The uniqueness of the strongly regular graph srg$(105, 32, 4, 12)$

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Abstract

We show that the strongly regular graph with parameters $v = 105$, $k = 32$, $\lambda = 4$ and $\mu = 12$ is uniquely determined by its parameters (upto isomorphism).

1 Introduction

In [4] E. van Dam and W. Haemers consider the question as to which graphs are determined by their spectrum. Restricted to regular graphs with three distinct eigenvalues the question is equivalent to asking which strongly regular graphs are uniquely determined by their parameters v, k, λ and μ. Apart from three infinite families there are only a few examples of parameter sets for which uniqueness is known, a list of which is given in their paper.

In this article we shall extend the list by one more example. In theorem 3 we prove that the strongly regular graph with parameters $v = 105$, $k = 32$, $\lambda = 4$ and $\mu = 12$ is unique. This graph has spectrum $\{[32]^1, [2]^{84}, [-10]^{20}\}$. The fact that the second eigenvalue is equal to 2 will play an important role in the proof. Note that the majority of the graphs in the list of van Dam and Haemers also have this property. Also remark that this graph occurs as a subgraph of the well-known McLaughlin graph [3], again a property that it shares with many others in the list.

2 Notation and terminology

Let Γ be any graph. We will silently identify Γ with its vertex set, and hence write $p \in \Gamma$ to indicate that p is a vertex of Γ and write $D \subseteq \Gamma$ when D is a subset of

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the vertex set of Γ. Also the subgraph of Γ induced by a subset D will often simply be denoted by D. We write $p \sim q$ to indicate that the vertices p and q of Γ are adjacent.

If $D \subseteq \Gamma$ then the *neighbourhood* $\Gamma(D)$ of D in Γ is the set of all vertices in $\Gamma - D$ that are adjacent to at least one vertex of D. We shall write $\Gamma_{\geq 2}(D) \stackrel{\text{def}}{=} \Gamma - D - \Gamma(D)$ for the set of all vertices in $\Gamma - D$ that are adjacent to none of the vertices of D. Also, we write $\Gamma(p)$ for $\Gamma(\{p\})$ and $\Gamma_{\geq 2}(p)$ for $\Gamma_{\geq 2}(\{p\})$.

Let $u_0, u_1, u_2 \in \mathbf{R}$. We define $\Gamma[u_0, u_1, u_2]$ to be the real symmetric matrix whose rows and columns are indexed by the vertices of Γ, with diagonal entries u_0, entries u_1 at positions p, q such that $p \sim q$, and entries u_2 at all other positions. $\Gamma[0, 1, 0]$ is the *adjacency matrix* of Γ, also denote by $A(\Gamma)$ or simply A when Γ is clear from context.

We have

$$\Gamma[u_0, u_1, u_2] = (u_0 - u_2)I + (u_1 - u_2)A(\Gamma) + u_2 J, \tag{1}$$

where I denotes the $v \times v$ identity matrix and J is the matrix with all elements equal to 1 (with $v = |\Gamma|$).

We refer to [1] for the definitions of *strongly regular graph*, *distance regular graph* and *intersection array*.

3 The known example

One strongly regular graph Γ^* with $v = 105$, $k = 32$, $\lambda = 4$ and $\mu = 12$ has already been known for some time. It can be described as follows : vertices are the 105 flags of $\mathrm{PG}_2(4)$ and two flags (a, A) and (b, B) are adjacent if and only if $a \neq b$, $A \neq B$, $a \in B$ or $b \in A$. Also the structure of the neighbourhood Δ^* of a vertex in Γ^* is easily described. Vertices of Δ^* are the 16 points and 16 lines of $\mathrm{AG}_2(4)$ with 1 parallel class $\mathcal{L}$ of $\mathrm{AG}_2(4)$ removed. Adjacency in Δ^* corresponds to incidence in $\mathrm{AG}_2(4)$. In other words, Δ^* is the incidence graph of a semi-affine plane.

The graph Δ^* can be partitioned into 8 *colour* classes (co-cliques) of size 4, as follows: 4 lines belong to the same class if and only if they are parallel in $\mathrm{AG}_2(4)$, 4 points belong to the same colour class if and only if in $\mathrm{AG}_2(4)$ they lie on the same line of $\mathcal{L}$. Two vertices are called *antipodal* if and only if they belong to the same colour class. Edges of Δ^* are called antipodal when each vertex of one edge is antipodal to a vertex of the other edge.

Two vertices of Δ^* will be said to be of the same *type* if and only if they both correspond to points of $\mathrm{AG}_2(4)$, or both correspond to lines of $\mathrm{AG}_2(4)$. Vertices of the same type are never adjacent. We shall extend this definition to colour classes: there are 4 classes of one type and 4 of the other.

It can easily be proved that Δ^* is a distance regular graph with intersection array $(4, 3, 3, 1; 1, 1, 3, 4)$. The eigenvalues of Δ^* are $4, 2, 0, -2$ and -4 with corresponding multiplicities $1, 12, 6, 12$ and 1. Moreover, any distance regular graph with this intersection array must be isomorphic to Δ^*.

4 Positive semidefinite matrices

A symmetric matrix $M \in \mathbf{R}^{n\times n}$ is called *positive semidefinite* when $xMx^T \geq 0$ for every vector $x \in \mathbf{R}^{1\times n}$. When $\Gamma[u_0, u_1, u_2]$ is positive semidefinite, Γ is said to admit a (u_0, u_1, u_2)-*representation.* Many of the proofs given in later sections rely heavily on the properties of positive semidefinite matrices which are given below. For proofs and further information we refer to [1, Chapter 3].

If M is a real symmetric positive semidefinite matrix and A is a real matrix with the same number of columns as M, then also AMA^T is positive semidefinite. Indeed $x(AMA^T)x^T = (xA)M(xA)^T \geq 0$ for every x. In particular, by choosing the appropriate matrix A we may easily prove that a matrix remains positive semidefinite when all of its elements are multiplied by the same positive number or when we multiply a row and at the same time the corresponding column by the same constant.

A symmetric matrix M is positive semidefinite if and only if all its eigenvalues are positive or zero. It follows that $\det M \geq 0$ when M is positive semidefinite.

If the rows and columns of M are indexed by vertices of a graph Γ, then for any vertex set $S \subseteq \Gamma$ we denote by $\mathsf{sub}(M; S)$ the square symmetric matrix obtained from M by removing all rows and columns that correspond to vertices not belonging to S. When M is positive semidefinite, then so is $\mathsf{sub}(M; S)$. In particular $\det(\mathsf{sub}(M; S)) \geq 0$ and every diagonal entry of M is positive or zero.

If $S_1, \ldots, S_k$ are mutually disjoint subsets of D, then $\mathsf{part}(M; S_1, \ldots, S_k)$ is a block matrix obtained from $\mathsf{sub}(M; S_1 \cup \cdots \cup S_k)$ by reordering and partitioning its rows and columns in such a way that the block at position i, j has rows corresponding to the elements of S_i and columns corresponding to the elements of S_j. A block matrix of this type is used in the following lemma, which is a generalization of [1, Proposition 3.7.1 (iii)].

Lemma 1. *Let M be a positive semidefinite matrix and let $\mathsf{part}(M; S_1, \ldots, S_k)$ be a partitioning of M into blocks M_{ij} of order $s_i \times s_j$, (with $s_i \stackrel{\text{def}}{=} |S_i|$). Let $v_1, \ldots, v_k$ denote real (row) vectors, where v_i has dimension s_i. Then the $k \times k$ matrix*

$$\overline{M} \stackrel{\text{def}}{=} \begin{pmatrix} v_1M_{11}v_1^T & v_1M_{12}v_2^T & \cdots & v_1M_{1k}v_k^T \\ v_2M_{21}v_1^T & v_2M_{22}v_2^T & \cdots & v_2M_{2k}v_k^T \\ \vdots & \vdots & \ddots & \\ v_kM_{k1}v_1^T & v_kM_{2k}v_2^T & \cdots & v_kM_{kk}v_k^T \end{pmatrix}$$

is positive semidefinite.

Proof. Let $a = (a_1, \ldots, a_k)$. Then

$$\begin{aligned} a\,\overline{M}\,a^T &= (a_1, \ldots, a_k)\,\overline{M}\,(a_1, \ldots, a_k)^T \\ &= \Big(a_1v_1 \,\Big|\, a_2v_2 \,\Big|\, \cdots \,\Big|\, a_kv_k \Big)\, M \begin{pmatrix} a_1v_1^T \\ \hline a_2v_2^T \\ \hline \vdots \\ a_kv_k^T \end{pmatrix} \geq 0, \end{aligned}$$

because M is positive semidefinite. ∎

In particular, the $k \times k$ matrix obtained by replacing every block by the arithmetic mean of its elements (which we will shall a *condensed* matrix) is positive semidefinite. Also the sum of the elements of M, or more generally, the sum of the elements of any principal submatrix $\mathsf{sub}(M,S)$ of M must be ≥ 0.

If we write M as

$$M = \begin{pmatrix} \mu & m \\ m^T & M' \end{pmatrix}$$

where $\mu \in \mathbf{R}$, $m \in \mathbf{R}^{1 \times n-1}$ and $M' \in \mathbf{R}^{n-1 \times n-1}$, then M is positive definite if and only if either $\mu = 0$, $m = 0$ and M' is positive semidefinite, or $\mu > 0$ and $M' - m^T m/\mu$ is positive definite. In what follows we shall use the terminology '*reduction of M with respect to the top left element*' when we apply this property.

5 Regular graphs with second largest eigenvalue equal to 2

When Γ is a regular graph of degree k, the adjacency matrix A of Γ has largest eigenvalue equal to k. In this section we shall consider regular graphs with the additional property that the second largest eigenvalue of A is equal to 2.

Lemma 2. *Let Γ be a regular graph of size v and degree $k > 2$, with second largest eigenvalue equal to 2. Then Γ admits an $(a+2, a-1, a)$-representation for $a = (k-2)/v$.*

Proof. This is an immediate consequence of [1, Proposition 3.5.3] and can easily be proved by considering the eigenvalues of $\Gamma[a+2, a-1, a]$. ∎

Lemma 3. *Let Γ be a graph which admits an $(a+2, a-1, a)$-representation for some $a \in \mathbf{R}$. Let D be an induced subgraph of Γ. Then the average degree d of D satisfies*

$$d \leq a|D| + 2. \tag{2}$$

In case of equality, either $\Gamma_{\geq 2}(D)$ is empty or $a = 0$.

Otherwise $\Gamma_{\geq 2}(D)$ admits an $(a'+2, a'-1, a')$-representation with

$$a' = \frac{a(2-d)}{a|D| + 2 - d}. \tag{3}$$

Proof. Consider the matrix $\mathsf{sub}(\Gamma[a+2, a-1, a]; D)$. This matrix is positive semidefinite and hence the sum of its elements must be positive or zero. This sum is equal to $a|D|^2 + 2|D| - d|D|$ and so $a|D| + 2 \geq d$.

Write $D' = \Gamma_{\geq 2}(D) = \{p_1, \ldots, p_m\}$ and consider $\mathsf{part}(\Gamma[a+2, a-1, a]; D, p_1, \ldots, p_m)$. This matrix is condensed to the following $m+1 \times m+1$ matrix :

$$\begin{pmatrix} a + (2-d)/|D| & a & \cdots & a \\ a & & & \\ \vdots & & D'[a+2, a-1, a] & \\ a & & & \end{pmatrix},$$

which must be positive semidefinite. If the top left element of this matrix is zero, then also a must be 0. Otherwise we reduce the matrix with respect to the top left element, yielding a $(a+2-\delta, a-1-\delta, a-\delta)$-representation for D' with

$$\delta = \frac{a^2}{a + (2-d)/|D|}$$

and hence $a' = a - \delta$ as stated. ∎

In particular, when D in the above lemma has average degree 2, then $\Gamma_{\geq 2}(D)$ admits a $(2, -1, 0)$-representation, or equivalently, $\Gamma_{\geq 2}(D)$ has largest eigenvalue ≤ 2.

Theorem 1. *Let Γ be a regular graph of size v and degree $k > 2$, with second largest eigenvalue equal to 2. Let D be an induced n-cycle of Γ. Then*

$$n \geq \frac{v}{2(k-1)}. \tag{4}$$

In case of equality, every vertex of $\Gamma - D$ is connected to either 0 or 1 elements of D, $\Gamma(D)$ is a co-clique and $\Gamma_{\geq 2}(D)$ has average degree equal to 2.

Proof. Let D be a subset of size n which induces an n-cycle on Γ. Because D has degree 2 we know that $|\Gamma(D)| \leq (k-2)n$ and hence that $\Gamma_{\geq 2}(D) = \Gamma - \Gamma(D) - D$ has size at least $v - (k-1)n$. By lemma 3, $\Gamma_{\geq 2}(D)$ must admit a $(2, -1, 0)$-representation and therefore have average degree ≤ 2.

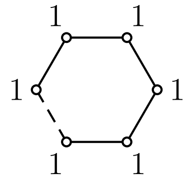
Now, count the number e of edges qr with $q \in \Gamma(D)$ and $r \in \Gamma_{\geq 2}(D)$ in two ways. For every $q \in \Gamma(D)$ we find at most $k-1$ adjacent vertices r, giving $e \leq (k-1)(k-2)n$. For every $r \in \Gamma_{\geq 2}(D)$ we find at least $k-2$ adjacent vertices q. Hence $e \geq (k-2)(v-(k-1)n)$. Therefore $(k-2)(v-(k-1)n) \leq (k-1)(k-2)n$ and hence $v \leq 2(k-1)n$. When $v = 2(k-1)n$ all inequalities in this argument must become equalities. ∎

6 Subgraphs with largest eigenvalue 2

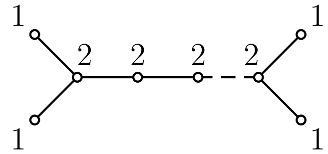
A graph admits a $(2, -1, 0)$-representation if and only if its largest eigenvalue is at most 2. These graphs have been classified upto isomorphism in [1, Theorem 3.2.5]. We reproduce this theorem below :

Theorem 2 (Smith). *The only connected graphs having largest eigenvalue 2 are the following graphs (the number of vertices is one more than the index given). For each graph, the corresponding eigenvector is indicated by the integers at the vertices.*

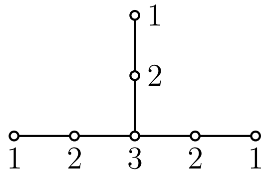
$\tilde{A}_n (n \geq 2)$



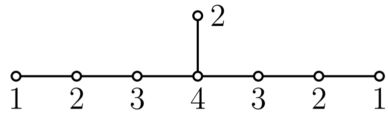
$\tilde{D}_n (n \geq 4)$



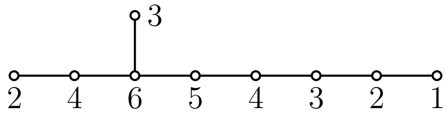
$\tilde{E}_6$



$\tilde{E}_7$



$\tilde{E}_8$



Moreover, each connected graph with largest eigenvalue < 2 is a subgraph of one of the above graphs, and each connected graph with largest eigenvalue > 2 contains one of these graphs.

For each connected graph Γ with largest eigenvalue 2, we obtain a *normalized eigenvector* y of Γ by dividing the eigenvector listed in theorem 2 by the sum of its coordinates. For such y we have $yIy^T = 1$, $yJy^T = 1$ and $yAy^T = 2yy^T = 2$, where A is the adjacency matrix of Γ. Then, applying (1), we find

$$y\,\Gamma[a+2, a-1, a]\,y^T = a. \tag{5}$$

Lemma 4. *Let Γ be a graph which admits an $(a+2, a-1, a)$-representation for some $a \in \mathbf{R}$. Let D be an induced subgraph of Γ with largest eigenvalue 2. Then $\Gamma_{\geq 2}(D)$ admits a $(2, -1, 0)$-representation.*

Proof. As in the proof of lemma 3, consider $\mathsf{part}(\Gamma[a+2, a-1, a]; D, p_1, \ldots, p_m)$ where $D' = \Gamma_{\geq 2}(D) = \{p_1, \ldots, p_m\}$. Now apply lemma 1 using vectors $v_1, v_2, \ldots = y, 1, 1, \ldots$, where y is a normalized eigenvector for D. By (5) we obtain the following $m+1 \times m+1$ matrix :

$$\begin{pmatrix} a & a & \cdots & a \\ a & & & \\ \vdots & & D'[a+2, a-1, a] & \\ a & & & \end{pmatrix}.$$

Reducing with respect to the top left element yields the positive semidefinite matrix $D'[2, -1, 0]$. ∎

Let D be a connected induced subgraph of Γ with largest eigenvalue equal to 2. Let y be a normalized eigenvector of D. For $p \in \Gamma - D$, we define the *weight* $w_D(p)$ of p with respect to D to be the sum $\sum_{i \sim p} y_i$ of all coordinates of y whose indices correspond to vertices of D adjacent to p. Note that $0 \leq w_D(p) \leq 1$.

Lemma 5. *Let D and D' be connected induced subgraphs of Γ, both with largest eigenvalue 2, such that no vertex of D is adjacent to any vertex of D'. If $p \in \Gamma - D - D'$, then $w_D(p) = w_{D'}(p)$.*

Proof. Consider the block matrix

$$\mathsf{part}(\Gamma[a+2, a-1, a]; D, D', p) = \left(\begin{array}{c|c|c} D[a+2, a-1, a] & aJ & \\ \hline aJ & D'[a+2, a-1, a] & \\ \hline & & a \end{array}\right),$$

where J denotes an all-1 matrix of appropriate size and the entries in the positions that are left blank are equal to either $a-1$ or a, depending on whether p is adjacent or not adjacent to the appropriate vertex of D or D'.

Applying lemma 1 with v_1 a normalized eigenvector of D, v_2 a normalized eigenvector of D' and $v_3 = 1$, we see that the following matrix must be positive semidefinite :

$$\begin{pmatrix} a & a & a - w_D(p) \\ a & a & a - w_{D'}(p) \\ a - w_D(p) & a - w_{D'}(p) & a \end{pmatrix},$$

and therefore also the following matrix, obtained from it by reducing with respect to the top left element :

$$\begin{pmatrix} 0 & w_D(p) - w_{D'}(p) \\ w_D(p) - w_{D'}(p) & \end{pmatrix}.$$

(The value of the bottom right element is irrelevant to this proof.)

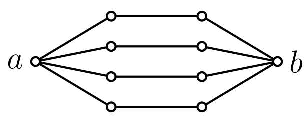
As the top left element of this last matrix is zero, it can only be positive semidefinite when the entire first row is zero. Hence $w_D(p) = w_{D'}(p)$. ∎

7 Strongly regular graphs with parameters $v = 105$, $k = 32$, $\lambda = 4$ and $\mu = 12$

From now on let Γ denote any strongly regular graph with parameters $v = 105$, $k = 32$, $\lambda = 4$ and $\mu = 12$. Using standard (algebraic) techniques (cf. [1]) we may easily prove that the eigenvalues of Γ are 32, 2 and -10 (with multiplicities 1, 84 and 20). Hence Γ is a graph with second largest eigenvalue equal to 2.

Fix a vertex ∞ of Γ and let $\Delta \stackrel{\text{def}}{=} \Gamma(\infty)$ be the subgraph of Γ induced on all vertices of Γ adjacent to ∞. From the parameters of Γ it follows that Δ is a regular graph of order 32 and degree 4. Also, as an induced subgraph of Γ, Δ is a graph with second largest eigenvalue at most 2. Applying theorem 1 to Δ, we find that Δ cannot contain any cycles of size less than $32/2(4-1) = 5.33\cdots$. Hence Δ has girth at least 6.

Lemma 6. Δ *cannot contain two vertices* a, b *connected by* 4 *paths of length* 3, *i.e.,* Δ *cannot contain an induced subgraph isomorphic to the graph depicted below :*



Proof. Assume that vertices a, b with these properties do exist. We partition Δ into four parts : the set $S = \{a, b\}$, the set S_1 consisting of the 8 vertices that lie on the paths of length 3 connecting a and b, the set S_2 of vertices adjacent to S_1 but not in $S \cup S_1$ and the set $S_3 = \Delta - S - S_1 - S_2$ of the remaining vertices of Δ.

Because the girth of Δ is at least 6, no element of S_2 can be adjacent to two different vertices of S_1. As the degree of Δ is 4 and every element of S_1 is adjacent to exactly one element of S and one element of S_1, it follows that $|S_2| = 16$ and hence $|S_3| = 32 - 2 - 8 - 16 = 6$.

By lemma 2, D admits an $(a+2, a-1, a)$-representation with $a = (4-2)/32 = 1/16$. Hence, the following matrix, obtained by condensing $\mathsf{part}(\Delta[a+2, a-1, a]; S, S_1, S_3)$, should be positive semidefinite :

$$\begin{pmatrix} a+1 & a-\frac{1}{2} & a \\ a-\frac{1}{2} & a+\frac{1}{8} & a \\ a & a & a+\frac{1}{3}-\frac{e}{18} \end{pmatrix},$$

where e denotes the number of edges in S_3. The determinant of this matrix must be non-negative. However, it turns out to be equal to $-(12+e)/2304$. ∎

A 4-*claw* is a graph obtained by joining 4 non-adjacent vertices with a common vertex, called the *center* of the 4-claw. A 4-claw is isomorphic to the graph $\tilde{D}_4$ of theorem 2 and hence has largest eigenvalue equal to 2. A normalized eigenvector of a 4-claw is of the form $(\frac{1}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ where the coordinate with value $\frac{1}{3}$ corresponds to the center of the 4-claw.

Note that the graph $X_p = p + \Delta(p)$ induced by a vertex p of Δ and its 4 neighbours in Δ is a 4-claw, as Δ does not contain any triangles.

Lemma 7. *Let X_p be a 4-claw of Δ with center p. Then $\Delta_{\geq 2}(X_p)$ consists of three disconnected components each of which is again a 4-claw.*

Proof. Because Δ has girth ≥ 6, it is easily seen that $\Delta(X_p)$ is a co-clique of size 12 with each vertex adjacent to exactly 1 vertex of $\Delta(p)$. Hence $\Delta_{\geq 2}(X_p)$ must have size 15 and counting edges with endpoints both in $\Delta(X_p)$ and $\Delta_{\geq 2}(X_p)$ we see that the average degree of $\Delta_{\geq 2}(X_p)$ must be $4 - 12 \cdot 3/15 = 8/5$. Therefore $\Delta_{\geq 2}(X_p)$ contains exactly 12 edges and the number of connected components of $\Delta_{\geq 2}(X_p)$ must be at least 3. In case of equality, each component must be a tree or an isolated vertex, but by lemma 6 isolated vertices in $\Delta_{\geq 2}(X_p)$ are not allowed.

Now consider an edge qr in $\Delta_{\geq 2}(X_p)$. Because Δ has girth ≥ 6, two vertices of $\Delta(X_p)$ that are adjacent to either q or r cannot be adjacent to the same vertex of $\Delta(p)$. As $|\Delta(p)| = 4$, we see that there can be at most 4 edges between $\{q, r\}$ and $\Delta_{\geq 2}(X_p)$, and hence that the sum of the degrees of q and r in $\Delta_{\geq 2}(X_p)$ must be at least $2 \cdot 4 - 4 = 4$. This severely restricts the number of possible tree components of $\Delta_{\geq 2}(X_p)$. For example : all leaves of such a tree must be adjacent to a vertex of that tree of degree at least 3.

As a consequence, each tree component of $\Delta_{\geq 2}(X_p)$ must have size at least 4. We can conclude that the 15 vertices of $\Delta_{\geq 2}(X_p)$ must be partitioned in three trees of sizes $5, 5, 5$, sizes $4, 5, 6$ or sizes $4, 4, 7$. (Non-tree components contain a cycle and therefore must have size at least 6. There is no room for such a component and three trees.)

By lemma 4, the tree components must have largest eigenvalue at most 2. Theorem 2, together with the fact that leaves of the tree should be adjacent to vertices of degree at least 3, proves that only graphs of type $\tilde{D}_n$ can possibly serve as components of $\Delta_{\geq 2}(X_p)$. By lemma 5, the weight of any vertex not in X_p or $\Delta_{\geq 2}(X_p)$ must be the same with respect to X_p as to $\tilde{D}_n$. For $q \in \Delta_2(p)$, we have $w_{X_p}(q) = 1/6$. Now, the weight of any vertex with respect to a graph of type $\tilde{D}_n$ must be an integral multiple of $1/(2n-2)$. Hence, $2n-2$ should be a multiple of 6. For $|\tilde{D}_n| \leq 7$ this only leaves $n = 4$ as a possibility, a 4-claw of size 5. This rules out the partitions of size $4, 5, 6$ and $4, 4, 7$. Hence $\Delta_{\geq 2}(X_p)$ is a union of 4-claws. ∎

Lemma 8. *Let Γ be a strongly regular graph with parameters $v = 105$, $k = 32$, $\lambda = 4$ and $\mu = 12$. Then the graph Δ induced on the neighbours of a fixed vertex of Γ must be isomorphic to the unique distance regular graph with intersection array $(4, 3, 3, 1; 1, 1, 3, 4)$.*

Proof. Let p be any vertex of Δ. Because the girth of Δ is at least 6, the subgraph of Δ induced by all vertices at distance ≤ 2 of p is uniquely determined. Lemma 7 shows that of the remaining vertices 3 are at distance 4 of p (the centers of the three 4-claws) and 12 are at distance 3 of p. Moreover, the 3 vertices at distance 3 form a co-clique, and so do the 12 vertices at distance 3, because the 4-claws are disconnected. This proves that Δ has intersection array $(4, 3, 3, 1; 1, 1, 3, 4)$ and hence is isomorphic to the graph Δ^* of section 3. ∎

8 The μ-sets of Δ

As before, fix a vertex ∞ of Γ and write $\Delta = \Gamma(\infty)$. By lemma 8 we may identify Δ with Δ^* of section 3. We shall now consider, for every $p \in \Gamma_{\geq 2}(\infty)$, the set $\mu(p) \stackrel{\text{def}}{=} \{q \in \Delta | p \sim q\}$. A set of this form shall be called a *μ-set* of Δ and the graph induced by $\mu(p)$ shall be called a *μ-graph*. Because of the parameters of the strongly regular graph Γ, we have $|\mu(p)| = \mu = 12$.

Lemma 9. *A μ-graph contains no vertex of degree > 1 and has at most 3 edges.*

Proof. If $q \in \mu(p)$ were adjacent to two different vertices a and b of $\mu(q)$, then $\Gamma(q)$ would contain a quadrangle ∞, a, q, b, which contradicts the fact that $\Gamma(q)$ must be isomorphic to Δ^*, of girth 6.

Condensing the matrix $\mathsf{part}(\Gamma[a+2, a-1, a]; \{\infty, p\}, p_1, \ldots, p_{12})$, with $a = (32-2)/105 = 2/7$ and $\mu(p) = \{p_1, \ldots, p_{12}\}$, yields

$$\begin{pmatrix} a+1 & a-1 \quad \cdots \quad a-1 \\ a-1 & \\ a-1 & \mu(p)[a+2, a-1, a] \\ a-1 & \end{pmatrix}$$

and reducing this with respect to the top left element yields a $(a'+2, a'-1, a')$-representation of $\mu(p)$ with

$$a' = a - \frac{(a-1)^2}{a+1} = \frac{3a-1}{a+1} = -1/9.$$

By lemma 3 the average degree of $\mu(p)$ can then be at most $2 + 12a' = 2 - 12/9 = 6/9 = 2/3$, and hence $\mu(p)$ contains at most $\frac{1}{2}(12 \cdot 2/3) = 4$ edges.

Now, assume that $\mu(p)$ indeed contains 4 edges. Partition $\mu(p)$ into the set D of 8 vertices of degree 1 and the set D' of 4 vertices of degree 0. This time we condense the matrix $\mathsf{part}(\mu(p)[a'+2, a'-1, a']; D, D')$ to obtain

$$\begin{pmatrix} a' + 1/8 & a' \\ a' & a' + 1/2 \end{pmatrix},$$

with determinant $(10a'+1)/16$ which is negative for $a' = -1/9$. Hence $\mu(p)$ contains at most 3 edges. ∎

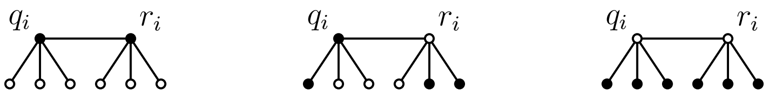
Lemma 10. *A μ-set $\mu(p)$ of Δ is one of the following :*

(Type A.) The disjoint union of 3 *edges and* 6 *isolated vertices, which are the* 3 *edges antipodal to a given edge qr together with the* 6 *vertices adjacent to either q or r and different from both. Δ has exactly* 64 *subsets of this type (one for each edge).*

(Type B.) A co-clique of size 12 *which is the union of* 3 *of the* 4 *colour classes of the same type. Δ has exactly* 8 *subsets of this type (one for each colour class).*

Proof. First, assume that $\mu(p)$ contains at least one edge $q_1 r_1$. Let $\{q_1, \ldots, q_4\}$ and $\{r_1, \ldots, r_4\}$ denote the colour classes to which q_1 and r_1 belong, indexed in such a way that the edges $q_i r_i$, $i = 1, \ldots, 4$ are mutually antipodal. Consider the 4-claws X_{q_i} and X_{r_i} in Δ. Note that $X(q_i r_i) \stackrel{\text{def}}{=} X_{q_i} \cup X_{r_i}$ contains 8 vertices, and that Δ is partitioned by the 4 sets $X(q_i r_i)$.

We easily compute the weight of p with respect to X_{q_1} and X_{r_1} to be equal to $1/2$. Then, by lemma 5, the weight of p with respect to any of the X_{q_i} and X_{r_i}, $i = 1, \ldots, 4$ must have the same value. This weight restriction leaves only 3 possible configurations for the intersection of $X(q_i r_i)$ with $\mu(p)$, as depicted below :



(Black vertices belong to $\mu(p)$, white vertices to $\Delta - \mu(p)$.)

Now, by lemma 9, $\mu(p)$ contains at most 3 edges. Also $|\mu(p)| = 12$ and every vertex of Δ (and hence of $\mu(p)$) belongs to exactly one of the $X(q_i r_i)$. It follows that the leftmost configuration must occur exactly 3 times, the middle one does not occur, and the rightmost occurs exactly once. In other words, $\mu(p)$ is of type A, where qr is the unique edge $q_i r_i$ which does not belong to $\mu(p)$.

Secondly, assume that $\mu(p)$ contains no edges, i.e., $\mu(p)$ is a co-clique. Because Δ is regular of degree 4, this means that there are $4 \cdot 12 = 48$ edges between $\mu(p)$ and $\Delta - \mu(p)$, and hence, on average a vertex q of $\Delta - \mu(p)$ is adjacent to $48/20 = 2.4$ vertices of $\mu(p)$. As a consequence, there must be at least one element q_1 in $\Delta - \mu(p)$ which is adjacent to at least 3 elements of $\mu(p)$. As before, let $\{q_1, \ldots, q_4\}$ denote the colour class of q_1.

The weight of p with respect to X_{q_1} is at least $1/2$, and by lemma 5, the same must hold for the weights with respect to any X_{q_i}. Because $|\mu(p)| = 12$ and $\mu(p)$ does not contain an edge, this is only possible when $\mu(p)$ does not contain any of the q_i, but every element of $\mu(p)$ is adjacent to one of the q_i. Hence, all vertices of $\mu(p)$ must be of the same type.

Finally, let $r_1 \in \mu(p)$, then the weight of p with respect to X_{r_1} is 1/3. Again, by lemma 5, the same must hold for any r_i in the colour class of r_1, and because all elements of $\mu(p)$ are of the same type, this implies that $r_i \in \mu(p)$. Hence, $r \in \mu(p)$ only if the entire colour class of r lies in $\mu(p)$. It follows that $\mu(p)$ is of type B. ■

9 The structure of Γ

The following lemma belongs to mathematical folklore :

Lemma 11. *Consider a real symmetric matrix M partitioned as follows :*

$$M = \left(\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right),$$

where A and C are symmetric. If M and A have the same rank, then $C = B^T A^\dagger B$, where $A^\dagger$ is a pseudo-inverse of A (satisfying $AA^\dagger A = A$). In other words, C is uniquely determined by A and B.

Proof. Because A and M have the same rank, the columns of B must be linear combinations of the columns of A, i.e., $B = AV$ for some matrix V, and then $B^T = V^T A$. Now, consider the matrix

$$M \cdot \left(\begin{array}{c|c} I & -V \\ \hline 0 & I \end{array}\right) = \left(\begin{array}{c|c} A & 0 \\ \hline B^T & C - V^T AV \end{array}\right).$$

This matrix has the same rank as M and hence as A, which is only possible when $C - V^T AV = 0$. Hence $C = V^T AV = V^T AA^\dagger AV = B^T A^\dagger B$. ■

A pseudo-inverse $A^\dagger$ of A can be obtained as follows : diagonalize A, say $A = X^{-1}DX$, and convert D to $D^\dagger$ by inverting every diagonal element which is non-zero. Then $A^\dagger = X^{-1}D^\dagger X$. Note that $A^\dagger = A^{-1}$ when A is nonsingular.

Lemma 12. *Let $a = 2/7$. Then* $\operatorname{rank}\Gamma[a+2, a-1, a] = \operatorname{rank}\Delta[a+2, a-1, a] = 20$.

Proof. In both cases the rank can be computed from the multiplicities of the eigenvalues of the corresponding adjacency matrices, which were already listed in sections 3 and 7.

The eigenvalues of $\Gamma[a+2, a-1, a] = 2I - A(\Gamma) + aJ$ are $2 - 32 + 105a = 0$ with multiplicity 1, $2 - 2 + 0a = 0$ with multiplicity 84 and $2 + 10 + 0a = 12$ with multiplicity 20. Hence $\operatorname{rank}\Gamma[a+2, a-1, a] = 20$. Similarly, the eigenvalues of $\Delta[a+2, a-1, a] = 2I - A(\Delta) + aJ$ are 50/7, 0, 2, 4 and 6, with multiplicities 1, 12, 6, 12 and 1. Hence also $\operatorname{rank}\Delta[a+2, a-1, a] = 30 - 12 = 20$. ■

Theorem 3. *All strongly regular graphs with parameters $v = 105$, $k = 32$, $\lambda = 4$ and $\mu = 12$ are isomorphic.*

Proof. Consider a strongly regular graph Γ with the given parameters. Choose any vertex $\infty \in \Gamma$ and let $\Delta = \Gamma(\infty)$ as before. Order the vertices of Γ in such a way that $A(\Delta)$ is a top left principal submatrix of $A(\Gamma)$.

Because of lemma 12 we may apply lemma 11 to the matrices $M = \Gamma[a+2, a-1, a]$ and $A = \Delta[a+2, a-1, a]$, with $a = 2/7$. Using the notation of lemma 11, this means that the entry C_{pq} is uniquely determined by A and the pth and qth columns of B. As an immediate corollary, we find that the pth and qth columns of B must be different when $p \neq q$, for otherwise $C_{pq} = C_{pp}$, contradicting the fact that diagonal entries of C have value $a+2$, while other elements have values a or $a-1$.

In graph theoretical terms this may be rephrased as follows : when $p, q \in \Gamma - \Delta$, then adjacency between p and q is uniquely determined by the adjacency between p and Δ and q and Δ. In other words, when $p, q \in \Gamma_{\geq 2}(\infty)$, then adjacency between p and q is uniquely determined by the values of $\mu(p)$ and $\mu(q)$. The corollary then reads : $\mu(p) \neq \mu(q)$ whenever $p \neq q$.

Lemma 10 tells us that there are at most 72 different possibilities for μ-sets in Δ. Also $|\Gamma_{\geq 2}(\infty)| = 72$. Hence, the vertices of $\Gamma_{\geq 2}(\infty)$ can be uniquely identified with the 72 sets listed in lemma 10. It follows that the structure of Γ is uniquely determined by the structure of Δ, which by lemma 8 is itself unique. ∎

Finally, we want to remark that in [2] the result of this theorem has been confirmed by means of an exhaustive computer search.

References

[1] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-regular graphs.* Springer-Verlag, Berlin, 1989.

[2] J. Degraer and K. Coolsaet. Classification of some strongly regular subgraphs of the McLaughlin graph. To appear in *Discrete Math.*

[3] J.-M. Goethals and J.J Seidel. The regular two-graph on 276 vertices. *Discrete Math.*, 12:143–158, 1975.

[4] E.R. van Dam and W.H. Haemers. Which graphs are determined by their spectrum? *Linear Algebra and its Applications*, 373:241–272, 2003.

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