Homology groups of translation planes and flocks of quadratic cones,I. The structure

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Abstract

The set of translation planes with spreads in $PG(3,q)$ admitting cyclic affine homology groups of order $q+1$ is shown to be equivalent to the set of flocks of quadratic cones in $PG(3,q)$. The analysis is general and considers analogous homology groups in $PG(3,K)$, for K an arbitrary field and corresponding partial flocks of quadratic cones in $PG(3,K)$.

1 Introduction.

There is tremendous interest in what might be called the geometry of flocks of quadratic cones in $PG(3,q)$. This geometry includes certain translation planes whose corresponding spreads are unions of q reguli sharing a common line (see Gevaert, Johnson, Thas [3]), certain generalized quadrangles of type (q^2,q) (see Thas [12]), and translation planes with spreads in $PG(3,q)$ admitting Baer groups of order q (see Johnson [7] and Payne and Thas [10]). In the last situation, there is a deficiency one partial flock of a quadratic cone due to the work of Johnson [7]. Furthermore, partial flocks of deficiency one may be extended uniquely to flocks by Payne and Thas [10]. There are also connections to sets of ovals, called 'herds' (see, e.g., [9]), the existence of which provides a more general extension theory when q is even (see Thas and Storme [11]). The reader interested in these and other connections is referred to the survey article by Johnson and Payne [9].

Recently, another sort of miraculous connection has emerged due to the work of Baker, Ebert and Penttila [1]. This is, that flocks of quadratic cones in $PG(3,q)$ are equivalent to the so-called 'regular hyperbolic fibrations with constant back half.'

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A 'hyperbolic fibration' is a set $\mathcal{Q}$ of $q-1$ hyperbolic quadrics and two carrying lines L and M such that the union $L \cup M \cup \mathcal{Q}$ is a cover of the points of $PG(3,q)$. (More generally, one could consider a hyperbolic fibration of $PG(3,K)$, for K an arbitrary field, as a disjoint covering of the points by a set of hyperbolic quadrics union two carrying lines.) The term 'regular hyperbolic fibration' is used to describe a hyperbolic fibration such that for each of its $q-1$ quadrics, the induced polarity interchanges L and M. When this occurs, and (x_1,x_2,y_1,y_2) represent points homogeneously, the hyperbolic quadrics have the form

$$V(x_1^2a_i + x_1x_2b_i + x_2^2c_i + y_1^2e_i + y_1y_2f_i + y_2^2g_i)$$

for $i = 1,2,\ldots,q-1$ (the variety defined by the fibration). When $(e_i,f_i,g_i) = (e,f,g)$ for all $i = 1,2,\ldots,q-1$, the regular hyperbolic quadric is said to have 'constant back half'.

The main theorem of Baker, Ebert and Penttila [1] is equivalent to the following.

Theorem 1. (Baker, Ebert, Penttila [1])

(1) *Let* $\mathcal{H} : V(x_1^2a_i + x_1x_2b_i + x_2^2c_i + y_1^2e + y_1y_2f + y_2^2g)$ *for* $i = 1,2,\ldots,q-1$ *be a regular hyperbolic fibration with constant back half.*

Consider $PG(3,q)$ *as* (x_1,x_2,x_3,x_4) *and let* C *denote the quadratic cone with equation* $x_1x_2 = x_3^2$.

Define

$$\pi_0 : x_4 = 0,\ \pi_i : x_1a_i + x_2c_i + x_3b_i + x_4 = 0 \quad \text{for } 1,2,\ldots,q-1.$$

Then

$$\{\pi_j, j = 0,1,2,\ldots,q-1\}$$

is a flock of the quadratic cone C.

(2) *Conversely, if* $\mathcal{F}$ *is a flock of a quadratic cone, choose a representation* $\{\pi_j, j = 0,1,2,\ldots,q-1\}$ *as above. Choose any convenient constant back half* (e,f,g), *and define* $\mathcal{H}$ *as* $V(x_1^2a_i + x_1x_2b_i + x_2^2c_i + y_1^2e + y_1y_2f + y_2^2g)$ *for* $i = 1,2,\ldots,q-1$. *Then* $\mathcal{H}$ *is a regular hyperbolic fibration with constant back half.*

Now for each of the $q-1$ reguli, choose one of the two reguli of totally isotropic lines. Such a choice will produce a spread and a translation plane. Hence, there are potentially 2^{q-1} possible translation planes obtained in this way.

In this article, we intend to connect flocks of quadratic cones with the translation planes obtained from regular hyperbolic quadrics with constant back halves in a more direct manner.

In particular, we prove:

Theorem 2. *Translation planes with spreads in* $PG(3,q)$ *admitting cyclic affine homology groups of order* $q+1$ *are equivalent to flocks of quadratic cones.*

In this setting, it is possible to prove that the component orbits of the cyclic homology group H form reguli.

More generally, we consider translation planes with spreads in $PG(3,K)$, for K an arbitrary field, that admit homology groups, one component orbit of which is a regulus in $PG(3,K)$. When this occurs, we obtain:

Theorem 3. *Let π be a translation plane with spread in $PG(3,K)$, for K a field. Assume that π admits an affine homology group H, so that some orbit of components is a regulus in $PG(3,K)$.*

(1) *Then π produces a regular hyperbolic fibration with constant back half.*

(2) *Conversely, each translation plane obtained from a regular hyperbolic fibration with constant back half admits an affine homology group H, one orbit of which is a regulus in $PG(3,K)$.*

The group H is isomorphic to a subgroup of the collineation group of a Pappian spread Σ, coordinatized by a quadratic extension field K^+, $H \simeq \langle h^{\sigma+1}; h \in K^+ - \{0\}\rangle$, where σ is the unique involution in $Gal_K K^+$.

(3) *Let $\mathcal{H}$ be a regular hyperbolic fibration with constant back half of $PG(3,K)$. The subgroup of $\Gamma L(4,K)$ that fixes each hyperbolic quadric of $\mathcal{H}$ and acts trivially on the front half is isomorphic to $\langle \rho, \langle h^{\sigma+1}; h \in K^+ - \{0\}\rangle\rangle$, where ρ is defined as follows: If $e^2 = ef + g$, f, g in K and $\langle e, 1\rangle_K = K^+$, then ρ is $\begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix}$, where $P = \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix}$.*

In particular, $\langle h^{\sigma+1}; h \in K^+ - \{0\}\rangle$ fixes each regulus and opposite regulus of each hyperbolic quadric of $\mathcal{H}$ and ρ inverts each regulus and opposite regulus of each hyperbolic quadric.

When K is a finite field, the associated homology group is cyclic of order $q+1$ and there is an associated flock of a quadratic cone by the work of Baker, Ebert and Penttila [1]. However, when K is an infinite field, this may no longer be true. The relevant theorem is as follows:

Theorem 4. (1) *A regular hyperbolic fibration with constant back half in $PG(3,K)$, K a field, with carrier lines $x=0, y=0$ may be represented as follows:*

$$V\left(x\begin{bmatrix} \delta & \mathcal{G}(\delta) \\ 0 & -\mathcal{F}(\delta) \end{bmatrix} x^t - y\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} y^t\right)$$

$$\text{for all } \delta \text{ in } \det K^+ = \left\{\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}^{\sigma+1}; u,t \in K, (u,t) \neq (0,0)\right\},$$

where

$$\left\{\begin{bmatrix} \delta & \mathcal{G}(\delta) \\ 0 & -\mathcal{F}(\delta) \end{bmatrix}; \delta \in \left\{\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}^{\sigma+1}; u,t \in K, (u,t) \neq (0,0)\right\}\right\} \cup \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$$

corresponds to a partial flock of a quadratic cone in $PG(3,K)$, for $\mathcal{F}$ and $\mathcal{G}$ functions on $\det K^+$.

(2) *The correspondence between any spread π corresponding to the hyperbolic fibration and the partial flock is as follows:*

If π is

$$x = 0, y = 0, y = x\begin{bmatrix} u & t \\ F(u,t) & G(u,t) \end{bmatrix},$$

then the partial flock is given by $\begin{bmatrix} \delta_{u,t} & \mathcal{G}(\delta_{u,t}) \\ 0 & -\mathcal{F}(\delta_{u,t}) \end{bmatrix}$, *with*

$$\begin{aligned}
\delta_{u,t} &= \det \begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}, \\
\mathcal{G}(\delta_{u,t}) &= (uG(u,t) + tF(u,t)) \cdot g + 2(uF(u,t) - tfG(u,t)), \\
-\mathcal{F}(\delta_{u,t}) &= \delta_{F(u,t),G(u,t)},
\end{aligned}$$

where

$$\delta_{F(u,t),G(u,t)} = \det \begin{bmatrix} F(u,t) & G(u,t) \\ fG(u,t) & F(u,t)+gG(u,t) \end{bmatrix} \in \det K^{+}.$$

(3) *The corresponding functions*

$$\phi_s(t) = s^2 t + s\mathcal{G}(t) - \mathcal{F}(t)$$

are injective for all s in K and for all $t \in \det K^{+}$.

Indeed, the functions restricted to $\det K^{+}$ *are surjective on* $\det K^{+}$*.*

(4) *Conversely, any partial flock of a quadratic cone in $PG(3,K)$, with defining set λ (i.e., so t ranges over λ and planes of the partial flock are defined via functions in t) equal to* $\det K^{+}$*, whose associated functions on* $\det K^{+}$*, as above, are surjective on* $\det K^{+}$ *(K^{+} some quadratic extension of K), produces a regular hyperbolic fibration in $PG(3,K)$ with constant back half.*

So, the question at least for finite translation planes is to find cyclic homology groups of order $q+1$ in translation planes with spreads in $PG(3,q)$. It is, of course, possible to find homology groups of order $q+1$ without these being cyclic. However, for certain orders, we may avoid this assumption. We are able to prove:

Theorem 5. *Let π be a translation plane with spread in $PG(3,q)$ that admits an affine homology group H of order $q+1$ in the translation complement. If any of the following conditions hold, π defines a regular hyperbolic fibration with constant back half and hence a corresponding flock of a quadratic cone:*

(1) *q is even,*
(2) *q is odd and $q \equiv 1 \bmod 4$,*
(3) *H is Abelian,*
(4) *H is cyclic.*

2 Homology Groups and Regular Hyperbolic Fibrations.

Lemma 1. (1) *Let $\mathcal{H}$ be a hyperbolic fibration of $PG(3,K)$, for K a field (a covering of the points by a set λ of mutually disjoint hyperbolic quadrics union two disjoint lines having an empty intersection with any of the quadrics). For each quadric in λ, choose one of the two reguli (a regulus or its opposite). The union of these reguli and the "carrying lines" form a spread in $PG(3,q)$.*

(2) *Conversely, any spread in $PG(3,K)$ that is a union of hyperbolic quadrics union two disjoint carrying lines (i.e. two lines not contained in any of the reguli) produces a hyperbolic fibration.*

Lemma 2. *Let π be a translation plane with spread in $PG(3,K)$, for K a field, that admits a homology group H, such that some orbit of components is a regulus in $PG(3,K)$. Let Γ be any H-orbit of components.*

Then there is a unique Pappian spread Σ containing Γ and the axis and coaxis of H.

Proof. Note that any regulus net and any component disjoint from the elements of that regulus may be embedded into a unique Pappian spread. Such a Pappian spread Σ may be coordinatized by a field extension of K, say K^+. Then there is a representation of that regulus within Σ as follows:

$$y = xm; m^{\sigma+1} = 1, \text{ for } m \in K^+.$$

Let the homology group H have coaxis $x = 0$ and axis $y = 0$ in the associated 4-dimensional vector space. Thus, we may represent the group H by

$$\left\langle \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}; T^{q+1} = I, T \in K^+ \right\rangle.$$

Let $y = xM$ be any component, where M is a non-singular matrix. We claim that there is unique associated Pappian spread containing

$$\left\{ x = 0, y = 0, y = xMT; T^{\sigma+1} = I \right\}.$$

To see this, change bases by $\begin{bmatrix} M^{-1} & 0 \\ 0 & I \end{bmatrix}$ to obtain Σ as

$$x = 0, y = xm; m \in K^+,$$

then change bases back to obtain the field MK^+M^{-1} and the associated Pappian spread $\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \Sigma$ containing the indicated set. Note that this is a regulus with $x = 0, y = 0$ adjoined in that Pappian spread. Since this Pappian spread may be coordinatized by an extension field of the kernel K, it follows that any such regulus (image under H) is a regulus in $PG(3,K)$. ∎

Lemma 3. *Under the assumptions of the previous lemma, we may represent the coaxis, axis and Γ as follows:*

$$x = 0, y = 0, y = xm; m^{\sigma+1} = 1; m \in K^+,$$

where m is in the field K^+, a 2-dimensional quadratic extension of K, and σ is the unique involution in $Gal_K K^+$.

A basis may be chosen so that Σ may be coordinatized by K^+ as $\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}$ *for all u, t in K, for suitable constants f and g.*

Proof. This follows directly from the previous lemma and the fact that the field coordinatizing a Pappian spread in $PG(3,K)$ may be defined by an irreducible quadratic over K. ∎

Lemma 4. *Under the previous assumptions, if* $\{1,e\}$ *is a basis for* K^+ *over* K *then* $e^2 = eg+f$, *and* $e^\sigma = -e+g$, $e^{\sigma+1} = -f$. *Furthermore,* $(et+u)^{\sigma+1} = 1$ *if and only if in matrix form* $et+u = \begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}$, *such that* $u(u+gt) - ft^2 = 1$.

The opposite regulus

$$y = x^\sigma m; m^{\sigma+1} = 1$$

may be written in the form

$$y = x\begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix}\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}; u(u+gt) - ft^2 = 1.$$

Proof. Choose a basis for the Pappian spread containing $x = 0, y = 0, y = x$, with the following form:

$$x = 0, y = 0, y = x\begin{bmatrix} u & t \\ tf & u+gt \end{bmatrix}; u,t \in K,$$

g, f appropriate constants, f non-zero.

In this context, we have a basis $\{1,e\}$ such that $e^2 = eg+f$. Notice that $et+u$ as a matrix is $\begin{bmatrix} u & t \\ tf & u+gt \end{bmatrix}$, in the basis set $\{\{1,e\},\{1,e\}\}$.

Since $e^{\sigma+1} \in K$, let $e^{\sigma+1} = \alpha$, and $e^\sigma + e = \rho$, for α, ρ in K, then $\alpha + e^2 = e\rho$, implying that $\rho = g$ and $\alpha = -f$.

Thus, $e^\sigma = -e+g$ and $e^{\sigma+1} = -f$.

We note that $(ex_2 + x_1)^\sigma = (x_1, x_2)\begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix}$.

If $et+u$ is such that $(et+u)^{\sigma+1} = 1$ then

$$\begin{aligned}(et+u)^{\sigma+1} &= 1 = (e^\sigma t+u)(et+u) \\ &= e^{\sigma+1}t^2 + e^\sigma ut + eut + u^2 \\ &= -ft^2 + (-e+g)ut + eut + u^2 \\ &= u^2 + utg - ft^2 = \det\begin{bmatrix} u & t \\ tf & u+gt \end{bmatrix}.\end{aligned}$$

Since $y = x^\sigma m = x\begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix} m$, it follows that the opposite regulus may be written as stated in the lemma. ∎

Lemma 5. *If* π *is a translation plane of order* q^2 *with spread in* $PG(3,K)$ *admitting a homology group* H *such that one component is a regulus in* $PG(3,K)$, *then, choosing the axis of* H *as* $y = 0$ *and the coaxis as* $x = 0$, *we have the following form for the elements of* H

$$\begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix}; T^{\sigma+1} = I.$$

Furthermore, we may realize the matrices T *in the form* $\begin{bmatrix} u & t \\ tf & u+gt \end{bmatrix}$ *such that* $u(u+gt) - t^2f = 1$.

Proof. This follows directly from the previous lemmas. ∎

We then obtain the following representation of the regulus corresponding to the orbit of $y = x$ under H.

Lemma 6. *The associated hyperbolic quadric corresponding to* $(y = x)H$: *has the following form:*

$$V[1, g, -f, -1, -g, f].$$

Using this homology group, we obtain:

Lemma 7. *It is possible to represent the spread as follows:*

$$x = 0, y = 0, y = xM_iT \quad \textit{for } i \in \rho, T^{\sigma+1} = I,$$

where $i \in \rho$*, some index set. We thus have a set of* ρ *reguli that have the property that* $x = 0$ *and* $y = 0$ *are interchanged by the polarity induced by the associated hyperbolic quadric.*

Proof. Since $GL(2, K^+)$, where K^+ is the quadratic extension field of K coordinatizing Σ, is triply transitive on the components of Σ, this implies that any regulus (orbit Γ of π) may be chosen to have the form $y = xm; m^{\sigma+1} = 1$. However, what we have not yet shown is that what we have called $x = 0, y = 0$ are the coaxis and axis of the homology group when it acts in π. In other words, when we choose Γ to the have the required form, do the axis and coaxis of the group H have the form $x = 0, y = 0$? We do have a homology group H^A that acts transitively on Γ when it sits in Σ.

So, we consider this from a different perspective. Suppose we have chosen the coaxis and axis as $x = 0, y = 0$ in Σ and then ask if Γ is one of the 'André' reguli; $y = xm; m^{\sigma+1} = \alpha$ for some α in K. We will show that this is, in fact, the case. But in order to do this, we require that we know the form of the Baer subplanes of any regulus of Σ.

We claim that if π_o is a Baer subplane of Σ that is disjoint from $x = 0$ and $y = 0$ then there is a unique pair m, n of elements of K^+ such that π_o is $y = x^\sigma m + xn$.

To see this, we note that since the group fixing $x = 0, y = 0$ is transitive on the remaining components of Σ and the kernel homology group of Σ is transitive on the points of components of Σ, we may choose π_o to contain $(1, 1)$, without loss of generality. Let π_o also contain the point (a, b) so that π_o is the set of points $\alpha(1, 1)+\beta(a, b)$, for $\alpha, \beta \in K$. Since π_o is not a component, then $a \neq b$ (as otherwise, π_o would be $y = x$). Hence, a and b are both in $K^+ - K$ since otherwise π_o would non-trivially intersect $x = 0$ or $y = 0$. We now want to solve the following system of equations uniquely for $m \neq 0$ and n:

$$m + n = 1,$$
$$a^\sigma m + an = b.$$

Since the determinant of the coefficient matrix is $a^\sigma - a \neq 0$, there is a unique solution for m and n. Note that if $m = 0$ then $a = b$, since n is not 0, a contradiction. Since $y = x^q m + xn$ is a 2-dimensional K-subspace sharing the basis for π_o, these two subspaces are equal. Hence, we may represent subplanes in the manner in question.

What this means is that we may choose the coaxis, axis and Γ as

$$x = 0, y = 0, y = xm; m^{\sigma+1} = 1,$$

without loss of generality.

The quadratic form for Γ is

$$V\left(x\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}x^t - y\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}y^t\right), \text{ where } x = (x_1, x_2), y = (y_1, y_2)$$

and (x_1, x_2, y_1, y_2) is the representation in $\{\{1, e\}, \{1, e\}\}$ on $x = 0, y = 0$.

Recall that now Γ is $y = x\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}; u(u+gt) - ft^2 = 1$. Now directly check that

$$x\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}x^t - x\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}^t x^t = 0.$$

Also, note that

$$x\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}x^t - x\begin{bmatrix} 1 & g \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}\begin{bmatrix} 1 & g \\ 0 & -1 \end{bmatrix}^t x^t$$
$$= x\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}x^t - x\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}^t x^t$$

and since $x\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}^t x^t = x\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}x^t$, we have $y = x^\sigma\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix} = x\begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix}\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}$, where $u(u+gt) - ft^2 = 1$, is the set of opposite lines and these are also isotropic under the indicated quadratic form. This proves the lemma. ■

Lemma 8. (1) *The spread for π has the following form:*

$$x = 0, y = 0, y = xM_i\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}; u(u+gt) - ft^2 = 1$$

and M_i a set of 2×2 matrices over K, where $i \in \rho$, some index set. Let

$$R_i = \left\{y = xM_iT; T^{\sigma+1} = 1\right\} \quad \textit{for } i \in \rho, \textit{ where } M_1 = I.$$

(b) *Then R_i is a regulus in $PG(3, K)$.*

(c)

$$R_i^* = \left\{y = xM_i\begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix}T; T^{\sigma+1} = I\right\}$$

represents the opposite regulus to R_i.

Proof. It remains to prove (c). We know that the opposite regulus to $R_1 = \{y = xT;$ $T^{\sigma+1} = 1\}$ is $\left\{y = x^\sigma T = x\begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix} T; T^{\sigma+1} = 1\right\}$. Consider $R_i = \{y = xM_iT;$ $T^{\sigma+1} = 1\}$ and change bases by $\tau_{M_i} : (x, y) \longmapsto (x, yM_i^{-1})$ to change R_i into R_1. Then R_1^* maps to R_i^* under $\tau_{M_i}^{-1}$. This implies that

$$R_i^* = \left\{y = xM_i \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix} T; T^{\sigma+1} = 1\right\}.$$

■

We now determine the associated quadratic forms.

Lemma 9. *The quadratic form for R_i is*

$$V\left(xM_i \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} M_i^t x^t - y \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} y^t\right).$$

Proof.

$$\begin{aligned} &xM_i \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} M_i^t x^t - xM_iT \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} T^t M_i^t x^t \\ &\qquad = xM_i \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} M_i^t x^t - xM_i \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} M_i^t x^t. \end{aligned}$$

This shows that the components of R_i are isotropic subspaces. Now note that the opposite regulus R_i^* to R_i has components

$$y = xM_i \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix} T; T^{\sigma+1} = 1.$$

Then

$$\begin{aligned} &xM_i \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} M_i^t x^t - xM_i \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix} T \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} T^t \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix}^t M_i^t x^t \\ &= xM_i \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} M_i^t x^t - xM_i \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix} \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix}^t M_i^t x^t \\ &= xM_i \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} M_i^t x^t - xM_i \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}^t M_i^t x^t. \end{aligned}$$

Note that

$$xM_i \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}^t M_i^t x^t \text{ is self-transpose and}$$

$$\text{thus equal to } xM_i \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}^t M_i^t x^t.$$

This proves the lemma. ■

Lemma 10. *The regulus*

$$R_i = \left\{ y = xM_iT;\, T^{\sigma+1} = 1 \right\} \quad \textit{for } i \in \rho, \textit{ where } M_1 = I$$

and its opposite regulus

$$R_i^* = \left\{ y = xM_i \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix} T; T^{\sigma+1} = 1 \right\}$$

are interchanged by the mapping

$$\rho = \begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix}, \textit{where } P = \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix}; \; (x, y) \longmapsto (x, y^\sigma).$$

Proof. $y = xM_iT$ maps to $y = xM_iTP = xM_iP(P^{-1}TP)$, and notice that $P^{-1}TP = T^\sigma$, if σ is an automorpism of K^+ and $T \in K$. Since $T^{\sigma(\sigma+1)} = T^{\sigma^2+\sigma} = T^{\sigma+1} = 1$, we have the proof. ∎

We may now state and prove our main theorem.

Theorem 6. *Let π be a translation plane with spread in $PG(3, K)$, for K a field. Assume that π admits an affine homology group H so that some orbit of components is a regulus in $PG(3, K)$.*

(1) *Then π produces a regular hyperbolic fibration with constant back half.*

(2) *Conversely, each translation plane obtained from a regular hyperbolic fibration with constant back half admits an affine homology group H, one orbit of which is a regulus in $PG(3, K)$.*

H is isomorphic to a subgroup of the collineation group of a Pappian spread Σ, coordinatized by a quadratic extension field K^+, $H \simeq \langle h^{\sigma+1}; h \in K^+ - \{0\}\rangle$, where σ is the unique involution in $Gal_K K^+$.

(3) *Let $\mathcal{H}$ be a regular hyperbolic fibration with constant back half of $PG(3, K)$, for K a field. The subgroup of $\Gamma L(4, K)$ that fixes each hyperbolic quadric of the regular hyperbolic fibration $\mathcal{H}$ and acts trivially on the front half is isomorphic to $\langle \rho, \langle h^{\sigma+1}; h \in K^+ - \{0\}\rangle\rangle$, where ρ is defined as follows: If $e^2 = ef + g$, f, g in K and $\langle e, 1\rangle_K = K^+$ then ρ is $\begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix}$, where $P = \begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix}$.*

In particular, $\langle h^{\sigma+1}; h \in K^+ - \{0\}\rangle$ fixes each regulus and opposite regulus of each hyperbolic quadric of $\mathcal{H}$ and ρ inverts each regulus and opposite regulus of each hyperbolic quadric.

Proof. It remains to prove (3). Choose $x = 0, y = 0$ as the two lines (components) interchanged by the associated polarity of each hyperbolic quadric of $\mathcal{H}$, a regular hyperbolic quadric with constant back half. Fix a quadric and choose either regulus of this quadric. We have seen above that there is a unique Pappian spread containing the regulus and $x = 0$, $y = 0$. Choose the quadratic extension field of K with basis $\{e, 1\}$, where $e^2 = ef + g$. Then the quadric has the following form:

$$V\left(x \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} x^t - y \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} y^t \right).$$

As any quadric of $\mathcal{H}$ has constant back half, we note that

$$\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}^t = \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}$$

where $u(u+gt) - ft^2 = 1$. Furthermore,

$$\begin{aligned} y\begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix}\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}\begin{bmatrix} 1 & o \\ g & -1 \end{bmatrix}^t y^t &= y\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix}^t y^t \\ &= y\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} y^t. \end{aligned}$$

Since the quadric has constant back half, we see that the mappings by elements of K^+ of determinant 1 of the first type occur as homology groups of each translation plane obtained from the hyperbolic fibration, each of whose orbits define reguli. If $y = xN_i$ for $i \in \tau$ is a regulus of some hyperbolic fibration in some spread, then since we now have a homology group acting on the spread, it follows that

$$\{y = xN_i \text{ for } i \in \tau\} = \left\{ y = xM_i\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}; u(u+gt) - ft^2 = 1 \right\},$$

for $i \in \rho$, where ρ is an appropriate index set.

We have noted previously that

$$\left\{ y = xM_i\begin{bmatrix} 1 & 0 \\ g & -1 \end{bmatrix}\begin{bmatrix} u & t \\ ft & u+gt \end{bmatrix}; u(u+gt) - ft^2 = 1 \right\}$$

is the associated opposite regulus and that the mapping ρ will interchange the regulus and the opposite regulus. Note that ρ is just the matrix version of x^σ.

Now let k be an element of $\Gamma L(4, K)$ that fixes each hyperbolic quadric of $\mathcal{H}$ and acts trivially on the front half of each quadric. Thus, we may represent k in the form

$$k : (x, y) \to (x, y^\tau Q),$$

where τ is an automorphism of the field K^+ coordinatizing Σ and Q is a non-zero element of K^+. We will see that if $Q^{\sigma+1} = 1$ or $\tau = \sigma$, the unique involution in $Gal_K K^+$, then k certainly satisfies the conditions. However, we see that we must have

$$y^\tau Q\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} Q^t y^{\tau t} = y\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} y^t \quad \forall\, y \in K^+.$$

Now we note that k will fix each hyperbolic quadric of $\mathcal{H}$. We also know that the mapping $(x, y) \to (x, y^\sigma)$ interchanges the regulus with its opposite regulus in each such quadric. Suppose that k fixes all reguli of the hyperbolic quadrics. Then k will be a collineation of each spread corresponding. This means that since $y = 0$ is fixed pointwise, the element k is an affine homology of any of the spreads. But this means that k is in $GL(4, K)$, when acting on the spread. This, in turn, implies that τ is 1 or σ. But, if $\tau = \sigma$, k does not act on a spread as a collineation. But then $Q\begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} Q^t = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $\tau = 1$, an easy calculation shows that $a = 1, d = -f$

and $b+c=g$. Working out the form of $Q\left[\begin{array}{cc}1 & g\\ 0 & -f\end{array}\right]Q^t$ shows that the element a is the determinant of Q. Hence, if $\tau=1$, we obtain the det 1 group; i.e., $Q^{\sigma+1}=1$.

So, we may assume that k does not fix all reguli of each quadric. If k fixes one regulus of a quadric, then g is a collineation group of an associated Pappian spread defined by the regulus and $x=0$ and $y=0$. The above remarks show that τ is still one in this setting. Thus, we may assume that k interchanges each regulus and opposite regulus of each hyperbolic quadric. If we follow the mapping $(x,y)\rightarrow(x,y^{\sigma})$ by k, it follows that $Q^{\sigma+1}=1$. But, then it also now is immediate that $\tau=\sigma$ or 1. This proves the theorem. ∎

In the finite case, we may improve this as follows:

Theorem 7. *Let π be a translation plane with spread in $PG(3,q)$. Assume that π admits a cyclic affine homology group of order $q+1$.*

Then π produces a regular hyperbolic fibration with constant back half.

Proof. By Jha and Johnson [6], any orbit of components Γ is a derivable net sitting in a Desarguesian spread Σ also containing $x=0,y=0$. This is a unique Desarguesian spread so the group H acts as a group of the Desarguesian spread. Let K^* denote the kernel homology group of π. Then, K^* fixes each component of the derivable net and $x=0,y=0$. It follows easily that K^* is a collineation group of Σ that fixes at least $q+3$ components. Clearly, this means that K^* fixes all components of Σ, as Σ is Desarguesian and K^* then sits in $\Gamma L(2,q^2)$. So K^* is a kernel homology group of Σ. But this means that Σ is coordinatized by a field extension of the kernel K isomorphic to $GF(q)$. Hence, Γ is a regulus in $PG(3,K)$ and our more general result applies. ∎

Now since a regular hyperbolic fibration with constant back half produces translation planes with spreads in $PG(3,q)$ admitting cyclic affine homology groups of order $q+1$, we obtain the following corollary.

Corollary 1. *Finite regular hyperbolic fibrations with constant back half are equivalent to translation planes with spreads in $PG(3,q)$ that admit cyclic homology groups of order $q+1$.*

3 Det K^+-Partial Flocks of Quadratic Cones.

We now consider more completely the forms of the quadrics associated with translation planes admitting a homology group as above.

3.1 The Forms:

Let π be a translation plane with spread in $PG(3,K)$ that produces a hyperbolic fibration with carriers $x=0,y=0$. Represent the spread of π as follows:

$$x=0,y=0,y=x\left[\begin{array}{cc}u & t\\ F(u,t) & G(u,t)\end{array}\right];u,t\in K,$$

for functions F and G on $K \times K$ to K.

Let $\delta_{u,t} = \det \begin{bmatrix} u & t \\ F(u,t) & G(u,t) \end{bmatrix}$. We may assume that when $u(u+gt)-ft^2 = 1$ then $F(u,t) = ft$ and $G(u,t) = u + gt$. We now compute

$$V\left(xM_i \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} M_i^t x^t - y \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} y^t\right).$$

The operand

$$xM_i \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} M_i^t x^t - y \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} y^t$$

for $M_i = \begin{bmatrix} u & t \\ F(u,t) & G(u,t) \end{bmatrix}$ is easily calculated as follows:

$$\begin{aligned} x \begin{bmatrix} u^2 + (ug-tf)t = \delta_{u,t} & uF(u,t) + (ug-tf)G(u,t) \\ F(u,t)u + (gF(u,t) - fG(u,t))t & F(u,t)^2 + (gF(u,t) - fG(u,t))G(u,t) \end{bmatrix} x^t \\ - y \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} y^t. \end{aligned}$$

We first show that the $(2,2)$-entry in the front half of the quadric is a function of $\delta_{u,t}$. If not, consider $y = xM_i$, corresponding to $\delta_{u,t}$, and $y = xM_j$, corresponding to δ_{u^*,t^*}, and assume that the corresponding $(2,2)$-elements are equal in both front halves. Then let $x = (0, x_2)$, and realize that then $(0, x_2, (0, x_2)M_i)$ is on the $\delta_{u,t}$-quadric and $(0, x_2, (0, x_2)M_j)$ is on the δ_{u^*,t^*}-quadric. This implies that

$$((0,x_2)M_i) \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} ((0,x_2)M_i)^t = ((0,x_2)M_j) \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} ((0,x_2)M_j)^t.$$

But this means that $(0, x_2, (0, x_2)M_i)$ is on both quadrics, a contradiction. Hence, this says that $\delta_{u,t}$ is a function of the $(2,2)$-entry, and since the argument is essentially symmetric, we have that the $(2,2)$-entry is a function of the $(1,1)$-entry, say $\mathcal{F}(\delta_{u,t})$. Similarly, if the sum of the $(1,2)$- and the $(2,1)$-entries of the front half is not a function of $\delta_{u,t}$ then there would be two distinct sums for a given $\delta_{u,t}$. But this again would say that $(x_1, 0, (x_1, 0)M_i)$ would be in two quadrics. Hence, the sum of the $(1,2)$- and $(2,1)$-elements is a function of $\delta_{u,t}$, say $\mathcal{G}(\delta_{u,t})$. Consider a corresponding translation plane with components $y = xM$ and $y = xN$. Notice that the $(2,2)$-element of the previous matrix for the front half is $\det \begin{bmatrix} F(u,t) & G(u,t) \\ fG(u,t) & F(u,t) + gG(u,t) \end{bmatrix}$.

Hence, we obtain the next result, noting that the mapping from K^+ to K, mapping k in K^+ to $\det k$ in K may not be onto.

We now prove the results on associated partial flocks of quadratic cones, the union of which is stated in the introduction.

Theorem 8. *A regular hyperbolic fibration with constant back half in $PG(3,K)$, K a field, with carrier lines $x=0, y=0$, may be represented as follows:*

$$V\left(x\begin{bmatrix}\delta & \mathcal{G}(\delta)\\ 0 & -\mathcal{F}(\delta)\end{bmatrix}x^t - y\begin{bmatrix}1 & g\\ 0 & -f\end{bmatrix}y^t\right)$$
$$\text{for all } \delta \text{ in } \det K^+ = \left\{\begin{bmatrix}u & t\\ ft & u+gt\end{bmatrix}^{\sigma+1}; u,t\in K, (u,t)\neq(0,0)\right\},$$

where

$$\left\{\begin{bmatrix}\delta & \mathcal{G}(\delta)\\ 0 & -\mathcal{F}(\delta)\end{bmatrix}; \delta\in\left\{\begin{bmatrix}u & t\\ ft & u+gt\end{bmatrix}^{\sigma+1}; u,t\in K, (u,t)\neq(0,0)\right\}\right\}\cup\left\{\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right\}$$

corresponds to a partial flock of a quadratic cone in $PG(3,K)$, and where $\mathcal{F}$ and $\mathcal{G}$ are functions on $\det K^+$.

Proof. Let λ be a subset of K such that for each t in $K-\{0\}$, the function $\phi_s(t) = s^2t + s\mathcal{G}(t) - \mathcal{F}(t)$ is injective for each element s in K. Then there is a corresponding partial flock of a quadratic cone in $PG(3,K)$. (See, e.g., Johnson and Lin [8]). The partial flock is a flock if and only $\phi_s(t)$ is bijective for all s in K. For $\lambda = \{\det K^+\}$, assume that $\phi_s(t)$ is not injective for some element s, so assume that $\phi_s(t) = \phi_s(t^*)$, for $t \neq t^*$. Then, consider $(s,1,y_1,y_2)$, where $y = (y_1,y_2)$ and this point is on the hyperbolic quadric corresponding to t. It is easy to check that this point would also be on the hyperbolic quadric corresponding to t^*, a contradiction.

This, combined with our previous comments, proves everything in part (1), with the exception of the surjectivity of the functions ϕ_s on $\det K^+$. We note that $y\begin{bmatrix}1 & g\\ 0 & -f\end{bmatrix}y^t$ maps $y=(y_1,y_2)$ onto $\det\begin{bmatrix}y_1 & y_2\\ fy_2 & y_1+gy_2\end{bmatrix}$. Hence, the functions listed must be surjective on $\det K^+$ in order that the hyperbolic fibration cover $PG(3,K)$. ∎

Our preliminary remarks prove the following theorem:

Theorem 9. *The correspondence between any spread π in $PG(3,K)$ corresponding to the hyperbolic fibration and the partial flock of a quadratic cone in $PG(3,K)$ is as follows:*

If π is

$$x=0, y=0, y=x\begin{bmatrix}u & t\\ F(u,t) & G(u,t)\end{bmatrix}$$

then the partial flock is given by $\begin{bmatrix}\delta_{u,t} & \mathcal{G}(\delta_{u,t})\\ 0 & -\mathcal{F}(\delta_{u,t})\end{bmatrix}$ *with*

$$\delta_{u,t} = \det\begin{bmatrix}u & t\\ ft & u+gt\end{bmatrix},$$
$$\mathcal{G}(\delta_{u,t}) = g(uG(u,t)+tF(u,t)) + 2(uF(u,t) - tfG(u,t)),$$
$$-\mathcal{F}(\delta_{u,t}) = \delta_{F(u,t),G(u,t)},$$

where

$$\delta_{F(u,t),G(u,t)} = \det \begin{bmatrix} F(u,t) & G(u,t) \\ fG(u,t) & F(u,t)+gG(u,t) \end{bmatrix} \in \det K^{+}.$$

Theorem 10. *If we have a hyperbolic fibration in $PG(3,K)$, there are corresponding functions given in the previous theorem such that the corresponding functions*

$$\phi_s(t) = s^2t + s\mathcal{G}(t) - \mathcal{F}(t)$$

are injective for all s in K and for all $t \in \det K^{+}$.

Indeed, the functions restricted to $\det K^{+}$ *are surjective on* $\det K^{+}$*.*

We now consider the converse.

Theorem 11. *Any partial flock of a quadratic cone in $PG(3,K)$, with defining set λ (i.e., so t ranges over λ and planes of the partial flock are defined via functions in t) equal to* $\det K^{+}$*, whose associated functions on* $\det K^{+}$*, as above, are surjective on* $\det K^{+}$ *(K^{+} some quadratic extension of K), produces a regular hyperbolic fibration in $PG(3,K)$ with constant back half.*

Proof. We have a partial flock of a quadratic cone, indexed by $\det K^{+}$, for some quadratic extension of K, where the functions ϕ_s are injective and surjective on $\det K^{+}$. Use K^{+} to define the elements f and g as above. For any given function $y = xM_i$, disjoint from $x = 0, y = 0$, we know that

$$xM_i \begin{bmatrix} 1 & g \\ 0 & -f \end{bmatrix} M_i^t x^t = x \begin{bmatrix} \delta_{u,t} & \mathcal{G}(\delta_{u,t}) \\ 0 & -\mathcal{F}(\delta_{u,t}) \end{bmatrix} x^t,$$

where $\mathcal{F}$ and $\mathcal{G}$ and the functions defined in the previous theorem, for some elements u,t. Construct a partial spread from $y = xM_i$ by applying the mappings $(x,y) \to (x,yQ)$, where $Q^{\sigma+1} = 1$, Q in K^{+}. It will then follow that this is a regulus and its opposite regulus defines a hyperbolic quadric, whose polarity interchanges $x = 0$ and $y = 0$.

We claim that, in this way, we obtain a spread and hence a hyperbolic fibration. Consider any point not on $x = 0, y = 0$. We may always start with the Pappian spread Σ coordinatized by K^{+} and choose $x = 0, y = 0, y = x$ in Σ to begin the process.

Each element $t \in \det K^{+}$, defines a hyperbolic quadric of the form

$$V_t = V(x_1^2t + x_1x_2\mathcal{G}(t) - x_2^2\mathcal{F}(t) - (y_1^2 + y_1y_2g - fy_2^2)),$$
$$\text{where } \mathcal{F}(1) = f \text{ and } \mathcal{G}(1) = g.$$

and one regulus R_1 together with $x = 0, y = 0$ define a unique Pappian spread Σ admitting K^{+}, as above, as a coordinatizing field. Suppose that a point (x_o, y_o) lies in V_t and V_{t^*}, for x_o and y_o both non-zero vectors. Then clearly $t = t^*$, as we have a partial flock of a quadratic cone. It remains to show that we have a cover of $PG(3,K)$. Choosing any regulus for each quadric certainly produces a partial spread. Furthermore, the partial spread admits the 'homology' group $(x,y) \to (x,yQ)$, such that $Q^{\sigma+1} = 1$. If there is a 2-dimensional subspace disjoint from this partial spread, we may write the subspace in the form $y = xM$, where M is a non-singular 2×2 matrix over K. We may then use this subspace to construct an extension to the partial flock, also admitting the homology group but still indexed by $\det K^+$, a contradiction. Hence, we have a maximal partial spread. Choose any point (x_1, x_2, y_1, y_2) such that $x_2 \neq 0$. Note that $y_1^2 + y_1 y_1 g - f y_2^2 = c$ is in $\det K^+$, as is x_2^2. Since the mappings ϕ_s are onto functions, there exists a unique t in $\det K^+$ such that $(x_1/x_2)^2 t + (x_1/x_2)\mathcal{G}(t) - \mathcal{F}(t) = c/x_2^2$. This means that, if $x_2 \neq 0$ then (x_1, x_2, y_1, y_2) is covered by the $\det K^+$-set of hyperbolic quadrics. Now assume that $x_2 = 0$. Then $x_1^2 t = c$ has a unique solution in $\det K^+$. Hence, this completes the proof of the theorem. ∎

4 Homology Groups in $GL(4, q)$.

Suppose that H is a homology group of order $q+1$ of a translation plane with spread in $PG(3, q)$. Let K^* denote the kernel homology group of the plane. Then HK^* fixes the axis and coaxis of the plane.

Lemma 11. *If q is even then H is cyclic.*

Proof. Under the stated condition, $H \cap K^*$ is trivial on a coaxis L, so $HK^* \mid L$ has order q^2-1 and induces a subgroup of $PGL(2, q)$ by HK^*/K^*, which is isomorphic to H. This subgroup contains no p-elements so must be a subgroup of a dihedral group of order $2(q + 1)$ or isomorphic to A_4, S_4 or A_5 (see, e.g., Huppert [5]). However, since the order is odd, H must be cyclic of order $q + 1$. ∎

Lemma 12. *Assume that q is odd and $q \equiv 1 \bmod 4$. Then H is cyclic.*

Proof. Under the stated conditions, $H \cap K^* \mid L$, the coaxis of H, has order 2. So $(q+1)/2$ is odd and > 2. Hence, $HK^*/K^* \mid L$, has order $(q+1)/2$ and is a subgroup of a dihedral group of order $2(q + 1)$ or isomorphic to A_4, S_4, A_5. Since this is an odd order subgroup, then $HK^*/K^* \mid L$ is cyclic of order $(q + 1)/2$. The Sylow u-subgroups S_u of odd order of H are cyclic and $S_u K^*/K^*$ and $S_v K^*/K^*$ are in a unique cyclic subgroup. Hence, $S_u K^* S_v K^* = S_u S_v K^*$ is a subgroup of HK^*. But this means that $S_u S_v K^*/K^* \simeq S_u S_v$ is a cyclic subgroup of H. Hence, there is a cyclic subgroup of H of odd order $(q + 1)/2$. Then there is a cyclic subgroup $O(H)$ of index two in H. There is a unique involution in H, which is normalized and hence centralized by $O(H)$, so H is cyclic. ∎

Lemma 13. *Let π be a translation plane of order q^2 admitting an Abelian homology group H. Then H is cyclic.*

Proof. The Sylow v-subgroups, for v odd, are cyclic (see p. 525 of Foundations [2]), and for v even, are cyclic or generalized quaternion. Since H is Abelian, it is a direct sum of its Sylow v-subgroups. Hence, H is cyclic. ∎

Combining the above three lemmas with our main results, we have:

Corollary 2. *Let π be a translation plane with spread in $PG(3,q)$ that admits an affine homology group H of order $q+1$ in the translation complement. If any of the following conditions hold, π constructs a regular hyperbolic fibration with constant back half and hence a corresponding flock of a quadratic cone.*

(1) *q is even,*

(2) *q is odd and $q \equiv 1 \bmod 4$,*

(3) *H is Abelian,*

(4) *H is cyclic.*

Remark 1. There are translation planes of order $q^2 = 7^2$ with spread in $PG(3,7)$ that admit quaternion homology groups of order $q+1=8$, due to Heimbeck [4].

5 Final Remarks.

We have shown that it is possible to construct partial flocks of quadratic cones from regular hyperbolic fibrations in $PG(3,K)$, where K is an arbitrary field. Furthermore, we have shown that translation planes with spreads in $PG(3,K)$, admitting 'regulus-inducing' homology groups produce regular hyperbolic fibrations. Given two functions on K, $f(t)$, $g(t)$, form the functions $\phi_s : \phi_s(t) = s^2t + sg(t) - f(t)$. A flock of a quadratic cone in $PG(3,K)$ is obtained if and only if the function ϕ_s is bijective for each s in K. Let K^+ be a quadratic extension of K (required for the construction of a flock) and let $\det K^+$ denote the set of determinants of elements of K^+, K^+ written as a 2×2 matrix field over K. We have seen that there is a corresponding regular hyperbolic fibration if and only if $\phi_s \mid \det K^+$ is surjective on $\det K^+$. Of course, this is trivially true in the finite case. We list this as an open problem, in the general case.

Problem 1. *Does every flock of a quadratic cone in $PG(3,K)$, for K an infinite field, produce a regular hyperbolic fibration?*

We have not considered the collineation group of the translation planes corresponding to regular hyperbolic fibrations. In particular, for finite translation planes, we have a variety of open problems.

Problem 2. *Let π be a translation plane with spread in $PG(3,q)$ that admits a cyclic affine homology group H of order $q+1$.*

(1) *Is H normal in the full collineation group of π?*

(2) *Is the full collineation group of π a subgroup of the group of the corresponding hyperbolic fibration?*

We notice that there are Heimbeck planes of order $7^2 = q^2$ that admit quaternion homology groups H of order $8 = q+1$, where H is not normal in the full collineation group of the plane. Hence, we may generalize the previous problem as follows

Problem 3. *Let π be a translation plane with spread in $PG(3,q)$ that admits at least three affine homology groups of order $q+1$. Completely classify the possible planes.*

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