two of its points are collinear). A subspace X is said to be **convex** if and only if the intermediate vertices of any shortest path in the point-collinearity graph connecting two vertices (points) of X necessarily belong to

X. The set of convex subspaces is closed under intersections (recall they are sets of points) and the intersection of all convex subspaces containing a set X is called **the convex closure** of X and is denoted here by $\langle\langle X\rangle\rangle$.

We presume the reader understands the notion of projective space and its projective rank. In this note we shall abuse notation slightly[1] by writing $PG(r, K)$ for the projective space of 1- and 2-dimensional vector subspaces of a vector space V of dimension $r + 1$ over a division ring K, and we write $PG(r)$ if the division ring is understood but unspecified.

A **gamma space** is a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ with the following property: if p is a point and L is a line, then p is collinear with 0, 1 or all points of L. This is equivalent to asserting that $p^\perp$ is a subspace, for any point p. A **projective gamma space** is a gamma space all of whose singular subspaces are projective spaces (with respect to the lines contained in them).

Suppose for the moment that $\Gamma = (\mathcal{P}, \mathcal{L})$ is a projective gamma space with the property that every line $L \in \mathcal{L}$ is contained in a projective plane subspace. For each point p, let $\mathcal{L}_p$ and Π_p denote the full collection of lines and planes incident with point p. Then, using the containment relation to define incidence, $\mathrm{Res}_\Gamma(p); = (\mathcal{L}_p, \Pi_p)$ becomes a "point-line geometry" (as defined above) which we call **the point-residual at** p.

A projective gamma space is said to be **locally connected** if and only if $\mathrm{Res}_\Gamma(p)$ is connected for each point p.[2]

A **polar space** is a gamma space in which $x^\perp \cap L$ is non-empty, for every point-line pair $(x, L) \in (\mathcal{P}, \mathcal{L})$. A polar space is said to be *non-degenerate* if and only if no point of it is collinear with all others.

Non-degenerate polar spaces possess these properties:

1. They are partial linear subspaces.

2. All singular subspaces are projective spaces.

3. If one maximal singular subspace is a $PG(r - 1, K)$ then it is true that every maximal singular subspace is a $PG(r - 1, K)$. The polar space is said to have **polar rank** r in this case. Otherwise, the polar space is said to have **infinite polar rank**.[3]

A convex subspace which is a nondegenerate polar space of polar rank at least two is called a **symplecton**.

2.2 The language of parapolar spaces

A **parapolar space** is a connected gamma space $\Gamma = (\mathcal{P}, \mathcal{L})$ having these two properties:

1. Every line lies in a symplecton.

[1] Normally $PG(r, K)$ refers to the full rank r geometry of all proper subspaces of V.

[2] Note that if a point lies in at least two lines but lies in no plane, then the point-residual at that point cannot be connected.

[3] In this case, maximal singular subspaces may possess ranks of varying (infinite) cardinalities.

2. If x and y are two non-collinear points with $x^{\perp} \cap y^{\perp} \neq \emptyset$, then either
   (a) $x^{\perp} \cap y^{\perp} = \{p\}$ ((x, y) is a **special pair**), or
   (b) $|x^{\perp} \cap y^{\perp}| > 1$ and there exists a symplecton S containing $\{x, y\}$ ((x, y) is a **polar pair**).

If $\Gamma = (\mathcal{P}, \mathcal{L})$ is a parapolar space, it has symplectic rank at least k if each symplecton has polar rank at least k.

A parapolar space $\Gamma = (\mathcal{P}, \mathcal{L})$ is called a **strong parapolar space** if and only it has no special pairs.

Here are some useful properties of any parapolar space Γ:

1. Γ is a partial linear space.

2. If Γ is locally connected, then either Γ is itself a polar space, or else no symplecton S possesses a "deep point" — that is, a point $s \in S$ such that $s^{\perp} \subseteq S$.

3. If Γ has symplectic rank at least three, every singular subspace generated by a line and a point not in it must lie in some symplecton and so is a projective plane. Thus all singular subspaces of Γ are projective spaces (of possibly infinite rank)[4].

4. If Γ has symplectic rank at least three, then every connected component of a point residual $\mathrm{Res}_{\Gamma}(p)$ is a strong parapolar space.

3 The geometry of a point-residual

Suppose Γ' were a parapolar space satisfying the hypotheses of Theorem 1, with $\mathcal{M}'$ and $d+1$ replacing the symbols $\mathcal{M}$ and d of the theorem (this will simplify notation for a point residual). Since $d + 1 > 0$, Γ' is locally connected. From the preceeding remarks about parapolar spaces any point-residual $\Gamma = \Gamma'_p$ of Γ' must at least satisfy the following:

3.1 The hypotheses

(A1) $\Gamma = (\mathcal{P}, \mathcal{L})$ is a strong parapolar space.

(A2) $\mathcal{M}$ is a collection of maximal singular spaces, and d is a non-negative integer such that
   (a) every point lies in a member of $\mathcal{M}$, and
   (b) for each non-incident pair $(x, M) \in \mathcal{P} \times \mathcal{M}$, we have $x^{\perp} \cap M \simeq PG(d, D)$ for some division ring D.

[4] This is not true for symplectic rank two. The product geometry $A_1 \times A_2$, where the A_i are linear spaces which are not projective (for example affine planes), is a parapolar space of symplectic rank exactly two whose singular subspaces are not projective.

REMARK: These hypotheses are weaker than those enjoyed by point residuals of Γ'. We do not assume in advance that all singular subspaces are projective or that their subspace posets satisfy the ascending chain condition. After all, the parapolar space $A_1 \times A_2$ mentioned in Subsection 2.2 actually satisfies our hypothesis (A1) and (A2).

3.2 Elementary results

Lemma 2. *If S is a symplecton and $M \in \mathcal{M}$, then $S \cap M = \emptyset$ or is a projective space of rank $d+1$. In the latter case, the intersection is a maximal singular subspace of S.*

Corollary 3. *Every symplecton of Γ has polar rank $d+2$. In particular, if $d > 0$, every singular subspace of Γ is a projective space.*

Lemma 4. *Fix $M \in \mathcal{M}$. For every subspace U of M which is a $PG(d)$, there exists a point x not in M such that $x^\perp \cap M = U$.*

Proof. Let $\mathcal{U}$ denote the collection of all subspaces of M which are of the form $x^\perp \cap M$ for some point $x \in \mathcal{P} - M$. Then $\mathcal{U}$ is a collection of proper subspaces of M, all of projective rank d. It suffices to show the following:

if $U \in \mathcal{U}$ and W is a subspace of M of projective rank d meeting U at a hyperplane, then $W \in \mathcal{U}$.

Suppose then that U and W are $PG(d)$-subspaces of M which generate a $PG(d+1)$ and $U \in \mathcal{U}$. Then by the definition of $\mathcal{U}$ there is a point x not in M with $x^\perp \cap M = U$. Choose $w \in W - U$. Then w is not collinear with x, and the convex closure $\langle\langle x, w\rangle\rangle$ is a symplecton R with $R \cap M = \langle U, W\rangle$ a maximal singular subspace of R. Since W is a hyperplane of $R \cap M$ and R is a polar space, there is a point y in R with $y^\perp \cap (M \cap R) = W$. But since $y^\perp \cap M$ has projective rank d by hypothesis, we have $y^\perp \cap M = W$ forcing $W \in \mathcal{U}$ as desired. The proof is complete.

Theorem 5. (A trichotomy) *One of the following must hold:*

1. $d = 0$.
2. $d = 1$.
3. *Every member of $\mathcal{M}$ has projective rank $d+1$ and is itself a maximal singular subspace of some symplecton.*

Proof. Suppose U_1 and U_2 are two $PG(d)$'s of a space $M \in \mathcal{M}$ whose intersection is non-empty and yet has projective rank at most $d-2$. By Lemma 4 there exist points u_i, $i = 1, 2$, such that ${u_i}^\perp \cap M = U_i$. Now let U_1' be the singular $PG(d+1)$ spanned by u_1 and U_1. Then for any point v in $U_1' - U_1$, we also have $v^\perp \cap M = U_1$, and this v could without loss of generality replace the original u_1. Now if u_2 were collinear with every such v, then ${u_2}^\perp$ would contain all U_1' since, lines being thick, the latter is generated by $U_1' - U_1$. But that would force U_1 to be contained in ${u_2}^\perp \cap M = U_2$, an absurdity. Thus there is some v in $U_1' - U_1$ which is not collinear with u_2, and we can take that one to be our u_1.

Then the convex closure of $\{u_1, u_2\}$ is a symplecton R which must contain the nonempty $U_1 \cap U_2$. Thus $R \cap M$ is non-empty. Then by Lemma 2 $R \cap M$ is a $PG(d+1)$ which is a maximal singular subspace of R. Since the u_i are in R, we see the the $u_i{}^{\perp} \cap M$ are $PG(d)$'s and so must be U_1 and U_2. But then U_1 and U_2 must meet at a $PG(d-1)$ since both lie in the $PG(d+1)$ known as $R \cap M$. That is contrary to hypothesis.

Thus we cannot encounter a non-empty intersection of $PG(d)'s$ in M which is not a hyperplane of each. But if $d \geq 2$, and M had rank exceeding $d+1$, this would indeed be a possibility. Thus the hypotheses in the previous sentence must be false, hence the conclusion of the Theorem holds.

4 The third case of the trichotomy

Lemma 6. *Suppose $d > 1$, and that there exists an element of $\mathcal{M}$ of projective rank $d+1$. Then Γ is itself a polar space.*

Proof. There exists an element M of projective rank $d+1$. Let S be a symplecton on a point p of M. Then $M \subseteq S$ (Lemma 2). Choose a point $s \in S - p^{\perp}$.

Assume by way of contradiction that $\mathcal{P} \neq S$. Then by the second item listed at the end of Subsection 2.2. $s^{\perp} - S$ is non-empty.

Now by (A2), for any $x \in s^{\perp} - S$, $x^{\perp} \cap M$ is a $PG(d)$ contained in the hyperplane $s^{\perp} \cap M$ of M. Thus $x^{\perp} \cap S = \langle s, M \cap s^{\perp} \rangle$. Since x was arbitrary

$$s^{\perp} - S \subseteq \left(M \cap s^{\perp}\right)^{\perp}.$$

Now taking R to be the symplecton which is the convex closure $\langle\langle x, N \rangle\rangle$, where N is a line of S on s not in the clique $x^{\perp} \cap S$, we see that $s^{\perp} \cap R$ is the union of the two cliques $R \cap S$ and $\left(M \cap s^{\perp}\right)^{\perp} \cap R$. Since R has polar rank at least three, that is impossible.

Thus $\mathcal{P} = S$, and the proof is complete.

This Lemma yields the immediate

Corollary 7. *In the third case of the trichotomy of Theorem 5, Γ is a polar space.*

From now on, we may assume in addition to (A1) and (A2), the Axiom

(A3) Γ *is not itself a polar space.*

With this assumption, the trichotomy of Theorem 5 and Lemma 6 leave just two cases:

Case 1: $d = 0$.

Case 2: $d = 1$ and every element of $\mathcal{M}$ has projective rank larger than two.

5 The second case of the trichotomy: the case that $d = 1$.

In this section we assume:

(H1) $d = 1$ *and every element of* $\mathcal{M}$ *has projective rank greater than* 2.

Now (A2)(b) , Lemma 2 and (H1) immediately imply

Lemma 8. (Basic consequences of Case 2) *The following statements hold*

1. *For every* $M \in \mathcal{M}$ *and every point* $x \in \mathcal{P} - M$*,* $x^{\perp} \cap M$ *is a line.*

2. *Every symplecton has polar rank three, and intersects any member of* $\mathcal{M}$ *at a plane or the empty set.*

5.1 The maximal singular subspaces.

We begin with a study of maximal singular subspaces. We shall discover that those not in $\mathcal{M}$ are actually planes.

Lemma 9. *The intersection of two distinct maximal singular spaces cannot properly contain a line.*

Proof. Any intersection forbidden by this lemma leads to a symplecton of polar rank four or more.

Lemma 10. *Suppose* π *is a plane in the symplecton* R*, which is disjoint from an* $\mathcal{M}$*-plane of* R*. Then* π *is a maximal singular subspace of* Γ*.*

Proof. Suppose π is not a maximal singular subspace. Then there is a point x in $\pi^{\perp} - \pi$; clearly $x \notin R$ and $x^{\perp} \cap R = \pi$. Suppose $M \cap R$ is the $\mathcal{M}$-plane disjoint from π. Then $x^{\perp} \cap M$ is a line L which is disjoint from R. Choose $u \in R \cap M$. Then $x^{\perp} \cap u^{\perp}$ is a generalized quadrangle containing line $N := u^{\perp} \cap \pi$, as well as N. Thus if $v \in L$, $v^{\perp} \cap N$ is a point w of π. But that is impossible as $v^{\perp} \cap R = R \cap M$ is a maximal singular subspace of R.

Lemma 11. *The following statements hold:*

(i) *Suppose* A *is a maximal singular space, and that* $M \cap A = \emptyset$ *for some element* $M \in \mathcal{M}$*. Then* A *is a plane.*

(ii) *Conversely, suppose* π *is a plane which is disjoint from some member of* M*. Then either* π *is a maximal singular subspace, or else it properly lies in a unique maximal singular subspace meeting every member of* $\mathcal{M}$*.*

Sketch of proof: (i) First A cannot be a line, since any line lies in some symplecton of rank three. The rest of the proof produces a symplecton Q such that $Q \cap M$ and $Q \cap A$ are opposite planes of the symplecton Q. Then one applies Lemma 10.

(ii) Suppose $\pi^{\perp} \cap M = \emptyset$, for some $M \in \mathcal{M}$. If there exists an element $y \in M$ such that $y^{\perp} \cap \pi$ is a line, then there is a symplecton R containing $\pi \cup \{y\}$ meeting R at the $\mathcal{M}$-plane $R \cap M$ and Lemma 10 may be applied. Otherwise, for distinct points $a, b \in \pi$, the lines $L_a := a^{\perp} \cap M$ and $L_b := b^{\perp} \cap M$ are disjoint. Choosing

$z \in L_b$ one discovers that the symplecton $S := \langle\langle z, a\rangle\rangle$ contains $\{b\} \cup L_a$ forcing $L_a \cap L_b = b^\perp \cap L_a$ nonempty, a contradiction.

Finally any maximal singular subspace properly containing π must intersect nontrivially each member of $\mathcal{M}$ by part (i).

Corollary 12. *Any two elements of $\mathcal{M}$ have a non-empty intersection.*

Proof. Since Lemma 6 and (A3) together force all elements of $\mathcal{M}$ to have projective dimension at least three, the result follows immediately from Lemma 11.

A little more work yields:

Lemma 13. *Any two members of $\mathcal{M}$ intersect in exactly a single point.*

5.2 The symplecta.

The following technical theorem is the crux of our final result.

Theorem 14. *Let π be a plane meeting M at a point for some element $M \in \mathcal{M}$. Suppose $\pi^\perp \cap M = \{p\}$. Then the following statements hold:*

1. *There is a unique symplecton R containing π.*
2. *Suppose $x \in M - R$. Then $x^\perp \cap \pi = \{p\}$. For any point $a \in \pi - \{p\}$ we have*

$$x^\perp \cap p^\perp \cap a^\perp - M \subseteq \pi^\perp.$$

3. *Any symplecton which intersects π at a line on p is oriflame.*

Proof. For any line pa of π on point p, $a^\perp \cap M$ is a line pu of M on p. Then, for any $b \in \pi - pa$, the symplecton $R = \langle\langle u, b\rangle\rangle$ meets M at a plane π_R (Lemma 8, part 2). Since $\pi^\perp \cap M = \{p\}$, we see that $b^\perp \cap M$ is a line pv of M distinct from pu. Thus the plane π_R is spanned by $\{u, v, p\}$.

Now suppose R' is another symplecton containing π. Then by Lemma 2 $R' \cap M$ is also a plane – but one which contains the lines $a^\perp \cap M$ and $b^\perp \cap M$ which span π_R. Since R and R' both contain π and π_R they must coincide.

Part 2. Let R be the unique symplecton on π, and set $\pi_R = R \cap M$ as above. Also let a, b and p span π as in the first part. Now choose $x \in M - \pi_R$. Then $x^\perp \cap R = \pi_R$ so $x^\perp \cap \pi = \{p\}$. Form the symplecton $S = \langle\langle x, a\rangle\rangle$. Then clearly $S \cap \pi = pa$. Finally choose a line pz in $(px)^\perp \cap (pa)^\perp$ which is not in M. This is possible since such a line pz is a line on p in the symplecton S, and all the lines and planes of S on p have the incidence structure of a generalized quadrangle. Now $z^\perp \cap M = px$ not in π_R. But if z were not collinear with b, we could form a symplecton $T := \langle z, b\rangle$ which would contain all of π. Then we must have $R = T$, by the uniqueness of R in part 1. But in that case, $z^\perp \cap \pi_R$ is a line, against $z^\perp \cap M = px$.

Thus z must be in $b^\perp$ and hence lies in $\pi^\perp$. From the choice of pz, Part 2 is proved.

Part 3. Suppose now S is a symplecton which intersects π at a line on p – say, the line pa. Then $S \cap M$ is a plane π_S containing $pu := a^\perp \cap M$. Now S cannot contain π_R since this would force $R = S$, while $S \cap \pi$ is only a line. Thus there is a line px in π_S that is not in π_R. Now the lines on p in $x^\perp \cap a^\perp$ are partitioned into two sets: (1) those on p lying in M (they are in the plane π_S), and (2), those not in M. By part 2, the latter belong to $b^\perp$ and so are in the clique $b^\perp \cap S$. Thus $(px)^\perp \cap (pa)^\perp$ is the union of two cliques. This means the generalized quadrangle of all lines and planes of S on p is a grid, and in turn S is an oriflame rank three polar space.

Corollary 15. *Every symplecton of* Γ *is oriflame.*

Proof. Let S be any symplecton. By Lemma 2 and (A1), there is an element $M \in \mathcal{M}$ such that $M \cap S = \pi_S$ is a plane. Now select a point $x \in S - \pi_S$. Let M_x be a member of $\mathcal{M}$ containing x. Then $x^\perp \cap M$ is a line, and since S is a polar space, $x^\perp \cap \pi_S$ is also a line contained in the former. Thus $x^\perp \cap M = x^\perp \cap \pi_S := N$, a line of π_S. Now there exists an element M_x of $\mathcal{M}$ which contains x. Then by Lemma 13, the intersection $M_x \cap M$ is a point p in $x^\perp \cap M = N \subseteq S \cap M = \pi_S$. Thus $\{p\} = M_x \cap M$ is on line N. Now select a line L of π_S which intersects N at exactly the point p. Since M is not a plane, there is a plane π of M which intersects $\pi_S = M \cap S$ at precisely the line L.

We claim that $\pi^\perp \cap M_x = \{p\}$. By way of contradiction suppose $z \in \pi^\perp \cap M_x - \{p\}$. Then by (A2)(b) and (H1), $z \in M_x \cap M$, forcing $z = p$, a contradiction.

Now (M_x, p, π, S) satisfies the hypotheses of (M, p, π, S) in Part 3 of Theorem 14. By Part 3. of Theorem 14, S is oriflame. The proof is complete.

5.3 More maximal singular subspaces.

Another consequence of Theorem 14 is the following:

Corollary 16. *Suppose* A *is a maximal singular subspace which is not a plane. Then for every point* $x \in \mathcal{P} - A$, *the set* $x^\perp \cap A$ *is a line.*

Now let $\mathcal{M}''$ be the class of all maximal singular spaces which are either in $\mathcal{M}$ or are not planes intersecting trivially some member of $\mathcal{M}$. Then by the preceeding Corollary, $\mathcal{M}''$ satisfies all the hypotheses on $\mathcal{M}$ appearing in axioms (A1) and (A2). That means that we could have taken $\mathcal{M} = \mathcal{M}''$ all along.

So, from this point onward, we assume

(H2) *If* A *is a maximal singular subspace not in the class* $\mathcal{M}$, *then* A *is a projective plane that is disjoint from at least one member of* $\mathcal{M}$.

Let $\mathcal{A}$ be the class of all maximal singular subspaces which are planes which are disjoint from some member of $\mathcal{M}$. Then (H3) can be taken to be the assertion that the set of all maximal singular subspaces of Γ can be partitioned as $\mathcal{M} + \mathcal{A}$.

The following results are now easy to prove:

Theorem 17. *(i) If* $(M, A) \in \mathcal{M} \times \mathcal{A}$, *then* $M \cap A$ *is the empty set or is a line.*

(ii) Any two distinct members of $\mathcal{A}$ *intersect at the empty set or at a single point.*

(iii) Every line of Γ *lies in a unique member of* $\mathcal{M}$.

5.4 The final result for Case 1 with (H1).

Theorem 18. $\Gamma = (\mathcal{P}, \mathcal{L})$ *is the Grassmannian of lines of some (possibly infinite-dimensional) projective space* $P = \mathbf{P}(V)$ *of projective dimension at least four.*

The proof establishes these points:

(1) *any two elements of* $\mathcal{M}$ *are incident with a unique point. That means* $(\mathcal{M}, \mathcal{P})$ *is a linear space.*

(2) *Any three members of* $\mathcal{M}$ *which are not incident with a common point are in fact incident with a unique element of* $\mathcal{A}$.

Then

(3) *The incidence system of elements of* $\mathcal{M}$ *and all points which are incident with a plane* A *is a projective plane.*

As a consequence of these observations, we also have

(4) *If two distinct elements of* $\mathcal{M}$ *are incident with an element* A *of* $\mathcal{A}$, *then the unique point of their intersection is incident with* A. *In other words, the set of elements of* $\mathcal{M}$ *which are incident with* A *is a subspace of the linear space* $(\mathcal{M}, \mathcal{P})$.

The four statements assert that $(\mathcal{M}, \mathcal{P})$ is a linear space in which any three "points" which are not incident with a common "line" are contained in a subspace which is a projective plane. This means $(\mathcal{M}, \mathcal{P})$ is a projective space (of unknown rank at least four) and the theorem is proved.

6 Assembling the proof of Theorem 1

In the case $d = 0$ of the trichotomy (Theorem 5) where we assume some point is on two members of $\mathcal{M}$, it is known from [8] that Γ is either a polar space, or is a product geometry $A_1 \times A_2$. The results of the previous section show that in the case $d > 0$, Γ is either a polar space, or is the Grassmannian of lines of a projective space $\mathbf{P}(V)$.

This means that in a geometry Γ satisfying the hypotheses of Theorem 1, each point residual is either (1) a polar space, (2) a product of two projective spaces of finite rank, or (3) a Grassmannian of lines of a projective space of finite rank. If (1) holds for even one point-residual, then Γ is a polar space (see the second item at the end of Subsection 2.2). Otherwise one shows that all point-residuals are either uniformly of type (2) or type (3). One now has all the ingredients of the theory of locally truncated geometries due to Ronan/Brouwer-Cohen, and the conclusion now follows.

In the case $d = 1$ (for a point-residual) which we have followed here, Γ is a strong parapolar space belonging to the locally-truncated diagram shown in Figure 1.

Thus one also obtains this Theorem:

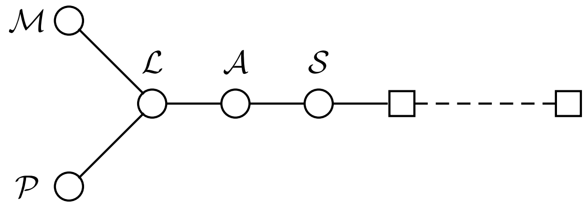

Figure 1: The locally truncated diagram for a conclusion to Theorem 1. The symbols $\mathcal{P}$,$\mathcal{L}$ and $\mathcal{M}$ have been given. $\mathcal{A}$ is another class of maximal singular spaces distinct from $\mathcal{M}$, while $\mathcal{S}$ is the class of symplecta. The square nodes do represent objects, but their residuals may not be as indicated by the diagram.

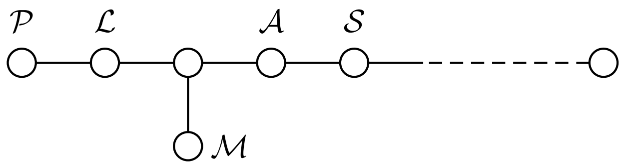

Figure 2: The diagram of the building geometries from which the conclusions of Theorem 19 are derived. The meanings of the node-labels are as given in Figure 1.

Theorem 19. *Suppose Γ is a locally connected parapolar space possessing a collection $\mathcal{M}$ of maximal singular subspaces satisfying these properties:*

1. *Every plane lies in a member of $\mathcal{M}$.*
2. *If $M \in \mathcal{M}$, and x is a point not in M then $x^\perp \cap M$ is either empty, a single point, or a $PG(3, D)$.*

Then Γ is either a polar space, or is a homomorphic image of a truncation of a building belonging to the diagram depicted in Figure 2. (The truncation is indicated by nodes which are labelled or which are bounded by labelled nodes.)

Among the conclusion geometries one finds the Lie incidence geometries of types $E_{6,1}$, $E_{7,1}$ and $E_{8,1}$ (in the Bourbaki numbering).

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