Cameron-Liebler line classes in $\mathrm{PG}(3,4)$

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Abstract

Cameron-Liebler line classes are sets of lines in $\mathrm{PG}(3,q)$ that contain a fixed number x of lines of every spread. Cameron and Liebler classified them for $x \in \{0,1,2,q^2-1,q^2,q^2+1\}$ and conjectured that no others exist. This conjecture was disproved by Drudge and his counterexample was generalised to a counterexample for any odd q by Bruen and Drudge.

In this paper, we give the first counterexample for even q, a Cameron-Liebler line class with parameter 7 in $\mathrm{PG}(3,4)$. We also prove the nonexistence of Cameron-Liebler line classes with parameters 4 and 5 in $\mathrm{PG}(3,4)$ and give some properties of a hypothetical Cameron-Liebler line class with parameter 6 in $\mathrm{PG}(3,4)$.

1 Introduction

Cameron-Liebler line classes were introduced by Cameron and Liebler [7] in an attempt to classify collineation groups of $\mathrm{PG}(n,q)$ that have equally many point orbits and line orbits. In their paper, they conjectured which groups these are. It is now known that the conjecture is true when the group is irreducible, but there is no classification yet of Cameron-Liebler line classes. In this paper, new existence and nonexistence results are proved.

Following Penttila [13], a *clique* in $\mathrm{PG}(3,q)$ is either the set of all lines through a point P, denoted by $\mathsf{star}(P)$, or dually the set of all lines in a plane π, denoted by $\mathsf{line}(\pi)$. The planar pencil of lines in a plane π through a point P is denoted by $\mathsf{pen}(P,\pi)$.

*Patrick Govaerts gratefully acknowledges the financial support of the Fonds Professor Frans Wuytack.

2000 *Mathematics Subject Classification :* 51E20, 51E21, 51E23.

Key words and phrases : Cameron-Liebler line classes, blocking sets.

There are many equivalent definitions for Cameron-Liebler line classes. Here three of them are listed: the first one because it is the most elegant one, the other ones because they will be useful later on.

Definition 1.1 (Cameron and Liebler [7], Penttila [13]) Let $\mathcal{L}$ be a set of lines in $\mathrm{PG}(3,q)$ and let $\chi_{\mathcal{L}}$ be its characteristic function. Then $\mathcal{L}$ is called a *Cameron-Liebler line class* if one of the following equivalent conditions is satisfied.
1. There exists an integer x such that $|\mathcal{L} \cap \mathcal{S}| = x$ for all spreads $\mathcal{S}$.
2. There exists an integer x such that for every incident point-plane pair (P, π)

$$|\mathsf{star}(P) \cap \mathcal{L}| + |\mathsf{line}(\pi) \cap \mathcal{L}| = x + (q+1)|\mathsf{pen}(P,\pi) \cap \mathcal{L}|. \tag{1}$$

3. There exists an integer x such that for every line l of $\mathrm{PG}(3,q)$

$$|\{m \in \mathcal{L} : m \text{ meets } l, m \neq l\}| = (q+1)x + (q^2-1)\chi_{\mathcal{L}}(l). \tag{2}$$

It follows from the proof of the equivalence of these properties that the number x in each of these statements is the same. It is called the *parameter* of the Cameron-Liebler line class.

The first definition implies that $x \in \{0, 1, 2, \ldots, q^2+1\}$. Cameron and Liebler [7] showed that a Cameron-Liebler line class of parameter x consists of $x(q^2+q+1)$ lines and that the only Cameron-Liebler line classes for $x = 1$ are the cliques and for $x = 2$ the unions of two disjoint cliques. They also noticed that the complement of a Cameron-Liebler line class with parameter x is a Cameron-Liebler line class with parameter q^2+1-x. So, it suffices to study Cameron-Liebler line classes with parameter $x \leq \lfloor (q^2+1)/2 \rfloor$. Thus, the case $q = 2$ was immediately solved. In their paper, Cameron and Liebler conjectured that no other Cameron-Liebler line classes exist.

Penttila [13] shows that for $q \neq 2$ there exist no Cameron-Liebler line classes with parameter $x = 3$ or $x = 4$, with possible exception of the cases $(x,q) \in \{(4,3),(4,4)\}$. Bruen and Drudge [5] prove the nonexistence of Cameron-Liebler line classes with parameter $2 < x \leq \sqrt{q}$. Drudge [8] excludes the existence of a Cameron-Liebler line class with parameter $x = 4$ in $\mathrm{PG}(3,3)$, and proves that for $q \neq 2$ there exist no Cameron-Liebler line classes with parameter $2 < x \leq \epsilon$, where $q+1+\epsilon$ denotes the size of the smallest nontrivial blocking sets in $\mathrm{PG}(2,q)$, see Section 3. He also gives a counterexample to the conjecture of Cameron and Liebler: a Cameron-Liebler line class with parameter $x = 5$ in $\mathrm{PG}(3,3)$, in this way settling the case $q = 3$. Bruen and Drudge [6] then construct a Cameron-Liebler line class with parameter $x = (q^2+1)/2$ for any odd q. Govaerts and Storme [10] improve on Drudge's bound on x for nonexistence of Cameron-Liebler line classes: for $q \neq 2$ there exist no Cameron-Liebler line classes with parameter $2 < x < 2\epsilon$. They improve on this result when q is a square or a cube.

In this paper, we will focus on the case $q = 4$ and we will present the first counterexample to Cameron and Liebler's conjecture for even q.

Theorem 1.2 *There exists a Cameron-Liebler line class with parameter* 7 *in* $\mathrm{PG}(3,4)$.

This theorem will be proved in Section 2. We will also study line classes with parameters 4, 5 and 6.

Theorem 1.3 1. *There is no Cameron-Liebler line class with parameter* 4 *in* PG$(3, 4)$.

2. *There is no Cameron-Liebler line class with parameter* 5 *in* PG$(3, 4)$.

3. *If* $\mathcal{L}$ *is a Cameron-Liebler line class with parameter* 6 *in* PG$(3, 4)$*, then every clique intersects* $\mathcal{L}$ *in* $3 \bmod 5$ *lines. Moreover, for each* $\beta \in \{3, 8, 13, 18\}$ *there exists a clique containing exactly* β *lines of* $\mathcal{L}$*. In a clique meeting* $\mathcal{L}$ *in* 18 *lines, the three lines not in* $\mathcal{L}$ *are not contained in a common planar pencil.*

To prove this theorem, we will need some results on (multiple) blocking sets in PG$(2, 4)$, see Section 3, and some lemmas on Cameron-Liebler line classes in PG$(3, q)$, see Section 4. Theorem 1.3 will then be proved in Section 5.

Remark 1.4 Theorem 1.3.1 was already proved by Govaerts in [9]. It completes the study of the case $x = 4$.

2 Proof of Theorem 1.2

We prove Theorem 1.2 by providing a construction.

Theorem 2.1 *Let* P *be a point of* PG$(3, 4)$ *and* π *a plane not containing* P*. Let* O *be a hyperoval in* π *and denote the cone with base* O *and vertex* P *by* $\mathfrak{C}$*. The set* $\mathcal{L}$ *consisting of all generators of* $\mathfrak{C}$*, all two-secants to* $\mathfrak{C}$ *skew to* O *and all lines in* π *external to* O *is a Cameron-Liebler line class with parameter* 7 *in* PG$(3, 4)$.

Proof With respect to the cone $\mathfrak{C}$ and the special plane π containing the hyperoval O, there are seven types of lines. Three of them consist of lines of $\mathcal{L}$: (1) generators of $\mathfrak{C}$, (2) 2-secants to $\mathfrak{C}$ skew to O and (3) lines in π skew to O. The other four consist of lines not in $\mathcal{L}$: (a) lines through P not contained in $\mathfrak{C}$, (b) 2-secants to $\mathfrak{C}$ that intersect O in one point, (c) lines skew to $\mathfrak{C}$ that are not contained in π and (d) 2-secants to O. It is not hard to check that every line of type (1), (2) or (3) intersects exactly 50 other lines of $\mathcal{L}$ and that every line of type (a), (b), (c) or (d) intersects exactly 35 lines of $\mathcal{L}$. By Definition 1.1.3, $\mathcal{L}$ is a Cameron-Liebler line class with parameter 7 in PG$(3, 4)$. ■

Remark 2.2 The obvious way to generalise the construction of Theorem 2.1 in PG$(3, 2^h)$ by taking a cone $\mathfrak{C}$ with vertex P and base a hyperoval O in a plane skew to P and taking the same set of lines $\mathcal{L}$ as in Theorem 2.1 does not produce a Cameron-Liebler line class unless $h = 2$. Neither does replacing the hyperoval O with any (other) maximal arc. This is easily checked by calculating the number of elements of such a set $\mathcal{L}$ and noting that it is never a multiple of $2^{2h} + 2^h + 1$ unless $h = 2$.

3 Multiple blocking sets in PG(2,4)

An m-fold blocking set in PG$(2, q)$ is a set of points that intersects every line in at least m points. It is called *minimal* if no point can be deleted from the set to obtain a smaller m-fold blocking set. A 1-fold (respectively 2-fold, 3-fold) blocking set is simply called a *blocking set* (respectively *double* blocking set, *triple* blocking set). A blocking set is called *trivial* if it contains a line.

As explained in Section 4, multiple blocking sets turn up in the study of Cameron-Liebler line classes. Information on small blocking sets will be of particular interest to us.

Theorem 3.1 1. **(Bruen [3])** *If B is a nontrivial blocking set in* $\mathrm{PG}(2,q)$*, then $|B| \geq q + \sqrt{q} + 1$. If equality is reached, then q is a square and B is a Baer subplane.*

2. **(Bruen [4], Ball [1])** *Let B be an m-fold blocking set in* $\mathrm{PG}(2,q)$*, $m > 1$. If B contains a line, then $|B| \geq mq + q - m + 2$. If B does not contain a line, then $|B| \geq mq + \sqrt{mq} + 1$.*

Corollary 3.2 1. *The smallest nontrivial blocking sets in* $\mathrm{PG}(2,4)$ *are Baer subplanes. They have size* 7*.*

2. *A double blocking set of* $\mathrm{PG}(2,4)$ *consists of at least* 12 *points.*

We will need some further information on small double blocking sets of $\mathrm{PG}(2,4)$.

Theorem 3.3 *Up to isomorphism, there are exactly three double blocking sets of size* 12 *in* $\mathrm{PG}(2,4)$*. If B is such a double blocking set, then either*

1. *B consists of the set of points of three nonconcurrent lines, or*

2. *there exist two lines l and m intersecting in a point P such that B consists of the set of points on l and m and three noncollinear further points, one on each of the three remaining lines through P, or*

3. *there exist three lines l_1, l_2 and l_3 through a point P and a fourth line l not through P such that B consists of the points of $l_i \setminus l$, $i = 1, 2, 3$, and the two points of l not on any of the lines l_i, $i \in \{1, 2, 3\}$.*

Proof We provide two proofs. The first one can be found below. The second one follows Remark 3.5; it is shorter than the first proof, but less accessible.

Suppose B is a double blocking set of size 12 in $\mathrm{PG}(2,4)$.

Case 1. Assume that B contains a line l. We will show that B contains a second line l'.

Consider a point P in $B \setminus l$. The 11 points of $B \setminus \{P\}$ lie on the five lines through P. So, there exists a line m through P containing at least four points of B. If m is contained in B, then take $l' = m$. So, suppose m contains a point $P' \notin B$.

Let Q be the intersection point of l and m, and let l_1, l_2 and l_3 be the lines through Q different from l and m. Let m_1, m_2, m_3 and m_4 be the lines different from m through P'. Then on each line m_i, there is exactly one more point of B. On one of the lines l_i, there are two more points of B, and the other two lines l_i contain one further point of B. Let l_1 be the line containing two more points of B and

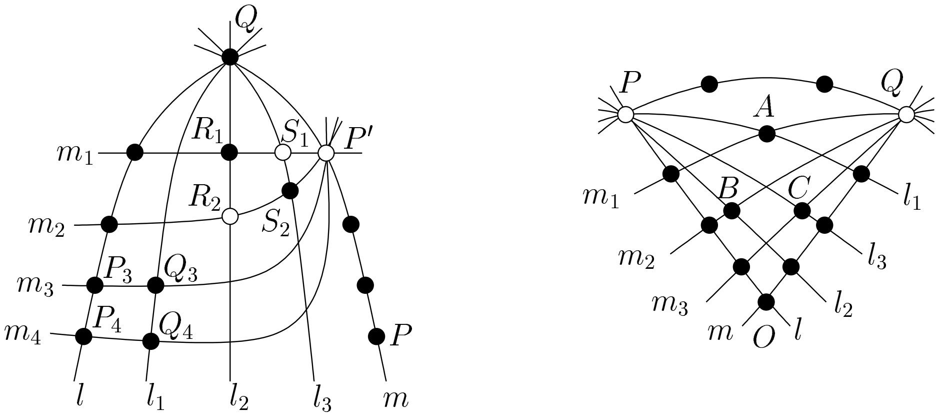

Figure 1: Notations for a double blocking 12-set (left) containing a line and (right) not containing a line.

let the lines m_i containing these points be m_3 and m_4. Name the lines m_1, m_2, l_2 and l_3 in such a way that the remaining points of B are $m_1 \cap l_2$ and $m_2 \cap l_3$. Call the points P_3, P_4, Q_3, Q_4, R_1, R_2, S_1 and S_2 as indicated in Figure 1. Consider the line P_4Q_3. It intersects l_2 in a point. This point is either R_1 or R_2. If it is R_1, then P_4Q_3 must also contain S_2 and a point of $m \setminus \{P'\}$. In this case, set $l' = P_4Q_3$. If it is R_2, then P_3Q_4 contains R_1, S_2 and a point of $m \setminus \{P'\}$, so we can set $l' = P_3Q_4$.

In any case, B contains a second line l', and B is of type 1 or 2 in the statement of the theorem.

Case 2. Assume that B contains no line. Consider a point $R \in B$. Since B contains 11 points different from R, there exists a line n through R containing at least four points of B. Since B contains no lines, n contains exactly four points of B. Let S be the point on n that does not lie in B. All lines through S different from n contain exactly two points of B. Let R' be a point of B not on n. As above, there exists a line n' through R' containing exactly four points of B. This line intersects n in a point different from S.

From the reasoning above, it follows that there exists a point $O \in B$ that lies on two lines l and m that contain exactly four points of B. Denote the point on l (respectively m) not in B by P (respectively Q). The lines through P (respectively Q) different from PQ and l (respectively PQ and m) are denoted by l_1, l_2, and l_3 (respectively m_1, m_2, and m_3). Clearly, PQ must contain two further points of B, and the lines l_i and m_j, $i, j = 1, 2, 3$, each one more point of B. Let A, B and C be the three extra points of B on the lines l_i and m_j. Name these points and lines in such a way that A (respectively B, C) lies on l_1 and m_1 (respectively l_2 and m_2, l_3 and m_3), see Figure 1.

Case 2.1. Assume that A, B and C are not collinear. In this case, none of the lines AB, BC and AC can contain O. Indeed, suppose for example that AB contains O. Then AB intersects m_3 in a point that cannot lie on l nor on PQ, but also neither on l_1 nor on l_2. Hence it lies on l_3. But then this point must be C, a contradiction. So, all three lines AB, BC and AC contain a point of $l \setminus \{O, P\}$ and a point of $m \setminus \{O, Q\}$. They also contain a point of $PQ \setminus \{P, Q\}$ and no two of them contain the same point of $PQ \setminus \{P, Q\}$. Since $PQ \setminus \{P, Q\}$ contains only three points and two of them belong to B, two of the three lines AB, BC and AC are contained in B, a contradiction.

Case 2.2. Assume that A, B and C are collinear. Consider the line AB (which equals the line AC). If it does not contain O, then it contains a point of $l \setminus \{O, P\}$ and a point of $m \setminus \{O, Q\}$, so that it is contained in B, a contradiction. Thus the line AB contains O and intersects PQ in a point that is not contained in B. The remaining two points of PQ lie in B. Hence, B is of the third type in the statement of the lemma.

It is easily seen that Case 2 is unique up to isomorphism, while Case 1 splits in two cases, depending on whether the three points of B outside $l \cup l'$ are collinear or not. ∎

Corollary 3.4 *Suppose that B is a 2-fold blocking set of size 12 in $\mathrm{PG}(2,4)$. Then, using the numbering from the theorem above, either B is of type 1 and has nine 2-secants, nine 3-secants and three 5-secants, or B is of type 2 and has ten 2-secants, six 3-secants, three 4-secants and two 5-secants, or B is of type 3 and has twelve 2-secants and nine 4-secants.*

Remark 3.5 In [11], Laskar and Sherk define the *type* of a double blocking set B in $\mathrm{PG}(2,q)$ as $(|B|; \tau_2, \tau_3, \ldots, \tau_{q+1})$ where τ_i, $i = 2, 3, \ldots, q+1$, denotes the number of i-secants to B. They determine all possible types of minimal double blocking sets in $\mathrm{PG}(2,4)$: $(12; 9, 9, 0, 3)$, $(12; 10, 6, 3, 2)$, $(12; 12, 0, 9, 0)$, $(13; 8, 4, 8, 1)$ and $(14; 7, 0, 14, 0)$. Hence, the intersection properties from Corollary 3.4 above were already proved in that paper.

Alternative proof of Theorem 3.3 Let B be a double blocking set of size 12 in $\mathrm{PG}(2,4)$. By Corollary 3.2, it is minimal. By Remark 3.5, it is of type $(12; 9, 9, 0, 3)$, $(12; 10, 6, 3, 2)$, or $(12; 12, 0, 9, 0)$. Clearly, if it is of type $(12; 9, 9, 0, 3)$, then it is of the first type (in the non-technical sense). If it is of type $(12; 10, 6, 3, 2)$, then it is of the second type. If it is of type $(12; 12, 0, 9, 0)$, then its complement is a set of 9 points such that each line of $\mathrm{PG}(2,4)$ intersects it in either 1 or 3 points. Such a set is called a *unital* and is well-known to be unique (in $\mathrm{PG}(2,4)$); it consists of the set of points of three nonconcurrent lines minus their intersection points. Hence B is of the third type of the statement. ∎

Lemma 3.6 *A double blocking set of size 13 in $\mathrm{PG}(2,4)$ contains a line.*

Proof If the blocking set is minimal, it contains a line by Remark 3.5.

If it is not minimal, it is obtained by adding a point to one of the double blocking sets of Theorem 3.3. The first two already contain a line and it is easy to check that adding a point to the third one always results in a new set containing a line. ∎

4 Three lemmas

Theorem 1.3 will be proved by studying how the lines of the Cameron-Liebler line class are distributed among the cliques of $\mathrm{PG}(3, q)$. To study the lines of the Cameron-Liebler line class in a clique, we follow Drudge's approach [8]. A clique $\mathcal{C}$ and its lines correspond to a projective plane and its points in the following way. If $\mathcal{C} = \mathsf{star}(P)$, then it suffices to take the quotient space with respect to P. If $\mathcal{C} = \mathsf{line}(\pi)$, then the dual plane can be considered. In this way, the lines of the line class in a clique correspond to a set of points B in a projective plane Π.

Lemma 4.1 *Let $\mathcal{L}$ be a Cameron-Liebler line class with parameter x in $\mathrm{PG}(3, q)$ and let $\mathcal{C}$ be a clique. Let Π denote the plane corresponding to $\mathcal{C}$ and B the set of points of Π corresponding to the lines of $\mathcal{C}$ in $\mathcal{L}$.*

1. **(Drudge [8])** *If $x < |\mathcal{C} \cap \mathcal{L}| \leq q + x$, then B is a blocking set in Π. If there exist no Cameron-Liebler line classes with parameter $x - 1$, then B is nontrivial.*

2. **(Govaerts and Storme [10])** *If $x + \alpha(q + 1) < |\mathcal{C} \cap \mathcal{L}|$, then B is an $(\alpha + 1)$-fold blocking set in Π.*

Lemma 4.2 (Govaerts and Storme [10]) *Let $\mathcal{L}$ be a Cameron-Liebler line class in $\mathrm{PG}(3, q)$, $q > 2$, with parameter $x > \epsilon$, where $q + 1 + \epsilon$ denotes the size of the smallest nontrivial blocking set in $\mathrm{PG}(2, q)$. Then there exists no clique $\mathcal{C}$ satisfying $x - \epsilon < |\mathcal{C} \cap \mathcal{L}| \leq q$ or $x < |\mathcal{C} \cap \mathcal{L}| \leq q + \epsilon$. If additionally $x \leq q$, then there exists no clique $\mathcal{C}$ satisfying $0 \leq |\mathcal{C} \cap \mathcal{L}| < \epsilon$.*

Remark 4.3 In [10], $x \leq q$ is assumed for each of the three intervals of Lemma 4.2. However, it is only used in the proof of the third one.

When knowing the number of lines of a Cameron-Liebler line class $\mathcal{L}$ in one clique, the following lemma gives severe restrictions on the possible intersections of other cliques with $\mathcal{L}$.

Lemma 4.4 *Suppose $\mathcal{L}$ is a Cameron-Liebler line class in $\mathrm{PG}(3, q)$ with parameter $x < q^2 + 1$. Then there exists an integer $0 \leq \alpha \leq x$ such that there exists a point through which there are exactly α lines of $\mathcal{L}$ and such that*

1. *for each point P: $|\mathsf{star}(P) \cap \mathcal{L}| \equiv \alpha \pmod{q + 1}$, and*

2. *for each plane π: $|\mathsf{line}(\pi) \cap \mathcal{L}| \equiv x - \alpha \pmod{q + 1}$.*

Proof As $x < q^2 + 1$, there exists a line l not in $\mathcal{L}$. By (2), exactly $x(q + 1)$ lines of $\mathcal{L}$ meet l. Hence there exists a point P on l through which there pass at most x lines of $\mathcal{L}$. Let $\alpha = |\mathsf{star}(P) \cap \mathcal{L}|$. Equation (1) shows that each plane π containing P satisfies $|\mathsf{line}(\pi) \cap \mathcal{L}| \equiv x - \alpha \pmod{q + 1}$. Again applying (1), now on the planes π through P and the points contained in them shows that for each point Q, $|\mathsf{star}(Q) \cap \mathcal{L}| \equiv \alpha \pmod{q + 1}$. A final application of (1) proves the lemma for all planes. ■

Remark 4.5 The preceding lemma shows that for Cameron-Liebler line classes, some sort of "mod$(q+1)$ property" is valid, similar to 1 mod p and t mod p results for small minimal 1-fold and t-fold blocking sets in $\mathrm{PG}(2,q)$, $q = p^h$, p prime, see [12, 14], and to 1 mod p results for ovoids on the parabolic quadrics $\mathrm{Q}(4,q)$ and $\mathrm{Q}(6,q)$, see [2]. However, the exact value for α is missing in Lemma 4.4.

In the theory of minimal t-fold blocking sets, t mod p results have proved to be very useful. Such results tell "how" a subspace intersects the minimal t-fold blocking sets: in $t \pmod p$ points. They make the blocking sets easier to handle and have made several classification theorems possible. We can only hope that similar results for other objects, in this case for Cameron-Liebler line classes, will prove to be equally fruitful.

5 Proof of Theorem 1.3

Theorem 1.3 will be proved by studying how the lines of the Cameron-Liebler line class are distributed among the cliques of $\mathrm{PG}(3,q)$.

We will use the following notation. Given a clique $\mathcal{C}$, the corresponding plane (see Section 4) will be denoted by Π and B will denote the set of points of Π corresponding to the lines of $\mathcal{L}$ in $\mathcal{C}$. A line l in Π determines a planar pencil $\mathcal{P}$ in $\mathcal{C}$. If $\mathcal{C} = \mathsf{star}(P)$ (respectively $\mathcal{C} = \mathsf{line}(\pi')$) and $\mathcal{P} = \mathsf{pen}(P,\pi)$ (respectively $\mathcal{P} = \mathsf{pen}(P',\pi')$), then let $\mathcal{C}^* = \mathsf{line}(\pi)$ (respectively $\mathcal{C}^* = \mathsf{star}(P')$).

5.1 Proof of Theorem 1.3.1

Suppose $\mathcal{L}$ is a Cameron-Liebler line class in $\mathrm{PG}(3,4)$ with parameter 4 and let $\mathcal{C}$ be a clique. By Lemma 4.2 and Corollary 3.2, $|\mathcal{C}\cap\mathcal{L}| \notin \{0,1,3,4\}$. By Lemma 4.4, $|\mathcal{C}\cap\mathcal{L}| \equiv 2 \pmod 5$. Hence $|\mathcal{C}\cap\mathcal{L}| \in \{2,7,12,17\}$.

Suppose that $|\mathcal{C}\cap\mathcal{L}| = 17$. Then, by (2), for each pencil $\mathcal{P}$ in $\mathcal{C}$ with corresponding clique $\mathcal{C}^*$:

$$17 + |\mathcal{C}^*\cap\mathcal{L}| = 4 + 5|\mathcal{P}\cap\mathcal{L}|. \tag{3}$$

Therefore $|\mathsf{l}\cap\mathsf{B}| \geq 3$ for each line l in Π. In Π, exactly four points do not belong to B. Take a line l containing at least two of these points. Then $|\mathcal{P}\cap\mathcal{L}| \leq 3$, hence $|\mathcal{P}\cap\mathcal{L}| = 3$. For this pencil $\mathcal{P}$, (3) yields $|\mathcal{C}^*\cap\mathcal{L}| = 2 < 3 = |\mathcal{P}\cap\mathcal{L}|$, a contradiction.

It can be concluded that $|\mathcal{C}\cap\mathcal{L}| \in \{2,7,12\}$. If $|\mathcal{C}\cap\mathcal{L}| = 2$, then B is a set of two points in Π. If $|\mathcal{C}\cap\mathcal{L}| = 7$, then B is a Baer subplane in Π by Lemma 4.1 and Corollary 3.2. If $|\mathcal{C}\cap\mathcal{L}| = 12$, then B is a double blocking set of size 12 in Π.

If $|\mathcal{C}\cap\mathcal{L}| = 12$ and B contains a line l, then $|\mathcal{P}\cap\mathcal{L}| = 5$ such that by (2), $|\mathcal{C}^*\cap\mathcal{L}| = 17$, a contradiction. So, if $|\mathcal{C}\cap\mathcal{L}| = 12$, then $\mathcal{C}\cap\mathcal{L}$ must be a blocking set of the third type of Theorem 3.3.

Let P be a point such that $|\mathsf{star}(P)\cap\mathcal{L}| = \alpha = 2$, see Lemma 4.4, and let $\mathcal{C} = \mathsf{star}(P)$. Let l be a tangent in Π. Then $|\mathcal{P}\cap\mathcal{L}| = 1$ and, by (2), $|\mathcal{C}^*\cap\mathcal{L}| = 7$. Hence B^* is a Baer subplane in Π^*. Take a 3-secant l^* in Π^*. Then $|\mathcal{P}^*\cap\mathcal{L}| = 3$ such that $|\mathcal{C}^{**}\cap\mathcal{L}| = 12$. Hence the double blocking set B^{**} in Π^{**} has a 3-secant: the line corresponding to $\mathcal{P}^*$. But this double blocking set is of the third type of Theorem 3.3, which has no 3-secants, see Corollary 3.4. We have obtained a contradiction. ∎

5.2 Proof of Theorem 1.3.2

Suppose that $\mathcal{L}$ is a Cameron-Liebler line class with parameter 5 in PG$(3,4)$ and let $\mathcal{C}$ be a clique.

By Lemma 4.2 and Corollary 3.2, $|\mathcal{C}\cap\mathcal{L}| \notin \{4,6\}$.

If $|\mathcal{C}\cap\mathcal{L}| = 1$, then there exists a line l in Π external to B. Hence $|\mathcal{P}\cap\mathcal{L}| = 0$ such that by (2), $|\mathcal{C}^*\cap\mathcal{L}| = 4$, a contradiction. So, $|\mathcal{C}\cap\mathcal{L}| \neq 1$.

If $|\mathcal{C}\cap\mathcal{L}| = 8$, then B is a nontrivial blocking set. Suppose that B has no 2-secants. Then, through a point R of B, there are two tangents, two 3-secants and one 4-secant to B. Let S_1, S_2 and S_3 be the three points of B different from R on the 4-secant and let T be a point of B different from R on one of the 3-secants to B. Now consider the lines $\mathsf{S}_1\mathsf{T}$, $\mathsf{S}_2\mathsf{T}$ and $\mathsf{S}_3\mathsf{T}$. They all contain at least two points of B, but cannot all contain more than two points of B, so one of them contains exactly two points of B, a contradiction. Therefore B has a 2-secant l. Then $|\mathcal{C}^*\cap\mathcal{L}| = 7$ such that, by Lemma 4.1 and Corollary 3.2, B^* is a Baer subplane admitting a 2-secant (the line corresponding to $\mathcal{P}$), a contradiction. Hence $|\mathcal{C}\cap\mathcal{L}| \neq 8$.

If $|\mathcal{C}\cap\mathcal{L}| = 2$, then any line l in Π tangent to B yields a clique $\mathcal{C}^*$ satisfying $|\mathcal{C}^*\cap\mathcal{L}| = 8$, a contradiction. If $|\mathcal{C}\cap\mathcal{L}| = 3$, then any line l in Π skew to B gives a clique $\mathcal{C}^*$ satisfying $|\mathcal{C}^*\cap\mathcal{L}| = 2$, a contradiction. So, $|\mathcal{C}\cap\mathcal{L}| \notin \{2,3\}$.

By the above, the integer α from Lemma 4.4 is either 0 or 5, such that, by Lemma 4.4, $|\mathcal{C}\cap\mathcal{L}| \in \{0,5,10,15,20\}$.

If $|\mathcal{C}\cap\mathcal{L}| = 10$, then B is a blocking set in Π. If there exists a tangent l to B, then the corresponding clique $\mathcal{C}^*$ satisfies $|\mathcal{C}^*\cap\mathcal{L}| = 0$, which is smaller than 1, contradicting $|\mathcal{P}\cap\mathcal{L}| = 1$. Hence B is a double blocking set of size 10 in PG$(2,4)$, contradicting Corollary 3.2. Therefore $|\mathcal{C}\cap\mathcal{L}| \neq 10$.

If $|\mathcal{C}\cap\mathcal{L}| = 5$, then B cannot contain a line l since then $|\mathcal{C}^*\cap\mathcal{L}|$ would equal 25. With this information, it is easy to see that B has a 2-secant l. But then $|\mathcal{C}^*\cap\mathcal{L}| = 10$, a contradiction. Hence $|\mathcal{C}\cap\mathcal{L}| \neq 5$.

If $|\mathcal{C}\cap\mathcal{L}| = 0$, then each clique $\mathcal{C}^*$ determined by a line l in Π contains exactly 5 lines of $\mathcal{L}$, a contradiction. So, $|\mathcal{C}\cap\mathcal{L}| \neq 0$.

By the above, each clique contains at least seven elements of $\mathcal{L}$, contradicting Lemma 4.4. We conclude that there is no Cameron-Liebler line class with parameter 5 in PG$(3,4)$. ■

5.3 Proof of Theorem 1.3.3

Suppose that $\mathcal{L}$ is a Cameron-Liebler line class with parameter 6 in PG$(3,4)$ and let $\mathcal{C}$ be a clique.

If $|\mathcal{C}\cap\mathcal{L}| = 11$, then B is a blocking set in Π. Since the smallest double blocking sets in PG$(2,4)$ have size 12, B has a tangent l. But then $|\mathcal{C}^*\cap\mathcal{L}| = 0$, a contradiction since $\mathcal{P}$ contains a line of $\mathcal{L}$. Hence $|\mathcal{C}\cap\mathcal{L}| \neq 11$.

Suppose $|\mathcal{C}\cap\mathcal{L}| = 5$. If B contains a line l, then $|\mathcal{C}^*\cap\mathcal{L}| = 26$, a contradiction. From this it is easy to see that B has a 2-secant l. But then $|\mathcal{C}^*\cap\mathcal{L}| = 11$, a contradiction. So, $|\mathcal{C}\cap\mathcal{L}| \neq 5$.

If $|\mathcal{C}\cap\mathcal{L}| = 1$ and l is a line in Π skew to B, then $|\mathcal{C}^*\cap\mathcal{L}| = 5$, a contradiction. Hence $|\mathcal{C}\cap\mathcal{L}| \neq 1$.

Suppose $|\mathcal{C}\cap\mathcal{L}| = 10$. Then B is a blocking set. As it has size 10, it cannot be a double blocking set. Hence it has a tangent l. But then $|\mathcal{C}^*\cap\mathcal{L}| = 1$, a contradiction. So, $|\mathcal{C}\cap\mathcal{L}| \neq 10$.

Suppose $|\mathcal{C}\cap\mathcal{L}| = 6$. If B has a tangent, then $\mathcal{C}^*$ contains exactly 5 lines of $\mathcal{L}$, a contradiction. If it has a 2-secant, then $\mathcal{C}^*$ contains exactly 10 lines of $\mathcal{L}$, a contradiction. Now suppose l is an r-secant for some $r \geq 3$ and let $\mathsf{P} \in \mathsf{l}\cap\mathsf{B}$. Since there are at most three points of B outside l there exists a tangent through P, contradicting the previous observations. Therefore $|\mathcal{C}\cap\mathcal{L}| \neq 6$.

If $|\mathcal{C}\cap\mathcal{L}| = 0$ and l is skew to B, then $|\mathcal{C}^*\cap\mathcal{L}| = 6$, a contradiction. Hence $|\mathcal{C}\cap\mathcal{L}| \neq 0$.

Suppose $|\mathcal{C}\cap\mathcal{L}| = 19$. Let l be a 3-secant to B, i.e., let l be the line joining the two points of $\Pi\setminus\mathsf{B}$. Then $|\mathcal{C}^*\cap\mathcal{L}| = 2$, contradicting the fact that l contains 3 points of B. Hence $|\mathcal{C}\cap\mathcal{L}| \neq 19$.

If $|\mathcal{C}\cap\mathcal{L}| = 12$, then B is a double blocking set. It cannot contain a line l, since otherwise $|\mathcal{C}^*\cap\mathcal{L}| = 19$. Hence B is of the third type of Theorem 3.3. If $|\mathcal{C}\cap\mathcal{L}| = 7$, then B is a Baer subplane.

Suppose $|\mathcal{C}\cap\mathcal{L}| = 9$. Then B is a nontrivial blocking set. Hence it admits no 0- or 5-secants. If it has a 2-secant l, then $|\mathcal{C}^*\cap\mathcal{L}| = 7$, implying that B^* is a Baer subplane. But a Baer subplane has no 2-secants, a contradiction. Now suppose that B has a 4-secant l and let Q be the point of l not in B. Let $\mathsf{P}\in\mathsf{B}\setminus\mathsf{l}$. The four lines joining P to a point of $\mathsf{B}\cap\mathsf{l}$ each contain one further point of B; call these points P_1, P_2, P_3 and P_4. Consider the lines QP_i, $i = 1,2,3,4$. These must be exactly three lines, since all lines through Q (one of these is l, another one is QP, three remain) contain (at least) one point of B. Hence, two lines from $\{\mathsf{QP}_i : 1 \leq i \leq 4\}$ contain exactly one point from the set $\{\mathsf{P}_i : 1 \leq i \leq 4\}$, the third one exactly two. Hence B has a 2-secant, contradicting the previous observations. It follows that B has only 1- and 3-secants. Hence B is a unital in $\mathrm{PG}(2,4)$. Let l be a 3-secant. Then $|\mathcal{C}^*\cap\mathcal{L}| = 12$, which is impossible since, by Corollary 3.4, the set B^* has no 3-secants. Therefore $|\mathcal{C}\cap\mathcal{L}| \neq 9$.

If $|\mathcal{C}\cap\mathcal{L}| = 2$ and l is a tangent to B, then $|\mathcal{C}^*\cap\mathcal{L}| = 9$, a contradiction. If $|\mathcal{C}\cap\mathcal{L}| = 4$ and l is a line skew to B, then $|\mathcal{C}^*\cap\mathcal{L}| = 2$, a contradiction. So, $|\mathcal{C}\cap\mathcal{L}| \notin \{2,4\}$.

By the above, the integer α from Lemma 4.4 is 3, such that, by the same lemma, each clique contains $3 \pmod 5$ lines of $\mathcal{L}$. Hence $|\mathcal{C}\cap\mathcal{L}| \in \{3,8,13,18\}$.

Let $\mathcal{C}$ be a clique containing 3 lines of $\mathcal{L}$, see Lemma 4.4. Let l be a tangent to B in Π. Then $|\mathcal{C}^*\cap\mathcal{L}| = 8$. The blocking set B^* has a 3-secant l^*. It gives a clique $\mathcal{C}^{**}$ containing 13 lines of $\mathcal{L}$. As B^{**} contains a line, see Lemma 3.6, there exists a clique containing 18 lines of $\mathcal{L}$ (the clique corresponding to this line in B^{**}).

Note that, if $\mathcal{C}$ is a clique containing 18 lines, then the three points of $\Pi\setminus\mathsf{B}$ are not collinear, since otherwise B admits a 2-secant, which gives a clique $\mathcal{C}^*$ containing exactly -2 lines of $\mathcal{L}$. ∎

Acknowledgement

We thank the anonymous referee for his careful reading of this article.

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