The two smallest minimal blocking sets of $Q(2n, 3)$, $n \geqslant 3$

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Abstract

We describe the two smallest minimal blocking sets of $Q(2n, 3)$, $n \geqslant 3$. To obtain these results, we use the characterization of the smallest minimal blocking sets of $Q(6, 3)$, different from an ovoid. We also present some geometrical properties of ovoids of $Q(6, q)$, q odd.

1 Introduction

Let $Q(2n, q)$, $n \geqslant 2$, be the non-singular parabolic quadric in $PG(2n, q)$. An *ovoid* of the polar space $Q(2n, q)$ is a set of points $\mathcal{O}$ of $Q(2n, q)$, such that every maximal singular subspace (or *generator*) of $Q(2n, q)$ intersects $\mathcal{O}$ in exactly one point. For $Q(2n, q)$, the generators are spaces of dimension $n - 1$. A *blocking set* of the polar space $Q(2n, q)$ is a set of points $\mathcal{K}$ of $Q(2n, q)$ such that every generator intersects $\mathcal{K}$ in at least one point. If $\mathcal{O}$ is an ovoid of $Q(2n, q)$, then $\mathcal{O}$ has size $q^n + 1$. So if $\mathcal{K}$ is a blocking set of $Q(2n, q)$ different from an ovoid, then $\mathcal{K}$ has size $q^n + 1 + r$, with $r > 0$. A blocking set $\mathcal{K}$ is called *minimal* if for every point $p \in \mathcal{K}$, $\mathcal{K} \setminus \{p\}$ is not a blocking set, or equivalently, if for every point $p \in \mathcal{K}$, there is a generator α such that $\alpha \cap \mathcal{K} = \{p\}$.

We suppose in this article that q is odd. We recall known results about ovoids of the parabolic quadric in 4, 6 and 8 dimensions.

Theorem 1. (Ball [1]) *Suppose that $\mathcal{O}$ is an ovoid of $Q(4, q)$, $q = p^h$, p prime, $h \geqslant 1$, then every elliptic quadric $Q^-(3, q)$ of $Q(4, q)$ intersects $\mathcal{O}$ in $1 \bmod p$ points.*

This result has interesting applications. One of them is the classification of all ovoids of $Q(4, q)$, q prime.

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Theorem 2. (Ball et al. [2]) *All ovoids of* $\mathrm{Q}(4,q)$*, q prime, are elliptic quadrics* $\mathrm{Q}^-(3,q)$*.*

When $q = p^h$, p an odd prime, $h > 1$, and $q = 2^{2h+1}$, $h \geq 1$, other classes of ovoids of $\mathrm{Q}(4,q)$ are known ([9, 15, 18, 19]).

The classification of the ovoids of $\mathrm{Q}(4,q)$, q prime, leads to the following theorem, using a result of [13].

Theorem 3. *When q is an odd prime, $q \geqslant 5$,* $\mathrm{Q}(6,q)$ *does not have ovoids.*

When $q = 3^h$, $h \geqslant 1$, $\mathrm{Q}(6,q)$ always has ovoids ([9, 16, 17]), and when q is even, then $\mathrm{Q}(6,q)$ does not have ovoids ([17]). For all other values of q, the existence or non-existence of ovoids of $\mathrm{Q}(6,q)$ is not known, although it is conjectured in [13] that $\mathrm{Q}(6,q)$ has ovoids if and only if $q = 3^h$, $h \geqslant 1$.

Finally, we recall the following theorem about ovoids of higher dimensional parabolic quadrics.

Theorem 4. (Gunawardena and Moorhouse [8]) *The parabolic quadric* $\mathrm{Q}(8,q)$*, q odd, does not have ovoids. This implies also that* $\mathrm{Q}(2n,q)$*, q odd, $n \geqslant 5$, does not have ovoids.*

We now recall known results about blocking sets different from ovoids. Suppose that $\alpha\mathcal{B}$ is a cone with vertex the k-dimensional subspace α and base some set $\mathcal{B}$ of points, lying in some subspace π, $\pi \cap \alpha = \emptyset$. Then the *truncated cone* $\alpha^*\mathcal{B}$ is defined as $\alpha\mathcal{B} \setminus \alpha$, hence, as the set of points of the cone $\alpha\mathcal{B}$ where the points of the vertex α are removed from. If α is the empty subspace, then $\alpha^*\mathcal{B} = \mathcal{B}$.

The case $q = 3$ of the following theorem was proved in [5]. The theorem for $q > 3$ odd prime was proved in [4]. We denote the polarity associated to the quadric by $\perp$.

Theorem 5. *The smallest minimal blocking sets of* $\mathrm{Q}(6,q)$*, q an odd prime, different from an ovoid of* $\mathrm{Q}(6,q)$*, are truncated cones* $p^*\mathrm{Q}^-(3,q)$*,* $p \in \mathrm{Q}(6,q)$*,* $\mathrm{Q}^-(3,q) \subseteq p^\perp \cap \mathrm{Q}(6,q)$*, and have size* q^3+q*.*

When $q > 3$ is an odd prime, this theorem generalizes to the following theorem.

Theorem 6. ([5]) *The smallest minimal blocking sets of* $\mathrm{Q}(2n,q)$*, $q > 3$ prime, $n \geqslant 4$, are truncated cones* $\pi^*_{n-3}\mathrm{Q}^-(3,q)$*,* $\pi_{n-3} \subseteq \mathrm{Q}(2n,q)$*,* $\mathrm{Q}^-(3,q) \subseteq \pi^\perp_{n-3} \cap \mathrm{Q}(2n,q)$*, and have size* $q^n + q^{n-2}$*.*

Ovoids of $\mathrm{Q}(6,q)$ can be used to construct smaller examples in higher dimension. For $q = 3$, the following result is known.

Theorem 7. ([5]) *The smallest minimal blocking sets of* $\mathrm{Q}(2n,q=3)$*, $n \geqslant 4$, are truncated cones* $\pi^*_{n-4}\mathcal{O}$*, $\mathcal{O}$ an ovoid of* $\mathrm{Q}(6,q=3)$*,* $\mathcal{O} \subset \pi^\perp_{n-4}, \pi_{n-4} \subset \mathrm{Q}(2n,q)$*, and have size* $q^n + q^{n-3}$*.*

Theorems 6 and 7 express the difference between $q > 3$ odd prime and $q = 3$. Furthermore, considering $\mathrm{Q}(2n, q = 3)$, $n \geqslant 4$, it is clear that a truncated cone $\pi^*_{n-3}\mathrm{Q}^-(3,q)$, contained in $\mathrm{Q}(2n,q)$, constitutes a minimal blocking set of size $q^n + q^{n-2}$. We show in this article that minimal blocking sets of $\mathrm{Q}(2n,3)$ of size k,

$q^n + q^{n-3} < k < q^n + q^{n-2}$ do not exist, and we characterize the minimal blocking sets of $\mathrm{Q}(2n, q = 3)$ of size $q^n + q^{n-2}$, as described in the following theorem. Finally, we show that minimal blocking sets of $\mathrm{Q}(2n, q = 3)$, $n \geqslant 3$, of size $q^n + q^{n-2} + 1$ do not exist.

Theorem 8. *The minimal blocking sets of* $\mathrm{Q}(2n,3)$, $n \geqslant 3$, *of size at most* 3^n+3^{n-2}, *are truncated cones* $\pi^*_{n-4}\mathcal{O}$, $\pi_{n-4} \subseteq \mathrm{Q}(2n,3)$, $\pi^{\perp}_{n-4} \cap \mathrm{Q}(2n,3) = \pi_{n-4}\mathrm{Q}(6,3)$, $\mathcal{O}$ *an ovoid of* $\mathrm{Q}(6,3)$, *and* $\pi^*_{n-3}\mathrm{Q}^-(3,3)$, $\pi_{n-3} \subseteq \mathrm{Q}(2n,3)$, $\pi^{\perp}_{n-3} \cap \mathrm{Q}(2n,3) = \pi_{n-3}\mathrm{Q}(4,3)$, $\mathrm{Q}^-(3,3) \subseteq \mathrm{Q}(4,3)$. *Furthermore, a minimal blocking set of size* $3^n + 3^{n-2} + 1$ *of* $\mathrm{Q}(2n,3)$ *does not exist.*

Finally, we mention that blocking sets of other classical polar spaces such as $\mathrm{Q}^-(2n+1,q)$ and $\mathrm{W}(2n+1,q)$ were studied by K. Metsch, [11, 12].

Before presenting the proof of the preceding theorem, we first mention some geometrical properties of ovoids of $\mathrm{Q}(6,q)$, q odd.

2 Geometrical results on ovoids of $\mathrm{Q}(6,q)$, q odd

For the next three lemmas, we suppose that $\mathrm{Q}(6,q)$ has ovoids. This implies that q is odd, since $\mathrm{Q}(6,q)$, q even, does not have ovoids [17], and this hypothesis is satisfied when $q = 3^h$, $h \geqslant 1$. Denote an ovoid of $\mathrm{Q}(6,q)$ by $\mathcal{O}$.

Lemma 1. *The ovoid* $\mathcal{O}$ *spans the* 6*-dimensional space* $\mathrm{PG}(6,q)$.

Proof. Let $\Omega = \langle \mathcal{O} \rangle$.

It is impossible that $\Omega \cap \mathrm{Q}(6,q)$ is a singular quadric. For, assume that $\langle \mathcal{O} \rangle \cap \mathrm{Q}(6,q) = \pi_s Q$, a cone with vertex π_s, an s-dimensional subspace, $s \geqslant 0$, and with base Q, a non-singular quadric of dimension at most 4. Then π_s projects $\mathcal{O}$ onto an ovoid of Q. However, no non-singular quadric of dimension at most four has ovoids of size $q^3 + 1$.

If $\Omega \cap \mathrm{Q}(6,q) = \mathrm{Q}(4,q)$, then $\mathcal{O}$ must necessarily be an ovoid of $\mathrm{Q}(4,q)$; impossible since $|\mathcal{O}| > q^2+1$. If $\langle \mathcal{O} \rangle \cap \mathrm{Q}(6,q) = \mathrm{Q}^+(5,q)$, then $\mathcal{O}$ must be an ovoid of $\mathrm{Q}^+(5,q)$; impossible since $|\mathcal{O}| > q^2 + 1$. Finally, $\langle \mathcal{O} \rangle \cap \mathrm{Q}(6,q) = \mathrm{Q}^-(5,q)$ is impossible, since $\mathrm{Q}^-(5,q)$ does not have ovoids [14]. ■

Lemma 2. *No elliptic quadric* $\mathrm{Q}^-(3,q)$ *is contained in* $\mathcal{O}$.

Proof. Suppose that some $\mathrm{Q}^-(3,q) \subseteq \mathcal{O}$. Consider a point $p \in \mathcal{O} \setminus \mathrm{Q}^-(3,q)$. The 4-space $\alpha = \langle p, \mathrm{Q}^-(3,q) \rangle$ intersects $\mathrm{Q}(6,q)$ in a parabolic quadric $\mathrm{Q}(4,q)$ or in a cone $r\mathrm{Q}'^-(3,q)$. If α intersects $\mathrm{Q}(6,q)$ in a $\mathrm{Q}(4,q)$ then it contains at least $q^2 + 2$ points of $\mathcal{O}$, a contradiction, since any $\mathrm{Q}(4,q)$ can intersect $\mathcal{O}$ in at most $q^2 + 1$ points, the number of points of an ovoid of $\mathrm{Q}(4,q)$. If α intersects $\mathrm{Q}(6,q)$ in a cone $r\mathrm{Q}'^-(3,q)$, then $\mathcal{O}$ contains at least two points spanning a line of $\mathrm{Q}(6,q)$, a contradiction. ■

The following lemma is an application of Theorem 1.

Lemma 3. *The ovoid* $\mathcal{O}$ *does not contain an ovoid* $\mathcal{O}'$ *of* $\mathrm{Q}(4,q)$, *with* $\mathrm{Q}(4,q)$ *contained in* $\mathrm{Q}(6,q)$.

Proof. Suppose the contrary, i.e., suppose that there is some ovoid $\mathcal{O}'$ of $\mathrm{Q}(4,q) \subseteq \mathrm{Q}(6,q)$, with $\mathcal{O}' \subseteq \mathcal{O}$. By the previous lemma, we may suppose that $\mathcal{O}'$ is not an elliptic quadric and hence, $\langle \mathcal{O}' \rangle$ is a 4-dimensional projective space α, such that $\alpha \cap \mathrm{Q}(6,q) = \mathrm{Q}(4,q)$. Since $\mathcal{O}$ spans the 6-dimensional space, we can choose a point $p \in \mathcal{O} \setminus \alpha$. Since $\alpha \cap \mathcal{O}$ contains an ovoid of $\mathrm{Q}(4,q)$, $p \notin \alpha^\perp$, hence $p^\perp \cap \mathrm{Q}(4,q) = \mathrm{Q}^\pm(3,q)$, or $p^\perp \cap \mathrm{Q}(4,q) = r\mathrm{Q}(2,q)$ which is a tangent cone to $\mathrm{Q}(4,q)$. If $p^\perp \cap \mathrm{Q}(4,q) = r\mathrm{Q}(2,q)$ or $p^\perp \cap \mathrm{Q}(4,q) = \mathrm{Q}^+(3,q)$, then $p^\perp$ contains a generator of $\mathrm{Q}(4,q)$ meeting $\mathcal{O}'$ and hence $p^\perp$ contains a point of $\mathcal{O}'$, a contradiction. If $p^\perp \cap \mathrm{Q}(4,q) = \mathrm{Q}^-(3,q)$, then Theorem 1 implies that $p^\perp$ contains a point of $\mathcal{O}'$, a contradiction. ∎

We call a hyperplane α of $\mathrm{PG}(6,q)$ *hyperbolic, elliptic* respectively, if $\alpha \cap \mathrm{Q}(6,q) = \mathrm{Q}^+(5,q)$, $\alpha \cap \mathrm{Q}(6,q) = \mathrm{Q}^-(5,q)$ respectively.

Corollary 1. *Any hyperbolic hyperplane α has the property that $\langle \alpha \cap \mathcal{O} \rangle = \alpha$.*

Proof. Suppose that α is a hyperbolic hyperplane. Then necessarily α intersects $\mathcal{O}$ in an ovoid $\mathcal{O}'$ of a $\mathrm{Q}^+(5,q)$. Since any ovoid of $\mathrm{Q}(4,q)$ is not contained in $\mathcal{O}$, the ovoid $\mathcal{O}'$ spans the 5-dimensional space α. ∎

It is known that $\mathrm{Q}(6,3)$ has, up to collineations, a unique ovoid [10]. In [20], one finds an explicit list, related to a chosen $\mathrm{Q}(6,3)$, of the coordinates in $\mathrm{PG}(6,3)$ of the points of this ovoid. With the aid of the software package `pg` [3], we can compute all hyperplanes of $\mathrm{PG}(6,3)$, select the elliptic hyperplanes from that list and check whether such an elliptic hyperplane is spanned by the points of the ovoid it contains. The software package `pg` is a package written in the language of the computer algebra system `GAP` [7]. Checking the mentioned property can be done with a few commands of the package `pg`. We found the following result.

Lemma 4. *Any elliptic hyperplane α of $\mathrm{PG}(6,3)$ has the property that $\langle \alpha \cap \mathcal{O} \rangle = \alpha$.*

We end this section with the following result. It was proved in [2], using Theorem 1.

Theorem 9. (Ball, Govaerts and Storme [2]) *Suppose that $\mathrm{Q}(6,q)$, $q = p^h$, $h \geqslant 1$, p an odd prime, has an ovoid $\mathcal{O}$. Then any elliptic hyperplane intersects $\mathcal{O}$ in $1 \bmod p$ points.*

3 Small minimal blocking sets of $\mathrm{Q}(2n,3)$, $n \geqslant 3$

We consider the parabolic quadric $\mathrm{Q}(2n,3)$, $n \geqslant 3$. Some lemmas are restricted to $n \geqslant 4$. In that case, we assume that the following hypothesis is true for $\mathrm{Q}(2k,3)$, $k = 3,\ldots,n-1$.

The minimal blocking sets of size at most $q^k + q^{k-2} + 1$ in $\mathrm{Q}(2k, q = 3)$ are truncated cones $\pi^*_{k-4}\mathcal{O}$, $\pi^\perp_{k-4} \cap \mathrm{Q}(2k, q = 3) = \pi_{k-4}\mathrm{Q}(6, q = 3)$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6, q = 3)$; and truncated cones $\pi^*_{k-3}\mathrm{Q}^-(3, q = 3)$, $\pi^\perp_{k-3} \cap \mathrm{Q}(2k, q = 3) = \pi_{k-3}\mathrm{Q}^-(3, q = 3)$, π_i an i-dimensional subspace contained in $\mathrm{Q}(2k, q = 3)$. These examples have respectively size $q^k + q^{k-3}$ and $q^k + q^{k-2}$.

To prove this hypothesis for $n = 4$, we will consider $\mathrm{Q}(6,3)$.

Suppose that $\mathcal{K}$ is a minimal blocking set of size at most $q^n + q^{n-2} + 1$ of $Q(2n, q = 3)$, $n \geqslant 3$. Since the smallest minimal blocking sets of $Q(2n, q = 3)$, $n \geqslant 4$, of size $q^n + q^{n-3}$, are already classified [5], we also assume that $|\mathcal{K}| \geqslant q^n + q^{n-3} + 1$ when $n \geqslant 4$.

The next two lemmas can be proved by techniques of [6].

Lemma 5. *For every point* $r \in \mathcal{K}$, $|r^\perp \cap \mathcal{K}| \leqslant q^{n-2} + 1$.

Lemma 6. *Consider a point* $r \in Q(2n, q) \setminus \mathcal{K}$, *then the points of* $r^\perp \cap \mathcal{K}$ *are projected from* r *onto a minimal blocking set* $\mathcal{K}_r$ *of* $Q(2n-2, q)$, *with* $Q(2n-2, q)$ *the base of the cone* $r^\perp \cap Q(2n, q)$.

We call a line of $Q(2n, q)$ meeting $\mathcal{K}$ in i points an *i-secant* to $\mathcal{K}$. The next lemma and its corollary are restricted to $n = 3$ but will be generalized to $n \geqslant 4$. We use the fact that a minimal blocking set of $Q(4, 3)$, different from an ovoid, contains at least $12 = q^2 + q$ points, with $q = 3$. This is proved in e.g. [5].

Lemma 7. *There are no lines of* $Q(6, 3)$ *meeting* $\mathcal{K}$ *in exactly* 2 *points.*

Proof. Suppose that L is a 2-secant to $\mathcal{K}$. Consider a generator π of $Q(6, 3)$ on L such that $\pi \cap \mathcal{K} = L \cap \mathcal{K}$; Lemma 5 implies that such a generator exists. Count the number of pairs (u, v), $u \in \pi \setminus L$, $v \in \mathcal{K} \setminus L$, $u \in v^\perp$. Since the projection of the set of points $u^\perp \cap \mathcal{K}$ from u is a minimal blocking set of $Q(4, 3)$, and since it cannot be an ovoid of $Q(4, 3)$, it must contain at least $q^2 + q$ points of $Q(4, 3)$. We obtain $q^2(q^2 + 1)$ as lower bound for this number. Using the size of $\mathcal{K}$, we find $(q^3 + q - 1)q = q^4 + q^2 - q$ as upper bound, hence, $q^2(q^2 + 1) \leqslant q^4 + q^2 - q$, a contradiction. ∎

Corollary 2. *Every generator* π *of* $Q(6, q = 3)$ *intersects* $\mathcal{K}$ *in* 1 *point, or in* 3 *or* 4 *collinear points.*

Proof. Since there are no 2-secants to $\mathcal{K}$, 2 points of $\mathcal{K}$ in π give rise to 3 or 4 collinear points of $\mathcal{K}$ in π. If there would be 3 points of $\mathcal{K}$ spanning π, then π would contain at least 7 points of $\mathcal{K}$, a contradiction with Lemma 5. ∎

To generalize these two propositions, we rely now on the induction hypothesis.

Lemma 8. *No generator* π_{n-1} *of* $Q(2n, q = 3)$, $n \geqslant 4$, *intersects* $\mathcal{K}$ *in exactly* 2 *points.*

Proof. Suppose that for some generator π_{n-1} of $Q(2n, q)$, $|\pi_{n-1} \cap \mathcal{K}| = 2$, where the two points of $\pi_{n-1} \cap \mathcal{K}$ lie on the line L. Count the number of pairs (u, v), $u \in \pi_{n-1} \setminus L$, $u \in v^\perp$, $v \in \mathcal{K} \setminus \pi_{n-1}$. Since no minimal blocking set of size at most $q^{n-1} + q^{n-3} + 1$ of $Q(2n-2, q)$ has a 2-secant, we find $|u^\perp \cap \mathcal{K}| \geqslant q^{n-1} + q^{n-3} + 2$. Hence, the lower bound on the number of pairs is $(q^{n-1} + \ldots + q^2)(q^{n-1} + q^{n-3})$. As upper bound, we find $(q^n + q^{n-2} - 1)(q^{n-2} + \ldots + q)$, which is smaller than the lower bound, a contradiction. ∎

Corollary 3. *No line* L *of* $Q(2n, 3)$, $n \geqslant 4$, *intersects* $\mathcal{K}$ *in exactly* 2 *points.*

Proof. Suppose that L is a 2-secant to $\mathcal{K}$. By the minimality of $\mathcal{K}$ and Lemma 5, there exists a generator π_{n-1} on L such that $L \cap \mathcal{K} = \pi_{n-1} \cap \mathcal{K}$, a contradiction. ∎

In three steps, we now prove Theorem 8 for $n = 3$.

Lemma 9. *Suppose that L is a line of* $Q(6,3)$ *meeting* $\mathcal{K}$ *in* 3 *or* 4 *points. Suppose that* π *is a generator of* $Q(6,3)$ *on* L*, then* $L \cap \mathcal{K} = \pi \cap \mathcal{K}$*, and* $|r^\perp \cap \mathcal{K}| \leqslant q^2+q+1$ *for every* $r \in \pi \setminus L$.

Proof. Let r_0 be one of the points of $\mathcal{K} \cap \pi$. Suppose that $r \in \pi \setminus L$. Then there exists a generator π' of $Q(6,3)$ through r meeting $\mathcal{K}$ only in r_0. The $q^2 - q$ lines of π' not through r_0 or r lie in q generators of $Q(6,3)$ different from π'. Hence, at least $q^3 - q^2$ points of $\mathcal{K}$ lie outside $r^\perp$, and so, $|r^\perp \cap \mathcal{K}| \leqslant q^2 + q + 1$. ■

Lemma 10. *Suppose that L is a* 3*-secant to* $\mathcal{K}$*, then the point* $r \in L \setminus \mathcal{K}$ *only lies on* 3*-secants to* $\mathcal{K}$ *and* $\mathcal{K} = r^*\mathcal{O}$*,* $\mathcal{O}$ *an ovoid of* $Q(4,3)$*, with* $Q(4,3)$ *the base of the cone* $r^\perp \cap Q(6,3)$.

Proof. Put $\mathcal{K} \cap L = \{r_1, r_2, r_3\}$ and $r \in L \setminus \mathcal{K}$. Since $|(r_1^\perp \cup r_2^\perp \cup r_3^\perp) \cap \mathcal{K}| \leqslant 3+1+1+1$, necessarily $|r^\perp \cap \mathcal{K}| \geqslant q^3 + q + 1 - 6 = q^3 - 2 > q^2 + q + 1$, so, using the proof of Lemma 9, r does not lie in a generator with 1 point of $\mathcal{K}$, so r only lies in generators containing at least 3 points of $\mathcal{K}$. Moreover, these 3 or 4 points are collinear with r by Corollary 2 and Lemma 9. If r projects the points of $r^\perp \cap \mathcal{K}$ onto an ovoid of $Q(4,3)$, then $|\mathcal{K}| = q(q^2+1)$; else $|\mathcal{K}| \geqslant q(q^2+2)$. Since $|\mathcal{K}| \leqslant q^3+q+1$, necessarily $\mathcal{K} = r^*\mathcal{O}$, $\mathcal{O}$ an ovoid of $Q(4,3)$, with $Q(4,3)$ the base of the cone $r^\perp \cap Q(6,3)$. ■

Theorem 10. *A minimal blocking set* $\mathcal{K}$ *of size* $|\mathcal{K}| \leqslant q^3 + q + 1$*,* $q = 3$*, of* $Q(6,3)$ *is an ovoid* $\mathcal{O}$ *or a truncated cone* $r^*\mathcal{O}$*,* $\mathcal{O}$ *an elliptic quadric* $Q^-(3,3) \subseteq Q(4,3)$*, with* $Q(4,3)$ *the base of the cone* $r^\perp \cap Q(6,3)$*. In particular, there does not exist a minimal blocking set of size* q^3+q+1 *on* $Q(6,3)$.

Proof. Assume that $\mathcal{K}$ is not an ovoid of $Q(6,3)$, then a line of $Q(6,3)$ is either a 1-, 3-, or 4-secant to $\mathcal{K}$. By Lemma 10, we can assume that there is no 3-secant to $\mathcal{K}$. So a line of $Q(6,3)$ containing at least 2 points of $\mathcal{K}$ contains 4 points of $\mathcal{K}$. Suppose that L is a 4-secant to $\mathcal{K}$. By Lemma 5, we find that $|\mathcal{K}| \leqslant 4$, since a point of $Q(6,3) \setminus L$ is perpendicular to at least one point of L. But $|\mathcal{K}| > q^3 + 1$, a contradiction. ■

Finally, we prove Theorem 8 in four steps.

Lemma 11. *Suppose that* π_{n-1} *is a generator of* $Q(2n,q)$ *such that* $|\pi_{n-1} \cap \mathcal{K}| = 1$*. For every* $r \in \pi_{n-1} \setminus \mathcal{K}$*, we have that* $|r^\perp \cap \mathcal{K}| \leqslant q^{n-1} + q^{n-2} + 1$.

Proof. Denote the unique point in $\pi_{n-1} \cap \mathcal{K}$ by s. The $q^{n-1} - q^{n-2}$ hyperplanes of π_{n-1}, not through r or s, all lie in q generators, different from π_{n-1}, all containing at least one point of $\mathcal{K}$. So at least $(q^{n-1} - q^{n-2})q$ points lie in $\mathcal{K} \setminus r^\perp$; so $|r^\perp \cap \mathcal{K}| \leqslant q^{n-1} + q^{n-2} + 1$. ■

Lemma 12. *Suppose that* $r \notin \mathcal{K}$*, and suppose that L is a line of* $Q(2n,3)$ *through* r *such that* $|L \cap \mathcal{K}| = 1$*. Then* $|r^\perp \cap \mathcal{K}| \leqslant q^{n-1} + q^{n-2} + 1$.

Proof. Consider a generator through the line $\langle r, s \rangle$, $s \in L \cap \mathcal{K}$, only containing the point $s \in \mathcal{K}$. Such a generator exists; or else $|s^\perp \cap \mathcal{K}| \geqslant q^{n-2} + 2$. The preceding lemma proves the assertion. ■

Lemma 13. *There does not exist a line of* $\mathrm{Q}(2n,3)$ *intersecting* $\mathcal{K}$ *in* 4 *points.*

Proof. Suppose that L is a line of $\mathrm{Q}(2n,3)$ meeting $\mathcal{K}$ in 4 points. By Lemma 5, we find that $|\mathcal{K}| \leqslant 4(q^{n-2}+1) < q^n+1$, a contradiction. ∎

Theorem 11. *The minimal blocking sets of* $\mathrm{Q}(2n,q=3)$, $n \geqslant 3$, *of size at most* $q^n+q^{n-2}+1$, *are truncated cones* $\pi^*_{n-4}\mathcal{O}$, $\pi^{\perp}_{n-4} \cap \mathrm{Q}(2n,q=3) = \pi_{n-4}\mathrm{Q}(6,q=3)$, $\mathcal{O}$ *an ovoid of* $\mathrm{Q}(6,3)$, *and* $\pi^*_{n-3}\mathrm{Q}^-(3,q=3)$, $\pi^{\perp}_{n-3} \cap \mathrm{Q}(2n,q=3) = \pi_{n-3}\mathrm{Q}(4,q=3)$, $\mathrm{Q}^-(3,q=3) \subseteq \mathrm{Q}(4,q=3)$. *Furthermore, a minimal blocking set of size* $q^n+q^{n-2}+1$ *of* $\mathrm{Q}(2n,q=3)$ *does not exist.*

Proof. Suppose that L is a line of $\mathrm{Q}(2n,3)$, which also is a 3-secant to $\mathcal{K}$. Put $L\cap\mathcal{K} = \{r_1,r_2,r_3\}$ and $r \in L\backslash\mathcal{K}$. Then $|(r_1^{\perp}\cup r_2^{\perp}\cup r_3^{\perp})\cap\mathcal{K}| \leqslant q^{n-2}+1+2(q^{n-2}-2) \leqslant q^{n-1}-3$. So $|r^{\perp}\cap\mathcal{K}| \geqslant q^n+q^{n-3}+1-(q^{n-1}-3) = 2q^{n-1}+q^{n-3}+4 > q^{n-1}+q^{n-2}+1$. So every generator through r meets $\mathcal{K}$ in at least 3 points, hence $|r^{\perp}\cap\mathcal{K}| \geqslant 3(q^{n-1}+1)$. The projection of $r^{\perp}\cap\mathcal{K}$ from r contains at least $q^{n-1}+q^{n-4}$ points; so since r lies on 3-secants to the projected points, necessarily $|r^{\perp}\cap\mathcal{K}| \geqslant 3(q^{n-1}+q^{n-4})$, by the induction hypothesis. The induction hypothesis implies also that $r^{\perp}\cap\mathcal{K}$ is projected onto a truncated cone $\pi^*_{n-5}\mathcal{O}$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6,q)$, or a truncated cone $\pi^*_{n-4}\mathrm{Q}^-(3,q)$, since the projection of $\mathcal{K}\cap r^{\perp}$ must be a minimal blocking set of the base $\mathrm{Q}(2n-2,3)$ of the cone $r^{\perp}\cap\mathrm{Q}(2n,3)$. It follows that $|r^{\perp}\cap\mathcal{K}| = q^n+q^{n-3}$ or, respectively, q^n+q^{n-2}. Hence, $r^{\perp}\cap\mathcal{K}$ contains a truncated cone $\pi^*_{n-4}\mathcal{O}$, $\pi^{\perp}_{n-4}\cap\mathrm{Q}(2n,q=3) = \pi_{n-4}\mathrm{Q}(6,q)$, $\mathcal{O}$ an ovoid of $\mathrm{Q}(6,q)$, or, respectively a truncated cone $\pi^*_{n-3}\mathrm{Q}^-(3,q)$. Since these structures are minimal blocking sets of $\mathrm{Q}(2n,q=3)$, we conclude that $\mathcal{K}$ is necessarily equal to one of these structures. ∎

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