Searching for maximal partial ovoids and spreads in generalized quadrangles

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1 Introduction

A (finite) *generalized quadrangle* (GQ) of order (s,t) is an incidence structure $S = (P,B,I)$, in which P and B are disjoint (non-empty) sets of objects, called *points* and *lines* respectively, and for which I is a symmetric point-line *incidence relation* satisfying the following axioms:

(i) Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

(ii) Each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y,M) \in P \times B$ for which $x\, I\, M\, I\, y\, I\, L$.

Interchanging points and lines in S yields a GQ of order (t,s), which is called the *dual* S^D of S. For the theory of generalized quadrangles, we refer to [12].

Here we will consider the classical generalized quadrangles (with q a power of a prime):

- The *quadrics* $Q(4,q)$ and $Q^-(5,q)$: let $Q = Q(4,q)$ (resp. $Q = Q^-(5,q)$) be a non-singular quadric of projective index 1 of the projective space $PG(4,q)$ (resp. $PG(5,q)$). Then the points of Q together with the lines on Q (subspaces of maximal dimension on Q) form a GQ of order $(s,t) = (q,q)$ (resp. (q,q^2)).

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- The *symplectic generalized quadrangle* $W(q)$: the points of the projective space $PG(3,q)$, together with the totally isotropic lines with respect to a symplectic polarity, form a GQ of order $(s,t) = (q,q)$. Note that $W(q)$ is isomorphic to the dual of $Q(4,q)$.
- The *Hermitian varieties* $H(3,q^2)$ and $H(4,q^2)$: let H be a non-singular Hermitian variety of the projective space $PG(3,q^2)$, resp. $PG(4,q^2)$. Then the points of H together with the lines on H form a GQ of order $(s,t) = (q^2,q)$, resp. (q^2,q^3). Note that $H(3,q^2)$ is isomorphic to the dual of $Q^-(5,q)$.

Two points p and q of S are called *collinear* if there is a line L in B incident with both. Two lines L and M of S are called *concurrent* if there is a point p in P incident with both. An *ovoid* of S is a set O of points of S such that each line of S is incident with a unique point of O, or equivalently, a set of $st+1$ pairwise non-collinear points. A *partial ovoid* is a set O' of points of S such that each line of S is incident with at most one point of O', i.e. a set of pairwise non-collinear points. Dually, a *spread* of S is a set R of lines of S such that each point of S is incident with a unique line of R, or equivalently, a set of $st+1$ pairwise non-concurrent lines. A *partial spread* of S is a set R' of lines of S such that each point of S is incident with at most one line of R', i.e. a set of pairwise non-concurrent lines. A partial ovoid (or spread) is called *maximal* or *complete* if it is not contained in a larger partial ovoid (or spread). We call two (partial) ovoids (or spreads) *equivalent* if there is an automorphism of S that transforms one into the other.

We denote an undirected graph by $G = (V,E)$, where V is the set of vertices and E is the set of edges. The *order* of a graph G is the number of its vertices, the *size* of G is its number of edges. Two vertices are said to be *adjacent* if they are connected by an edge. A *clique* is a set of pairwise adjacent vertices; an *independent set* is a set of pairwise non-adjacent vertices. A clique in a graph G is an independent set in its complement $\overline{G}$. A *maximal* clique is a clique that is not contained in a larger clique. A *maximum clique* is a clique of maximum cardinality in the graph. Maximal and maximum independent sets are defined in the same way. Searching for maximal and maximum cliques and independent sets are well known to be NP-hard problems [6].

With a generalized quadrangle S a so-called *collinearity graph* G_S can be associated, where the points of S correspond to the vertices of G_S and two vertices are adjacent if and only if the corresponding points are collinear. An ovoid of S is a maximum independent set of size $st+1$ in G_S, or equivalently, a maximum clique in its complement $\overline{G_S}$. A spread of S is a maximum independent set of size $st+1$ in the collinearity graph G_{S^D} of the dual S^D. Maximal partial ovoids and spreads are maximal independent sets in G_S or G_{S^D}.

In this paper we tackle the problems of finding the largest maximal partial ovoid or spread in a generalized quadrangle, of exploring the spectrum of sizes for which maximal partial ovoids or spreads exist, and of classifying up to equivalence all maximal partial ovoids or spreads of a certain size in a generalized quadrangle.

The paper is organized as follows. Section 2 describes the algorithms. A first class of algorithms, described in subsection 2.1, are exhaustive search algorithms, where we use standard backtracking clique searching algorithms and add pruning strategies based on specific properties of the collinearity graphs. This approach leads to exact answers concerning e.g. the size of the largest maximal partial ovoid or spread, or the classification of all maximal partial ovoids and spreads of a given size. Another class of algorithms, described in subsection 2.2 are heuristic algorithms, based on techniques from combinatorial optimization. This approach turns out to be very effective for exploring the spectrum of sizes for which maximal partial ovoids or spreads exist. Finally in section 3 we present some new results obtained by our computer searches. These include new exact values, improving on earlier theoretical bounds, for the size of the largest maximal partial ovoids (resp. spreads) in several of the classical GQs which do not have an ovoid (resp. spread), as well as their complete classification up to equivalence. For $Q^-(5,q)$ and $H(3,q^2)$, we extended the known spectra of sizes for which maximal partial ovoids exist.

2 Search algorithms for maximal partial ovoids and spreads

2.1 Exhaustive search algorithms

The basic form of most published algorithms (e.g. [4]) for the maximal or maximum clique problem is a backtracking search which tries to extend a partial clique by adding the vertices of a set A of remaining allowed vertices in a systematic way. When reaching a point where the set A is empty, a new maximal clique has been found.

Pruning strategies are used to avoid going through every single clique of the graph. E.g. when searching for maximum cliques, a straightforward idea is to backtrack when the set A becomes so small that even if all its vertices could be added to form a clique, the size of that clique would not exceed the size of the largest clique found so far. Other pruning strategies involve vertex colorings: in a vertex coloring adjacent vertices must be assigned different colors, so if a graph or an induced subgraph can be colored with c colors, then the graph or subgraph cannot contain a clique of size $c+1$.

Recently Östergård [11] presented a new maximum clique algorithm that allows to introduce a new pruning strategy. Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of the graph, let $S_i = \{v_i, \ldots, v_n\}$ and let $c(i)$ denote the size of the largest clique in S_i. For any $1 \leq i \leq n-1$, either $c(i) = c(i+1)$ or $c(i) = c(i+1)+1$. Moreover $c(i) = c(i+1)+1$ if and only if there is a clique of size $c(i+1)+1$ in S_i that contains v_i. The algorithm starts with $c(n) = 1$ and computes $c(i)$, $i = n-1, \ldots, 1$ by searching for such a clique. Finally the size of a maximum clique is given by $c(1)$. The values of $c(i)$ can be used for pruning the search as follows. When searching for a clique of size larger than s, the search can be pruned if $j + c(i) \leq s$, where j denotes the size of the current partial clique and i is the index of the next vertex v_i to be added to the current partial clique.

Furthermore the special structure of a collinearity graph allows us to add some specific pruning strategies to the standard algorithms.

Since the classical generalized quadrangles have automorphism groups that act transitively on the pairs of non-collinear points, every (partial) ovoid is equivalent to a (partial) ovoid containing a given pair of non-collinear points. In terms of the clique finding algorithm, this means that it is possible to restrict the search to cliques containing a certain fixed edge. This gives a straightforward and already very effective way to reduce the search space.

More advanced isomorph-rejection techniques, such as the techniques described in [13], allow to reduce the search space even further. Having determined the set stabilizer of the current partial clique in a step of the search process, it suffices to try only one point of each orbit for extending the current partial clique in the next recursive steps. We use *nauty* [9] to compute the set stabilizer.

For some types of generalized quadrangles optimal vertex colorings can be constructed from theoretical arguments. For instance, classical constructions for ovoids in $Q(4,q)$ and $H(3,q^2)$ are known [12]. The points of an ovoid in such a GQ S correspond to lines of s spread in its dual S^D, hence to a partition of the vertices of $\overline{G_{S^D}}$ into classes of mutually non-adjacent vertices. In other words, this is a partition of $\overline{G_{S^D}}$ into color classes that can be used for pruning the search in a maximum clique algorithm in $\overline{G_S}$. It can be proven that the obtained vertex coloring is optimal, i.e., uses a minimum number of colors.

In some situations the pruning in a clique finding algorithm in a collinearity graph can be improved by keeping track of the incidence structure of the GQ as well as the graph. For instance, when classifying the ovoids in a GQ or when checking whether a GQ has an ovoid, the following idea proves to be useful. Consider a step in the recursive process where the current partial ovoid gives rise to a line for which only one point still belongs to the allowed set. If that point is not added to the current partial ovoid, then the resulting partial ovoid can never be extended to an ovoid, so we can prune these possibilities and force the point to be added to the current partial ovoid.

2.2 Heuristic completion strategies

A simple greedy algorithm builds a maximal clique step by step, by adding vertices from a set of allowed vertices, until this set is empty. Several strategies are possible for choosing a vertex to be added in each step.

For instance, adding a vertex that leaves the largest number of vertices in the allowed set, will tend to build large maximal cliques. Note that this strategy is equivalent to the forward-looking approach of choosing a point of minimal relevance described in [7]. A similar strategy, which is inspired by the pruning strategies using colorings, consists of adding the vertex that leaves the largest number of colors in the allowed set for the next step. This also results in large maximal cliques.

On the other hand, choosing the vertex that leaves the least number of vertices or the least number of colors in the set of allowed vertices, is expected to result in small maximal cliques.

Starting from a maximal clique obtained by one of the above approaches, a simple restart strategy removes some of the vertices of the clique and again adds vertices until the clique is maximal. Both the removing and the adding can be done either randomly or following one of the above heuristics.

3 Results

3.1 Largest maximal partial ovoids and spreads

For $W(q)$ (q odd), $Q^-(5,q)$ and $H(4,q^2)$, it is known that no ovoids exist. Some theoretical upper bounds on the size of a maximal partial ovoid are known.

For $W(q)$ (q odd), Tallini obtained the following upper bound:

Theorem 1 (Tallini [14]). *If O' is a partial ovoid of $W(q)$, q is odd, then $|O'| \leq q^2+1-q$.*

For $Q^-(5,q)$, the best known upper bounds are by Thas and by Blokhuis and Moorhouse:

Theorem 2 (Thas [15]). *If O' is a partial ovoid of $Q^-(5,q)$, then $|O'| \leq q^3+1-q(q-1)$.*

Theorem 3 (Blokhuis and Moorhouse [3]). *If K is a k-cap of a quadric in $PG(n,q)$ with $q=p^h$ and p prime, then*

$$k \leq \left[\binom{p+n-1}{n} - \binom{p+n-3}{n}\right]^h + 1.$$

For $H(4,q^2)$, there is an upper bound by Moorhouse:

Theorem 4 (Moorhouse [10]). *If K is a k-cap of a Hermitian variety in $PG(n,q^2)$, with $q=p^h$ and p prime, then*

$$k \leq \left[\binom{p+n-1}{n}^2 - \binom{p+n-2}{n}^2\right]^h + 1.$$

In Table 1, we present some new results obtained by the exhaustive search algorithms and pruning techniques described in section 2.1. We have implemented all algorithms in Java, and use the Java Native Interface (JNI) to call *nauty* [9] for the isomorph-rejection steps.

For each quadrangle, Table 1 gives the parameters (s,t), the order $|G|$ of the corresponding collinearity graph, the value of $st+1$ (which would be the size of the ovoid or spread) and the specific value of the best known theoretical bound. We present the value for the size $|O'|$ or $|R'|$ of the largest partial ovoid or spread, found by the program – these values were found by exhaustive search and hence are the largest possible sizes for maximal partial ovoids or spreads in the respective GQs. Note that in all cases the size of the largest partial ovoid or spread is smaller than the best known theoretical bound. Moreover we obtained a complete classification of all non-equivalent largest partial ovoids or spreads; in the table the number $\#O'$ or $\#R'$ of non-equivalent largest partial ovoids or spreads is given.

Also given in Table 1 is the exact value of 29 for the size of the largest maximal partial spread in $H(4,4)$, thus confirming Brouwer's unpublished result that $H(4,4)$ has no spread (which would have size 33).

Largest maximal partial ovoids

GQ	(s,t)	$\|G\|$	$st+1$	bound	$\|O'\|$	$\#O'$
$W(5)$	(5,5)	156	26	21 [14]	**18**	**2**
$W(7)$	(7,7)	400	50	43 [14]	**33**	**1**
$Q^-(5,4)$	(4,16)	325	65	37 [3]	**25**	**3**
$H(4,4)$	(4,8)	165	33	25 [10]	**21**	**1**

Largest maximal partial spreads

GQ	(s,t)	$\|G\|$	$st+1$	bound	$\|R'\|$	$\#R'$
$H(4,4)$	(4,8)	297	33	32	**29**	**6**

Table 1: Size of the largest maximal partial ovoids/spreads, and the number of non-equivalent largest partial ovoids/spreads, for some generalized quadrangles, obtained by exhaustive search.

3.2 Spectrum of sizes for maximal partial ovoids in $Q^-(5,q)$

Theorems 2 and 3 give upper bounds on the size of maximal partial ovoids in $Q^-(5,q)$. Next to these upper bounds, there is a lower bound by Ebert and Hirschfeld:

Theorem 5 (Ebert and Hirschfeld [5]). *Let K be a complete cap of $Q^-(5,q)$. Then $|K| \geq 2q+1$. If $q \geq 4$, then $|K| \geq 2q+2$.*

Ebert and Hirschfeld also constructed a maximal partial ovoid of size 42 in $Q^-(5,5)$.

In [1], Aguglia, Cossidente and Ebert construct complete spans (i.e. maximal partial spreads) of size q^2+1 in $H(3,q^2)$, which correspond of course to maximal partial ovoids in its dual $Q^-(5,q)$. They also mention results from (extensive but not exhaustive) computer searches for possible values for the size of such complete spans, which seem to indicate that both lower and upper bounds on the size should be quadratic in q, rather than the linear lower bound from Theorem 5 and the cubic upper bounds from Theorems 2 and 3. For $q=4$, they found complete spans of sizes between $17=q^2+1$ and $25=(q+1)^2$, for $q=5$ between $26=q^2+1$ and 39. For $q=7,8$, they found complete spans of size less than q^2+1.

In Table 2, we present results obtained by the heuristic algorithms described in section 2.2. Again we have implemented these algorithms in Java.

For each $Q^-(5,q)$ considered, Table 2 gives the value of q, the parameters (s,t) of the GQ, the order $|G|$ of the corresponding collinearity graph, the specific value LB of the lower bound from Theorem 5, earlier results described in [1] and the specific value UB of the best of the two upper bounds from Theorems 2 and 3. Finally, the last column lists the sizes for which our program found maximal partial ovoids of that given size. The notation $a..b$ means that for all values in the interval $[a,b]$, a maximal partial ovoid of that size has been found.

In all cases, we have extended the spectrum found in [1]. Note that for all $Q^-(5,q)$ considered we found maximal partial ovoids with size less than q^2+1. These results seem to confirm the quadratic upper bound as suggested in [1], but seem to indicate a subquadratic lower bound.

For $Q^-(5,4)$, we also found that no maximal partial ovoids of size 10, 11, 12 or 14 exist; these results were obtained by an exhaustive search for maximal cliques of

q	(s,t)	$\|G\|$	LB [5]	Earlier results [1]	UB	New results
4	(4,16)	325	10	$\in [17, 25]$	37 [3]	**13,15..25**
5	(5,25)	756	12	$\in [26, 39]$	106 [15, 3]	**18,20..44,48**
7	(7,49)	2752	16	$\in [46, 60]$	302 [15]	**32..92,95,96,98**
8	(8,64)	4617	18	$\in [57, 74]$	217 [3]	**41..121,123,125,126**
9	(9,81)	7300	20	–	401 [3]	**52..146**
11	(11,121)	15984	24	–	1222 [15]	**68..212,214,216**
13	(13,169)	30772	28	–	2042 [15]	**89..265, 267,268,272,273**

Table 2: Spectrum of sizes for maximal partial ovoids in $Q^-(5,q)$, for small values of q, obtained by heuristic search. The notation $a..b$ means that all values in the interval $[a,b]$ have been found.

given size. Taking into account the result given in Table 1, that the largest maximal partial ovoid in $Q^-(5,4)$ has size 25, this means that the obtained values 13 and 15 through 25 form the complete spectrum for $Q^-(5,4)$.

A maximal partial ovoid of size 96 in $Q^-(5,7)$ can also be obtained by the following construction. Consider the quadric $x_0^2 + x_1^2 + \ldots + x_5^2 = 0$. Consider the set P' of points of the form $(3, \pm1, \pm1, \pm1, \pm1, \pm1)$ and every cyclic permutation of the coordinates. The set P' has cardinality 192. The points of P' with an even number of minus-signs form a maximal partial ovoid of size 96 in $Q^-(5,7)$, while the points of P' with an odd number of minus-signs form another maximal partial ovoid of size 96.

3.3 Spectrum of sizes for maximal partial ovoids in $H(3,q^2)$

$H(3,q^2)$ is known to have ovoids of size q^3+1, and recently it has been proven that the largest complete strictly partial ovoid has size $q^3 - q + 1$ [8].

Aguglia, Ebert and Luyckx [2] have shown that, for q even, the smallest maximal partial ovoid has size q^2+1. They also show that no maximal partial ovoids of size q^2+2 exist, and that, for q odd, a maximal partial ovoid in $H(3,q^2)$ has size at least q^2+3. Moreover, in [2], the authors present constructions for a maximal partial ovoid of size 61 in $H(3,25)$ and one of size 155 in $H(3,49)$; these are the smallest maximal partial ovoids known up to now.

In [7], Giuzzi describes computational methods to construct maximal partial ovoids of the Hermitian surface and presents results obtained by this approach.

In Table 2, we present results obtained by the heuristic completion algorithms described in section 2.2. Again we have implemented these algorithms in Java.

For each $H(3,q^2)$ considered, Table 3 gives the value of q, the parameters (s,t) of the GQ, the order $|G|$ of the corresponding collinearity graph, earlier results from [7] and [2]. The last column lists the sizes for which our program found maximal partial ovoids of that given size. Again the notation $a..b$ means that for all values in the interval $[a,b]$, a maximal partial ovoid of that size has been found.

In all cases, we have found the ovoids of size q^3+1 and the largest strictly partial ovoids of size q^3-q+1. For $q=5,7$, we extended the previously known spectrum

q	(s,t)	$\|G\|$	LB [2]	Earlier results	New results
3	(9,3)	280	12	16..25,28 [7]	**16..25,28**
4	(16,4)	1105	17	–	**17,21,25,29..61,65**
5	(25,5)	3276	28	78..119,121,126 [7] 61 [2]	**56..121,126**
7	(49,7)	17200	52	195..337,344 [7] 155 [2]	**142..337,344**
8	(64,8)	33345	67	–	**121,153,166,167,174,175,179, 180,186,190,192..505,513**

Table 3: Spectrum of sizes for maximal partial ovoids in $H(3,q^2)$, for small values of q, obtained by heuristic search.

as obtained in [7], and we found smaller maximal partial ovoids than the smallest previously known ones, constructed in [2].

For $H(3,16)$, we found a maximal partial ovoid whose size meets the lower bound of $q^2+1=17$. We also found maximal partial ovoids of size $21=q^2+q+1$, $25=q^2+2q+1$, $29=q^2+(q-1)q+1$, and then all sizes until $61=q^3-q+1$.

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