A combinatorial approach to Coxeter groups

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1 Introduction

Let $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix over a set I. A Coxeter system of type M is a pair (W, S) consisting of a group W and a set $S = \{s_i \mid i \in I\} \subseteq W$ such that $(s_i s_j)^{m_{ij}} = 1_W$ for all $i, j \in I$ and such that the set of these relations yields a presentation of W; hence such that $W \cong \langle S \mid ((s_i s_j)^{m_{ij}})_{i,j \in I} \rangle$ for short. The aim of the present paper is to give a combinatorial proof of the following

Fundamental Fact: Let M be a Coxeter diagram over a set I, let (W, S) be a Coxeter system of type M where $S = \{s_i \mid i \in I\}$. Then the order of the product $s_i s_j$ is equal to m_{ij} for all $i, j \in I$.

This is well known and obtained by a 2-line argument as a very first observation about the geometric representation of a Coxeter group. It seems therefore appropriate to explain why we are nevertheless interested in replacing this short geometric argument by a combinatorial proof which takes about 20 pages. There are in fact two main reasons.

In [7] Tits solved the word problem for Coxeter groups. In that paper he indicated how to use his result in order to produce a 'combinatorial' proof of the classification of finite Coxeter groups. In [3] it is shown, how to employ folding techniques in order to give a short proof of the classification based on Tits' original idea. This gives hence a proof which does not rely on classification of the positive definite bilinear forms associated to Coxeter matrices (which basically corresponds to Coxeter's original proof in [1] and [2]). However, this proof still relies on the geometric representation of a Coxeter group because the fact above is used in [7] for the solution of the word problem. We also mention that in [6] and [8] the theory of Coxeter groups is developed to a large extent in a purely combinatorial set-up. As both references use the geometric representation only to prove the fact above, the question about a

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purely combinatorial theory of Coxeter groups arises naturally. Our paper provides an affirmative answer to that question.

Whereas our first reason refers to 'mathematical curiosity', the second is more serious and related to current research. The fundamental fact above expresses the fact that a certain amalgam of dihedral groups does not collapse. We expect that our techniques can be modified in such a way that they also apply to amalgams of groups of Lie type of rank 2. A proof of the fact that those amalgams do not collapse would be most interesting for proving the existence of certain groups of Kac-Moody type, for which the present existence proofs are quite involved.

The reader may have noticed that we talked about a 20-pages proof, whereas the present paper has less than 20 pages. This is due to the fact that we do not give the complete proof here. For instance, we only consider the case of simply laced diagrams with no triangles. The general case can be dealt by using this result and the well-known folding techniques for Coxeter groups. We provide more information about this reduction in the last section. Even for the case of simply laced diagrams without triangles (which is treated in Section 3) not all details are given. We believe that the reader will have no problem to reconstruct them. We remark nevertheless that all details can be found in [4].

The strategy of our proof of the fundamental fact is as follows: it suffices to produce a group generated by a set of involutions $\Sigma := \{\sigma_i \mid i \in I\}$ such that the order of $\sigma_i\sigma_j$ is m_{ij} for all $i, j \in I$. We define M-homotopy classes of words and M-reduced words in the free monoid as they are defined in Tits' solution of the word problem for Coxeter groups. Let X denote the set of M-homotopy classes of reduced words. We define the set Σ as a subset of $\mathrm{Sym}(X)$. In order to define the involutions $\sigma_i \in \mathrm{Sym}(X)$ we will show that for each M-reduced word f and each $i \in I$ either the word fi is M-reduced or there exists an M-reduced word f' such that $f'i$ is M-homotopic to f. This will require most of the work. Also, a rank 2 version of this fact is needed in order to show that the orders of the products $\sigma_i\sigma_j$ are the desired ones.

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2 Definitions and notation

The purpose of this section is to fix notation and to give some basic definitions concerning free monoids and Coxeter matrices.

We denote the set of non-negative integers by $\mathbb{N}$. If X is a subset of $\mathbb{N}$, then the smallest natural number in X is denoted by $\min X$ and the greatest natural number in X by $\max X$ (if it exists).
Given a set M, then $\mid M \mid$ denotes its cardinality.

Given a group G and H a subset of G, then the subgroup of G generated by H is denoted by $\langle H \rangle$. Moreover we write $H \leq G$ if H is a subgroup of G and $H \trianglelefteq G$ if H is a normal subgroup of G. The neutral element of G is denoted by 1_G and if g is an element of G, its order in G is denoted by $o(g)$.

Free monoids

Let I be a set.

A **word** over I of length $k > 0 \in \mathbb{N}$ is a sequence $i_1 \dots i_k$ with $i_\lambda \in I$ for $1 \leq \lambda \leq k$. By definition there is a unique word of length 0 over I which is denoted by $\emptyset$ and which is called the **empty word**. We denote the set of words over I by $F(I)$.

We define a multiplication on $F(I)$ as follows: let $f = i_1 \dots i_k$ and $g = j_1 \dots j_l$ be words in $F(I)$, then $fg = i_1 \dots i_k j_1 \dots j_l$. This multiplication is associative and its neutral element is the empty word. The **free monoid over I** is $F(I)$ endowed with this multiplication and it is also denoted by $F(I)$.

The length function on $F(I)$ is the mapping $l : F(I) \longrightarrow \mathbb{N}$ assigning to each word over I its length. We have in particular $l(\emptyset) = 0$ and $l(fg) = l(f) + l(g)$ for $f, g \in F(I)$.

A **head** of a word $f \in F(I)$ is a word $h \in F(I)$ such that there exists a word $g \in F(I)$ with $f = hg$. In the same way, a **tail** of a word $f \in F(I)$ is a word $t \in F(I)$ such that there exists a word $g \in F(I)$ with $f = gt$.

The symbol $p_m(i,j)$ denotes the word $\dots ijij$ of length m.

Coxeter matrices

Let I be a set.

A **Coxeter matrix** over I is a symmetric matrix $M = (m_{ij})_{i,j \in I}$ whose entries are in the set $\mathbb{N} \cup \{\infty\}$ such that $m_{ii} = 1$ for each $i \in I$ and $m_{ij} \geq 2$ for all $i \neq j \in I$. With such a matrix one associates a **Coxeter diagram** $\Gamma(M)$ as follows: $\Gamma(M)$ is a labeled graph with vertex set I and edge set consisting of all unordered pairs of $\{i,j\}$ such that $m_{ij} \geq 3$. Each edge is labeled by the corresponding m_{ij}. Since M and $\Gamma(M)$ carry the same information, we do not distinguish these two notions.

A **system of involutions** is a pair (W, S) consisting of a group W and a set $S \subset W$ of involutions. If (W, S) is a system of involutions the matrix $(o(st))_{s,t \in S}$ is a Coxeter matrix over the set S. It is called the **type** of (W, S).

In the remainder of this subsection we fix a Coxeter matrix $M = (m_{ij})_{i,j \in I}$ over a set I and for $i, j \in I$, $p(i,j)$ denotes the word $p_m(i,j)$ where $m = m_{ij}$.

Two words $f, g \in F(I)$ are **elementary M-homotopic** if they are of the form $f = f_1 p(i,j) f_2$ and $g = f_1 p(j,i) f_2$ for some $f_1,\ f_2 \in F(I)$ and some $i, j \in I$ such that $m_{ij} \neq \infty$. We denote that relation by $\cong_M$.

Two words $f, g \in F(I)$ are **M-homotopic** if there exists a sequence $f = f_0, f_1, \dots, f_k = g$ such that, for $1 \leq i \leq k$, f_{i-1} and f_i are elementary M-homotopic. This relation is an equivalence relation denoted by $f \simeq_M g$. The equivalence class of a word $f \in F(I)$ is denoted by $[f]_M$.

A word is **M-reduced** if it is not M-homotopic to a word of the form $g = g_1 i i g_2$ where $g_1, g_2 \in F(I)$ and $i \in I$.

We remark some basic facts which often will be used without further reference. If two words are M-homotopic, then their lengths are equal and they contain the same letters. In particular, if $i_1, \dots, i_k \in I$ are pairwise distinct, then $f = i_1 \dots i_k \in F(I)$ is M-reduced. If $f \in F(I)$ is M-reduced, its heads and tails are M-reduced too.

3 Simply laced Coxeter diagrams without triangles

Throughout this section M is a simply laced Coxeter diagram without triangles. These conditions mean the following:

- $m_{ij} \in \{1, 2, 3\}$ for all $i, j \in I$;
- If $i, j, k \in I$ are pairwise distinct, then there exists a subset $\{\lambda, \mu\}$ of $\{i, j, k\}$ such that $m_{\lambda\mu} = 2$.

M-reduced words

In the following we denote the relations "elementary M-homotopic" and "M-homotopic" respectively by $\cong$ and $\simeq$. In the same way, the M-homotopy-class of a word f is denoted by $[f]$. As in the previous section we denote the word $p_m(i, j)$ with $m = m_{ij}$ by $p(i, j)$. Moreover, the word $p_m(i, j)$ with $m = m_{ij} - 1$ is denoted by $q(i, j)$. Hence $p(i, j) = ij$ and $q(i, j) = j$ if $m_{ij} = 2$ and $p(i, j) = jij$ and $q(i, j) = ij$ if $m_{ij} = 3$.

Proposition 3.1. *Let $f \in F(I)$ be an M-reduced word. Then:*

1. *If $f \simeq f'i \simeq g'k$, with $f',\ g' \in F(I)$, then there exists a word $h \in F(I)$ such that $f \simeq hp(k, i)$.*
2. *If $f \simeq f_1 f_2 \simeq g_1 f_2$, with $f_1,\ f_2,\ g_1 \in F(I)$, then $f_1 \simeq g_1$.*

Proof. We apply induction on the length $l(f) = n$ of the word f.

If $l(f) = 0$ or $l(f) = 1$, then both assertions of the proposition are trivial.

If $l(f) = 2$, then f is a word of the form ab where $a \neq b \in I$. We distinguish two cases: either $m_{ab} = 2$ or $m_{ab} = 3$. In the first case, the equivalence class of f consists of the two words ab et ba. In the second case, the only element in the equivalence class of f is the word ab. In both cases the assertions are obvious.

If $l(f) = 3$, then f is a word of the form abc with $a \neq b \neq c \in I$ since f is M-reduced.
If $a = c$, the case $m_{ab} = 2$ is impossible because f is supposed to be M-reduced. Therefore we have $m_{ab} = 3$ if $a = c$ and the equivalence class of f consists of the two words aba and bab. The assertions follow.
Suppose now that $a \neq c$. Below we list the M-equivalence classes of abc depending on the subdiagram induced on the set $\{a, b, c\}$. In each case the assertions of the proposition follow immediately.

- If $m_{ab} = m_{ac} = m_{bc} = 2$, we get $[f] = \{abc,\ acb,\ bac,\ bca,\ cba,\ cab\}$.
- If $m_{ab} = m_{ac} = 2$ and $m_{bc} = 3$, we get $[f] = \{abc,\ bac,\ bca\}$.
- If $m_{ab} = m_{bc} = 2$ and $m_{ac} = 3$, we get $[f] = \{abc,\ bac,\ acb\}$.
- If $m_{bc} = m_{ac} = 2$ and $m_{ab} = 3$, we get $[f] = \{abc,\ acb,\ cab\}$.
- If $m_{ab} = m_{ac} = 3$ and $m_{bc} = 2$, we get $[f] = \{abc,\ acb\}$.
- If $m_{bc} = m_{ac} = 3$ and $m_{ab} = 2$, we get $[f] = \{abc,\ bac\}$.

- If $m_{ab} = m_{bc} = 3$ and $m_{ac} = 2$, we get $[f] = \{abc\}$.

We consider now the case $l(f) = n \geq 4$ and assume that the assertions hold for all $f' \in F(I)$ with $l(f') < n$.

We first prove Assertion 1.

Let f', g', i and k be as in Assertion 1, i.e. $f \simeq f'i \simeq g'k$. We claim that there is a 'direct' M-homotopy between $f'i$ and $g'k$, i.e. a sequence of elementary M-homotopic words starting with $f'i$ and ending with $g'k$ such that the sequence of the ending letters changes at most once.

Suppose first that we are given an M-homotopy from $f'i$ to $g'k$ such that the sequence of ending letters changes twice - from i to j and then from j to k.

We first consider the case $i = k$. An M-homotopy from $f'i$ to $g'i$ is of the form

$$f'i \cong \ldots \cong f''p(j,i) \cong f''p(i,j) \cong \ldots \cong f'''p(i,j) \cong f'''p(j,i) \cong \ldots \cong g'i$$

where i is a tail of each word of the first and third indicated sub-homotopy and where j is a tail of the sub-homotopy in the middle. So there is an M-homotopy from $f''p(i,j)$ to $f'''p(i,j)$ consisting of elementary M-homotopic words having j as last letter. Therefore $f''q(j,i) \simeq f'''q(j,i)$. Those words, seen as heads of f, are M-reduced words of length $n-1$. Using the induction hypothesis, we see that $f'' \simeq f'''$. By 'extending' an M-homotopy from f'' to f''' to an M-homotopy from $f''p(j,i)$ to $f'''p(j,i)$ and replacing the sub-homotopy in the middle of the homotopy above by this homotopy we get a 'direct' homotopy as claimed.

We now consider the case $i \neq k$. There are several possible sub-diagrams induced on the set $\{i,j,k\}$ and we treat the different cases separately. As for the case $l(f) = 3$ above, there are — up to symmetry — seven cases to consider. Here we treat only two of them. The others can be dealt by analogous arguments (see [4]).

1. $m_{ij} = m_{ik} = m_{jk} = 2$ $\qquad \overset{i}{\bullet} \quad \overset{j}{\bullet} \quad \overset{k}{\bullet}$

   We consider an M-homotopy from $f'i$ to $g'k$ of the form

   $f'i \cong \ldots \cong f''ji \cong f''ij \cong \ldots \cong f'''kj \cong f'''jk \cong \ldots \cong g'k$

   where i,j and k are respectively tail of each word in the first, the second and the last sub-homotopy. There is an M-homotopy from $f''ij$ to $f'''kj$ all whose words have tail j. Therefore $f''i \simeq f'''k$. Those two words, seen as heads of the M-reduced word f, are themselves M-reduced and of length $n-1$. By induction there exists a word $h \in F(I)$ such that $f''i \simeq hki \cong hik \simeq f'''k$.

   As $f''i$, hki, hik and $f'''k$ are M-reduced words of length $n-1$ and as $f''i \simeq hki$ and $hik \simeq f'''k$, we get by induction $f'' \simeq hk$ and $hi \simeq f'''$.

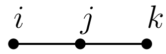
   Let

   $$f'' = x_0 \cong x_1 \cong \ldots \cong x_\lambda = hk$$
   $$hi = y_0 \cong y_1 \cong \ldots \cong y_\mu = f'''$$

   be M-homotopies.

   We have a direct M-homotopy from $f'i$ to $g'k$ as follows:

   $f'i \simeq f''ji = x_0 ji \cong \ldots \cong x_\lambda ji = hkji \cong hjki \cong hjik \cong hijk = y_0 jk \cong \ldots \cong y_\mu jk = f'''jk \simeq g'k.$

2. $m_{ij} = 3 = m_{jk}$ et $m_{ik} = 2$ 

   We consider an M-homotopy from $f'i$ to $g'k$ of the form

   $f'i \cong \ldots \cong f''iji \cong f''jij \cong \ldots \cong f'''jkj \cong f'''kjk \cong \ldots \cong g'k$

   where i,j and k are respectively tail of each word in the first, the second and the last sub-homotopy. There is a M-homotopy from $f''jij$ to $f'''jkj$ consisting of words having j as a tail. Therefore $f''ji \simeq f'''jk$. Those two words - seen as heads of the M-reduced word f - are themselves M-reduced and of length $n-1$. By induction there exists a word $h \in F(I)$ such that $f''ji \simeq hki \cong hik \simeq f'''jk$.

   As $f''ji$, hki, hik and $f'''jk$ are M-reduced words of length $n-1$ and as $f''ji \simeq hki$ and $hik \simeq f'''jk$, we get $f''j \simeq hk$ and $hi \simeq f'''j$ by induction. Those words, seen as heads of the M-reduced word f, are themselves M-reduced and of length $n-2$.

   By induction, there exist $h', h'' \in F(I)$ such that $f''j \simeq h'jkj \cong h'kjk \simeq hk$ and $hi \simeq h''iji \cong h''jij \simeq f'''j$. All these words are M-reduced of length $n-2$. By induction we get $f'' \simeq h'jk$, $h'kj \simeq h$, $h \simeq h''ij$ and $h''ji \simeq f'''$. Note that $h'kj \simeq h \simeq h''ij$ implies $h'kj \simeq h''ij$, which means that in the following we are able to realize a M-homotopy from $f'i$ to $g'k$ without using the word h. Moreover $h'kj$ and $h''ij$ are M-reduced words (seen as heads of the M-reduced word f) and of length $n-3$. The induction provides $h'k \simeq h''i$.

   These are M-reduced words (seen as heads of the M-reduced word f) of length $n-4$. By induction there exists a word $h''' \in F(I)$ such that $h'k \simeq h'''ik \cong h'''ki \simeq h''i$. By induction, we get $h' \simeq h'''i$ and $h'''k \simeq h''$

   Let

   $$f'' = x_0 \cong x_1 \cong \ldots \cong x_\alpha = h'jk$$

   $$h' = a_0 \cong a_1 \cong \ldots \cong a_\epsilon = h'''i$$

   $$h'''k = b_0 \cong b_1 \cong \ldots \cong b_\lambda = h''$$

   $$h''ji = t_0 \cong t_1 \cong \ldots \cong t_\delta = f'''$$

   be M-homotopies.

   We have a direct M-homotopy from $f'i$ to $g'k$ as follows:

   $f'i \simeq f''iji = x_0iji \cong \ldots \cong x_\alpha iji = h'jkiji = a_0jkiji \cong \ldots \cong a_\epsilon jkiji = h'''ijkiji \cong h'''ijikji \cong h'''jijkji \cong h'''jikjki \cong h'''jkijki \cong h'''jkijik \cong h'''jkjijk \cong h'''kjkijk = b_0jkijk \cong \ldots \cong b_\lambda jkijk = h''jkijk \cong h''jikjk = t_0kjk \cong \ldots \cong t_\delta kjk = f'''kjk \simeq g'k.$

This finishes the proof of the claim.

Suppose that $f'i = f_0, f_1, \ldots, f_m = g'k$ is an M-homotopy. If the number of changes of the last letter in this homotopy is at most 2, then Assertion 1 follows directly from the claim. If it is at least 3 we apply the claim to find a new M-homotopy where the number of these changes is strictly smaller and Assertion 1 follows in this way by an obvious induction.

We come to the proof of Assertion 2.

Let $f \simeq f_1 f_2 \simeq g_1 f_2$, where $f_1,\ f_2,\ g_1 \in F(I)$. We have to prove that $f_1 \simeq g_1$. If $l(f_2) = 1$, then $f_1 \simeq g_1$ has been proved in the previous part (case $i = k$). Suppose $l(f_2) > 1$.

For some $i \in I$ and some $f_2' \in F(I)$, we have $f_2 = f_2' i$. So $f \simeq f_1 f_2' i \simeq g_1 f_2' i$. By Assertion 1, there exists a direct M-homotopy from $f_1 f_2' i$ to $g_1 f_2' i$. In other words there exists an M-homotopy consisting of elementary M-homotopic words such that each one ends with the letter i. We deduce $f_1 f_2' \simeq g_1 f_2'$. By induction, we finally get $f_1 \simeq g_1$. This finishes the proof of Assertion 2. ∎

Proposition 3.2. *Let $f \in F(I)$. If f is M-reduced and fi is not, then there exists an M-reduced word $f' \in F(I)$ such that $f \simeq f'i$.*

Proof. The proof uses induction on $l(f) = n$. For $l(f) = 0$ and $l(f) = 1$, the assertion is trivial. Suppose $n \geq 2$.

Suppose - by way of contradiction - that f is M-reduced, that fi is not M-reduced and that there is no M-reduced $f' \in F(I)$ such that $f \simeq f'i$.

Since fi is not M-reduced, it is M-homotopic to a word of the form $g_1 aag_2$ with $g_1,\ g_2 \in F(I)$ and $a \in I$:

$$fi = f_0 i_0 \cong f_1 i_1 \cong f_2 i_2 \cong \ldots \cong f_k i_k = g_1 aag_2.$$

So either f_k is not M-reduced (if $l(g_2) > 0$), or it is M-reduced and there exists $f_k' \in F(I)$ with $f_k \simeq f_k' i_k$ (which implies $a = i_k$).

We set $l := \max\{0 \leq j < k \mid f_j$ is M-reduced and there is no $f_j' \in F(I)$ with $f_j \simeq f_j' i_j\}$. The natural number l is well-defined by the hypothesis made on f_k.

We consider the word f_{l+1} and distinguish two cases.

1. The word f_{l+1} is not M-reduced.

   - Suppose $m_{i_l i_{l+1}} = 1$.

     We have $i_l = i_{l+1}$ and $f_l i_l \cong f_{l+1} i_{l+1}$. We deduce $f_l \cong f_{l+1}$ and the word f_l is M-reduced by hypothesis. So the word f_{l+1} is M-reduced and we obtain a contradiction.

   - Suppose $m_{i_l i_{l+1}} = 2$.

     We have $i_l \neq i_{l+1}$ and $f_l i_l = h i_{l+1} i_l \cong h i_l i_{l+1} = f_{l+1} i_{l+1}$. Considered as a head of f_l, the word h is M-reduced of length $n-1$. Since $hi_l = f_{l+1}$ and since f_{l+1} is supposed not to be M-reduced, we have that hi_l is not M-reduced. The induction hypothesis provides the existence of a word $h' \in F(I)$ such that $h \simeq h' i_l$.

     So we get $f_l = h i_{l+1} \simeq h' i_l i_{l+1} \cong h' i_{l+1} i_l$, which means that f_l is M-homotopic to a word ending with the letter i_l. Referring to the construction of l, we obtain a contradiction.

   - Suppose $m_{i_l i_{l+1}} = 3$.

     We have $i_l \neq i_{l+1}$ and $f_l i_l = h i_l i_{l+1} i_l \cong h i_{l+1} i_l i_{l+1} = f_{l+1} i_{l+1}$. Considered as a head of f_l, the word h is M-reduced of length $n-2$ and $h i_{l+1} i_l = f_{l+1}$ is not M-reduced.

Suppose that hi_{l+1} is not M-reduced. Then the induction provides the existence of a word $h' \in F(I)$ such that $h \simeq h'i_{l+1}$.
This implies $f_l = hi_li_{l+1} \simeq h'i_{l+1}i_li_{l+1} \cong h'i_li_{l+1}i_l$ and referring to the construction of l, we obtain a contradiction.

So we have that hi_{l+1} is an M-reduced word of length $n-1$ and that $hi_{l+1}i_l$ is not M-reduced. The induction provides the existence of a word $h'' \in F(I)$ such that $hi_{l+1} \simeq h''i_l$. By Proposition 3.1, there exists a word $h''' \in F(I)$ such that $hi_{l+1} \simeq h'''i_{l+1}i_li_{l+1} \cong h'''i_li_{l+1}i_l \simeq h''i_l$. Then, still by Proposition 3.1, there exists a word $h''' \in F(I)$ such that $h \simeq h'''i_{l+1}i_l$. This implies $f_l = hi_li_{l+1} \simeq h'''i_{l+1}i_li_li_{l+1}$. Since f_l is M-reduced, we obtain a contradiction.

2. The word f_{l+1} is M-reduced: there exists a word $f'_{l+1} \in F(I)$ such that $f_{l+1} \simeq f'_{l+1}i_{l+1}$ (by the construction of l).

   By construction, we know that $f_li_l \cong f_{l+1}i_{l+1}$.

   - Suppose $m_{i_li_{l+1}} = 1$.
     We have $i_l = i_{l+1}$ and $f_li_l \cong f_{l+1}i_l$.
     We deduce that $f_l \cong f_{l+1} \simeq f'_{l+1}i_{l+1} = f'_{l+1}i_l$. Referring to the construction of l, we obtain a contradiction.

   - Suppose $m_{i_li_{l+1}} = 2$.
     We have $i_l \neq i_{l+1}$ and $f_li_l = hi_{l+1}i_l \cong hi_li_{l+1} = f_{l+1}i_{l+1}$. Considered as a head of f_l, the word h is M-reduced of length $n-1$. Since $hi_l = f_{l+1}$ and since f_{l+1} is supposed to be M-reduced, we get that hi_l is M-reduced.

     By hypothesis we have $hi_l = f_{l+1} \simeq f'_{l+1}i_{l+1}$, with $f'_{l+1} \in F(I)$. By Proposition 3.1 there exists a word $h' \in F(I)$ such that $hi_l \simeq h'i_{l+1}i_l \cong h'i_li_{l+1} \simeq f'_{l+1}i_{l+1}$. Then, still by Proposition 3.1, we have $h \simeq h'i_{l+1}$. Replacing in the expression of f_l, we get $f_l = hi_{l+1} \simeq h'i_{l+1}i_{l+1}$. This contradicts the fact that f_l is M-reduced.

   - Suppose $m_{i_li_{l+1}} = 3$.
     We have $i_l \neq i_{l+1}$ and $f_li_l = hi_li_{l+1}i_l \cong hi_{l+1}i_li_{l+1} = f_{l+1}i_{l+1}$. Considered as a head of f, the word h is M-reduced of length $n-2$. Since $f_{l+1} = hi_{l+1}i_l$ and since f_{l+1} is M-reduced, the word $hi_{l+1}i_l$ is M-reduced.

     By hypothesis we have $hi_{l+1}i_l = f_{l+1} \simeq f'_{l+1}i_{l+1}$, with $f'_{l+1} \in F(I)$. By Proposition 3.1 there exists a word $h'' \in F(I)$ such that
     $hi_{l+1}i_l \simeq h''i_li_{l+1}i_l \cong h''i_{l+1}i_li_{l+1} \simeq f'_{l+1}i_{l+1}$. Then, again by Proposition 3.1, we have $h \simeq h''i_l$. Replacing in the expression of f_l, we get $f_l = hi_li_{l+1} \simeq h''i_li_li_{l+1}$. This contradicts the fact that f_l is M-reduced.

■

Lemma 3.1. *Let f be an M-reduced word in $F(I)$ and let $i \neq j \in I$ such that fi and fj are M-reduced. Then $fp(i,j)$ is an M-reduced word. In particular, the words f, fi, fj, fij and fji are M-reduced.*

Proof. The last statement is a consequence of the first in view of the fact that heads of M-reduced words are M-reduced.

The proof of the first statement is by contradiction.

1. If $m_{ij} = 2$ then $fp(i,j) = fij$.

   Suppose that fij is not M-reduced. By Proposition 3.2, there exists a word f' M-reduced in $F(I)$ such that $fi \simeq f'j$. Since fi is M-reduced by hypothesis, the word $f'j$ is M-reduced. By Proposition 3.1, there exists a word $h \in F(I)$ such that $fi \simeq hji \cong hij \simeq f'j$. Again by Proposition 3.1, there exists a word $h \in F(I)$ such that $f \simeq hj$. But this yields $fj \simeq hjj$ and contradicts the hypothesis that fj is M-reduced.

2. If $m_{ij} = 3$ then $fp(i,j) = fjij$.

   Similarly as in the previous case one proves that that fji is M-reduced. Suppose now that $fjij$ is not M-reduced. By Proposition 3.2, there is an M-reduced word f'' in $F(I)$ such that $fji \simeq f''j$. As we have just proved that fji is M-reduced, the word $f''j$ is M-reduced too. By Proposition 3.1, there exists a word $h' \in F(I)$ such that $fji \simeq h'iji \cong h'jij \simeq f''j$. Again by Proposition 3.1, there exists a word $h' \in F(I)$ such that $f \simeq h'i$. This leads to $fi \simeq h'ii$ and contradicts the hypothesis that fi is M-reduced.

■

Proof of the fundamental fact

Lemma 3.2. *Let $f, g \in F(I)$ be two M-reduced words and let $j \in I$.*

- *If $f \simeq g$ and if fj is M-reduced, then gj is M-reduced and $gj \simeq fj$.*
- *If $f \simeq g$ and if fj is not M-reduced, then gj is not M-reduced. Moreover if $f', g' \in F(I)$ are such that $f \simeq f'j$ and $g \simeq g'j$, then $f' \simeq g'$.*

Proof. The first assertion is obvious. The second follows from Propositions 3.1 and 3.2. ■

We put $X := \{[f] \mid f \in F(I) \text{ is } M\text{-reduced}\}$. and we define the mapping σ_j from X to X by,

$$\sigma_j([f]) := \begin{cases} [fj] & \text{if } fj \ M\text{-reduced} \\ [f'] & \text{if } fj \text{ not } M\text{-reduced and if } [f'] \in X \text{ such that } [f] = [f'j] \end{cases}$$

Lemma 3.3. *The mapping σ_j is well defined and it is an involutory permutation of X.*

Proof. The first assertion follows from Lemma 3.2. Using Proposition 3.2 one verifies that σ_i^2 is the identity on X. As $\sigma_i([\emptyset]) = [i] \neq [\emptyset]$ it follows that σ_i is an involution for all $i \in I$. ■

Lemma 3.4. *Let $f \in F(I)$ be an M-reduced word and let $i \neq j \in I$. Then the $\langle \sigma_i \sigma_j \rangle$-orbit of $[f]$ has length m_{ij}. In particular, the order of $\sigma_i \sigma_j$ is m_{ij}.*

Proof. Suppose first that f is such that the words fi and fj are both M-reduced. Then it is easily verified (using the last statement of Lemma 3.1) that $\{[f], [fij]\}$ (resp. $\{[f], [fij], [fji]\}$) is the $\langle \sigma_i \sigma_j \rangle$-orbit of $[f]$ if $m_{ij} = 2$ (resp. $m_{ij} = 3$). In particular, the orbit has length m_{ij}.

Suppose now that f is arbitrary. There exists an M-reduced word g such that gi and gj are both M-reduced and such that $[f]$ is in the $\langle \sigma_i, \sigma_j \rangle$-orbit of $[g]$. As $\langle \sigma_i, \sigma_j \rangle$ normalizes $\langle \sigma_i \sigma_j \rangle$, we conclude that the $\langle \sigma_i \sigma_j \rangle$-orbit of $[f]$ has length m_{ij}. ∎

By the universal property of the Coxeter system (W, S) of type M, there is a unique homomorphism from W to $\mathrm{Sym}(X)$ mapping s_i onto σ_i for all $i \in I$. By the previous lemma this implies that m_{ij} divides the order of $s_i s_j$ for all $i, j \in I$. We conclude that $o(s_i s_j) = m_{ij}$ for all $i, j \in I$ which completes the proof of the fundamental fact for the diagram M.

4 The general case

The reduction of the general case to the case of simply laced diagrams without triangles is quite standard. For instance it has been applied for Artin groups by L. Paris in [5].

A covering of a graph $\Gamma = (V, E)$ is a pair (Γ', φ) consisting of a graph $\Gamma' = (V', E')$ and an epimorphism φ from Γ' such that for each $v' \in V'$ the restriction of φ induces a bijection between all neighbours of v' and all neighbours of $\varphi(v')$.

Let M be a simply laced Coxeter diagram over a finite set I whose graph is Γ, let (Γ', φ) be a finite covering of Γ, let M' be the simply laced Coxeter diagram corresponding to Γ' over the finite set I' and let $(W', \{s_x \mid x \in I'\})$ be a system of involutions of type M'. Define for $i \in I$ the involution $s_i \in W'$ as the product over all s_x where x runs through $\varphi^{-1}(i)$. As all those s_x commute, their ordering in the product does not play any role which shows that s_i is well defined. It is straightforward to check that the system $(\langle s_i \mid i \in I \rangle, \{s_i \mid i \in I\})$ is a system of type M.

On the other hand, for each finite graph there exists a finite covering without triangles. Hence systems of involutions of arbitrary simply laced type can be constructed combinatorially from those without triangles using the fact described in the previous paragraph.

Each finite Coxeter diagram can be obtained by a 'folding' of a finite simply laced diagram. This is proved in [3] where 'foldings' correspond to 'admissible partitions' in loc. cit.. Along with such foldings one can produce systems of involutions of arbitrary finite Coxeter diagram as 'subsystems' of finite simply laced ones in a similar way as above. In order to treat the case where I contains infinitely many elements, one considers limits.

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