Primitive Permutation Groups of Unitary type with a regular Subgroup

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1 Introduction.

In [2], see [3] as well, I started the classification of the triples (G, Ω, X) such that G contains a regular subgroup X. There I did not make any assumptions on the structure of X.

The pairs (G, Ω) such that G contains a cyclic regular subgroup have been classified by Feit [7] (the insoluble ones), see also [15, Theorem 1.49], and by J. F. Ritt [19, p. 27] (the soluble ones) – see also G. A. Jones [13]. C. H. Li [15] classified those (G, Ω) such that G contains an abelian regular subgroup.

The motivation of our project is that such a classification will have many applications. For instance it will imply a complete list of the primitive graphs which are Cayley graphs (for an introduction to Cayley graphs see for instance [8] and for more applications see for instance [6,13]).

Recall the subdivision of the primitive permutation groups into different types, see for instance [16]. Every primitive permutation group (G, Ω) of affine type, of diagonal action type or of twisted product action type possesses a regular subgroup, as well as every primitive permutation group which is the product of primitive permutation groups of diagonal action type.

Therefore it remains to determine those triples (G, Ω, X) with G almost simple or (G, Ω) of product action type. In the latter case the socle of G is the direct product $T_1 \times \cdots \times T_n$ of isomorphic to non-abelian simple groups and $\Omega = \Omega_1 \times \cdots \times \Omega_n$ such that the normalizer $N_G(T_i)$ of T_i in G induces a primitive permutation group of almost simple type on Ω_i. If $N_G(T_i)$ contains a subgroup which is regular on Ω_i, then G also has a subgroup which is regular on Ω. It remains to study the difficult problem that G has a regular subgroup but $N_G(T_i)$ does not.

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In this paper here we focus on G being almost simple. The aim of the paper is to present the results obtained so far and moreover, to finish the case that $\mathrm{soc}(G) \cong \mathrm{U}_n(q)$. It is worth noting that we found only two infinite families of examples if G is a classical group, see Table 1 in Section 2 and Tables 3,4,5 in Section 3. Moreover it is interesting that in all the examples for $\mathrm{soc}(G) \cong \mathrm{L}_n(q)$, except in the examples in lines 7,11 and 13 of Table 3, the stabilizer G_ω of an element ω in Ω contains a Singer cycle, see Table 3. In this paper we prove the following theorem.

Theorem 1. *Let (G, Ω) be a primitive permutation group with $T = \mathrm{soc}(G) \cong \mathrm{U}_n(q), n \geq 3$ and with Ω the set of non-isotropic 1-spaces of the natural module for T. Then G has no regular subgroup.*

This result, together with [3, Theorem 8] then yields

Theorem 2. *Let (G, Ω) be a primitive permutation group with $T = \mathrm{soc}(G) \cong \mathrm{U}_n(q)$. Suppose there is a subgroup X of G which acts regularly on the set Ω. Then (T, Ω, X) are as in Example (a), (b), (c), (d) or (e) of Section 2.*

Let us have a closer look at the method of proof used in this paper. Let G be an almost simple group acting faithfully and primitively on a set Ω and suppose that G contains a regular subgroup X. Then for every point ω in Ω the subgroup X is a complement to the stabilizer G_ω of ω in G. In the proof of Theorem 1 T already acts primitively on Ω. Therefore there is a subgroup $T \leq G^\star \leq G$ such that $G^\star$ has a maximal subgroup B of G containing X but not containing T, see Corollary 5.5. Accordingly, we assume $G = G^\star$ and we can use the results obtained in [17] on maximal factorizations. We then study closely the obtained factorizations.

An outline of the structure of the paper is as follows. In the next section we present Examples (a), (b), (c), (d) and (e). Moreover, for (G, Ω) as in these examples, we determine up to isomorphism all the subgroups of G which act regularly on Ω. Section 3 contains the results obtained in [2] and [3] and all the examples (G, Ω, X) which are known to us for G a classical group (see Tables 3,4,5). Section 3 and 4 are preparations to prove Theorem 1. In Section 3 I will recall the definition of a Zsigmondy prime, which is essential in the proof of Theorem 1. Some simple facts about factorizations and the existence of $G^\star$ are discussed in the forth section. In the last section Theorem 1 is proved.

The author learned from C.E. Praeger that she as well as M.W. Liebeck and J. Saxl are also working on the classification of the primitive permutation groups with a regular subgroup.

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2 Examples with G a unitary group.

Let V be a vector space of dimension n over the finite field $GF(q^2)$, and let $(,)$ be a nondegenerate sesquilinear form from $V \times V$ to $GF(q^2)$, that is $(,)$ is left linear, the field $GF(q^2)$ possesses an involutory automorphism τ and

$$(v, u) = (u, v)^\tau = (u, v)^q \quad \text{for all } u, v \in V.$$

In this section $T \cong \mathrm{U}_n(q)$ will be a group of isometries of $(V, (,))$ and G a group with $T \leq G \leq \mathrm{Aut}(T)$.

Example (a). (See also [2, Example (a)]) Let $(n, q) = (4, 2)$. Then there is a subgroup X of T which acts regularly on the set Ω of maximal totally isotropic subspaces of V. We show the following stronger result:

Lemma 2.1. *Let (T, Ω) be as in Example (a). Then a subgroup X of $G = \mathrm{Aut}(T)$ is regular, if and only if X is extraspecial of order 27, that is if X is as in Line a of Table 1.*

Proof. We have $|\Omega| = 27$. Let P be a Sylow 3-subgroup of G and let A be the stabilizer of a maximal totally isotropic subspace of V. Then $P \cong \mathbb{Z}_3 \wr \mathbb{Z}_3$ and if $a \in A$ has order 3, then $a^G \cap P$ is a set of three generators and their inverses in the unique abelian subgroup of P of order 27, see [5, pp. 26,27]. Therefore the maximal subgroups of P avoiding a^G are precisely the extraspecial ones. ∎

Example (b). (See also [2, Example (m)]. Let $(n, q) = (3, 8)$ and let $H = T : 3^2$ be a subgroup of index 2 in $\mathrm{Aut}(T)$. We claim that there is a subgroup X in T which acts regularly on the set Ω of isotropic 1-spaces of V. Let A be the stabilizer of an isotropic 1-space in H and let X be the normalizer in H of a subgroup of order 19 of T. Then $X \cong \mathrm{Frob}(19 : 9) \times 3$ and it is a supplement to A in H according to [17, Table 3]. Since $|X| = |H : A|$, it is a complement to A in H and acts regularly on Ω as claimed. Immediately from [5, p. 66] we obtain the following.

Lemma 2.2. *Let (T, Ω) be as in Example (b). If X is a regular subgroup of $\mathrm{Aut}(T)$, then $G := TX$ and X are as listed in Line b of Table 1.*

Example (c) (See also [2, Example (n)]). Let $(n, q) = (3, 8)$ and let $H = T : 3^2$ a subgroup of index 2 in $\mathrm{Aut}(T)$. Every subgroup A of G isomorphic to $19 : 9 \times 3$ is a maximal subgroup of G. So, G acts primitively on the set Ω of cosets of A in G and by Example (b) the stabilizer of an isotropic 1-space in G acts regularly on Ω. Using [5, p. 66] we get

Lemma 2.3. *Let (T, Ω) be as in Example (c). If X is a regular subgroup of G with $T \leq G \leq \mathrm{Aut}(T)$, then X is the stabilizer in G of an isotropic 1-space, that is, $G := TX$ and X are as Line c of Table 1.*

Example (d). (See also [2, Example (o)]). Let $(n, q) = (4, 3)$ and let $T \leq H \leq \mathrm{Aut}(T)$ such that $H = T : 2$ and $|\mathrm{PGU}_4(3) : H| = 2$. Let A be a maximal subgroup of H such that $A \cap T \cong \mathrm{L}_3(4)$. Then $A \cong \mathrm{P\Sigma L}_3(4) = (A \cap T) : f$ with f a field automorphism of $A \cap T$, see [5, p. 53]. Let Ω be the set of cosets of $A \cap T$ in T and let K be the normalizer of A in $\mathrm{Aut}(T)$. Then $[\mathrm{Aut}(T) : K] = [K : H] = 2$. According to Theorem A of [17] the stabilizer B in H of a totally isotropic line in the natural T-module is a supplement to A in H, i.e. $H = AB$, where $B \cong 3^4 : (2 \times \mathrm{A}_6)$. Moreover, $A \cap B \cong \mathrm{A}_6$, see [17, Lemma p.113]. Hence, $X = O_{3,2}(B)$ is a complement to $A \cap B$ in B and therefore X is regular on Ω.

Lemma 2.4. *Let (T, Ω) be as in Example (d). If X is a regular subgroup of $\mathrm{Aut}(T)$, then $|O_3(X)| = 3^4$ and $O_2(X) = 1$. In particular, X and $G := TX$ are as in Line d of Table 1.*

Proof. We have $|X| = |\Omega| = 3^4 \cdot 2$. This implies $|O_3(X)| = 3^4$. Notice that T has just one class of involutions. Therefore, X does only contain outer involutions of $\mathrm{Aut}(T)$ by Lemma 5.1. According to [5, p. 54], those outer involutions which act on Ω do not centralize a subgroup of order 3^4. Thus $O_2(X) = 1$ and X is as in Line 4 of Table 1. We derive from [5, p. 52] that $G = T : 2$ is the group introduced above. ∎

Example (e). Let $(n, q) = (4, 8)$ and $G = \mathrm{Aut}(\mathrm{U}_4(8))$ and let Ω be the set of maximal totally isotropic subspaces of the natural T-module V. Let A be the stabilizer of an element of Ω in H. Then according to [30, Theorem A] $G = AB$ with $B \cong (9 * 3^{\cdot}\mathrm{U}_3(8)).3.3$ the stabilizer of a non-isotropic 1-space U of V and $A \cap B$ is a subgroup of index 9 in the stabilizer B_W in B of some maximal totally isotropic subspace W of $U^{\perp}$. This yields that $A \cap B \cong 2^{3+6} : \mathbb{Z}_{63} : \mathbb{Z}_3$. Let X be the normalizer of a subgroup of order 19 in B. Then $X \cong 9^{\cdot}(\mathrm{Frob}(19 : 9) \times 3)$. Set $Y = X \cap Z(B \cap T)$, so $Y \cong \mathbb{Z}_9$. As $Y \trianglelefteq B$, B acts on the set of fixed points of Y on Ω. This shows that Y acts semiregularly on Ω and that $Y \cap A = 1$. As also $X/Y \cap AY/Y = 1$ by Example (b), it follows that X is a complement to A in G and therefore regular on Ω.

Lemma 2.5. *Let (T, Ω) be as in Example (e). If X is a regular subgroup of $\mathrm{Aut}(T)$, then X is the normalizer of an element of order 19 in the stabilizer of a non-isotropic 1-space of V and $G := TX$ and X are as in Line e of Table 1.*

Proof. According to [30, Theorem A] X is contained in such a stabilizer. As $|X| = 3^5 \cdot 19$, it follows that $O_{19}(X) \neq 1$ and this yields the assertion. ∎

Table 1. Examples with T a unitary group.

No	T	G	T_ω	Ω	X
b	$\mathrm{U}_3(8)$	$T : 3^2$	P_1	iso. pts	$19 : 9 \times 3$
c	$\mathrm{U}_3(8)$	$T : 3^2$	$19 : 9 \times 3$		P_1
a	$\mathrm{U}_4(2)$	T	P_2	tot. iso. lns	$3^{1+2}_+, 3^{1+2}_-$
d	$\mathrm{U}_4(3)$	$T : 2$	$\mathrm{L}_3(4)$		$3^4 : 2, O_3(X) : 2$
e	$\mathrm{U}_4(8)$	$\mathrm{Aut}(T)$	P_1	tot. iso. lns	$9^{\cdot}(19 : 9 \times 3)$

3 The previous results.

In this section we present the results obtained in [2, Chapter 7].

3.1 The alternating and symmetric groups.

Let us recall a special class of strongly 3-transitive groups: Let p be an odd prime and $e = 2m$ even. Then $M(p^e + 1)$ is the following group: Consider $K = \mathrm{PGL}_2(p^e)\langle\alpha\rangle$ with α a field automorphism of order 2. Then $K/\mathrm{L}_2(p^e)$ is elementary abelian of order 4 and therefore contains three subgroups of order 2. The group $M(p^e + 1)$ is the subgroup of K which is neither $\mathrm{L}_2(p^e)\langle\alpha\rangle$ nor $\mathrm{PGL}_2(p^e)$. Notice that this group is called $M(p^e)$ in [12, p. 163].

Theorem 3. *[2, Theorem 8] Let (G, Ω) be a primitive permutation group with $\mathrm{soc}(G) = T \cong \mathrm{A}_n$, $n \geq 5$ and suppose that there is a subgroup X of G which acts regularly on Ω. Then one of the following holds, where ω is an element in Ω and $\Delta = \{1, \ldots, n\}$. Conversely if (G, Ω) is a primitive permutation group satisfying one of the listed conditions, then G has a regular subgroup X.*

(a) $G = \mathrm{A}_n$.

- *(a.a)* $\Omega = \Delta$ *and* $G_\omega = \mathrm{A}_{n-1}$.
- *(a.b)* G_ω *is sharply k-transitive on Δ and X is the pointwise stabilizer of a k-subset of Δ, for some $k \in \{3, 4, 5\}$, and one of the following holds.*
  - *(a.b.a)* $n = p^2 + 1$, *with p a prime congruent to 3 modulo 4, $k = 3$ and* $G_\omega \cong M(p^2 + 1)$;
  - *(a.b.b)* $n = 11$, $k = 4$ *and* $G_\omega \cong \mathrm{M}_{11}$;
  - *(a.b.c)* $n = 12$, $k = 5$ *and* $G_\omega \cong \mathrm{M}_{12}$.
- *(a.c)* G_ω *is k-homogeneous, but not k-transitive on Δ, for some $k \in \{2, 3, 4\}$, and one of the following holds. In the last two items p is a prime congruent to 3 modulo 4, but $p \neq 3, 7, 11, 23$.*
  - *(a.c.a)* $n = 9, k = 4$, $G_\omega \cong \mathrm{P\Gamma L}_2(8)$ *and* $X \cong \mathrm{S}_5$;
  - *(a.c.b)* $n = 33, k = 4$, $G_\omega \cong \mathrm{P\Gamma L}_2(32)$ *and* $X \cong (\mathrm{A}_{29} \times \mathrm{A}_3) : 2$;
  - *(a.c.c)* $n = p + 1$, $k = 3$, $G_\omega \cong \mathrm{L}_2(p)$ *and* $X \cong \mathrm{S}_{p-2}$;
  - *(a.c.d)* $n = p$, $k = 2$, $G_\omega \cong \mathrm{Frob}(p : (p - 1)/2)$ *and* $X \cong \mathrm{S}_{p-2}$.
- *(a.d)* Ω *is the set of k-subsets of Δ, for some $k \in \{2, 3\}$, and one of the following holds. In the last item q a is prime power congruent to 3 modulo 4.*
  - *(a.d.a)* $n = 8$, $k = 3$ *and* $X \cong \mathrm{AGL}_1(8)$;
  - *(a.d.b)* $n = 32$, $k = 3$ *and* $X \cong \mathrm{A\Gamma L}_1(32)$;
  - *(a.d.c)* $n = q$, $k = 2$ *and* $X \cong \mathrm{AGL}_1(q)/\langle -1\rangle \cong \mathrm{Frob}(q : (q - 1)/2)$.
- *(a.e)* $n = 8$, $G_\omega \cong 2^3 : L_3(2)$, $|\Omega| = 15$ *and* $X \cong \mathbb{Z}_{15}$.

(b) $G = \mathrm{S}_n$.

(b.a) $G_\omega \cap \mathrm{A}_n$ *is a subgroup of index* 2 *in* G_ω *and is as in (a.a), (a.b.a) or as a group listed in (a.d).*

(b.b) G_ω *is sharply* k*-transitive on* Δ*, for some* $k \in \{2,3\}$*,* $X \cong \mathrm{S}_{n-k}$ *and one of the following holds. In both cases* p *is a prime and* $p \geq 5$*.*

(b.b.a) $n = p$*,* $k = 2$ *and* $G_\omega \cong \mathrm{Frob}(p : (p-1))$*;*

(b.b.b) $n = p+1$*,* $k = 3$ *and* $G_\omega \cong \mathrm{PGL}_2(p)$*.*

(b.c) $n = 6$*,* $G_\omega \cong \mathrm{PGL}_2(5)$ *is transitive on* Δ *and* X *is a subgroup of* G *of order* 6*;*

Remark. If (G,Ω) is as in (b.b.a) or (b.b.b), but $p \neq 7, 11, 23$ in the latter case, then (A_n, Ω) is as in item (a.c.c) or (a.c.d) of the theorem, respectively.

3.2 The sporadic groups.

Theorem 4. *[2, Theorem 9] Let* (G,Ω) *be a primitive permutation group with* $\mathrm{soc}(G) = T$ *a sporadic simple group. Suppose that there is a subgroup* X *of* G *which acts regularly on* Ω*. Let* A *be the stabilizer in* G *of an element in* Ω*. Then* (G,A,X) *are as follows. Conversely if* (G,Ω) *is a primitive permutation group satisfying one of the listed conditions, then* G *has a regular subgroup.*

Table 2. Examples with T a sporadic group.

No	T	G	$A \cap T$	Ω	X
1	M_{11}	T	M_{10}	*set of points of Steiner system* $\mathcal{S} = S(4,5,11)$ *related to* T	$\mathbb{Z}_{11}$
2		T	$\mathrm{M}_9.2 \cong 3^2 : SD_{16}$	*set of duads of Steiner system* $\mathcal{S}$	$\mathrm{Frob}(11:5)$
3	M_{12}	T	M_{11}	*set of points of Steiner system* $\mathcal{S} = S(4,5,11)$ *related to* T	$2^2 \times 3, \mathrm{A}_4, 2 \times \mathrm{S}_3$
4		T	$\mathrm{L}_2(11)$		$3^2 : SD_{16}$
5	M_{22}	$\mathrm{Aut}(T)$		*set of points of Steiner system* $S(4,6,22)$ *related to* T	$\mathrm{Frob}(11:2)$
6	M_{23}	$T = \mathrm{Aut}(T)$	M_{21}	*set of points of Steiner system* $\mathcal{S} = S(4,7,23)$ *related to* T	$\mathbb{Z}_{23}$
7		T	$\mathrm{M}_{21}:2$	*set of duads of* S	$\mathrm{Frob}(23:11)$
8		T	$\mathrm{Frob}(23:11)$		$\mathrm{M}_{21}:2, 2^4:\mathrm{A}_7$

No	T	G	$A \cap T$	Ω	X
9		T	$2^4 : \mathrm{A}_7$	*set of blocks of* $\mathcal{S}$	$\mathrm{Frob}(23 : 11)$
10	M_{24}	$T =$ $\mathrm{Aut}(T)$	M_{23}	*set of points of Steiner system* $S(5,8,24)$ *related to* T	$D_8 \times 3, (2^2 \times 3) : 2$ $\mathrm{S}_4, D_{24}, 2 \times \mathrm{A}_4$
11			$\mathrm{L}_2(23)$		$2^4 : \mathrm{A}_7, \mathrm{M}_{21} : 2$
12	J_2	$\mathrm{Aut}(T)$	$\mathrm{U}_3(3)$	*set of points of the rank three graph for* T	$X \not\leq T, X \cong 5^2 : 4$ *each eigenvalue of* $X/O_5(X)$ *on* $O_5(X)$ *generates* $GF(5)^*$
13	HS	$\mathrm{Aut}(T)$	M_{22}	*set of points of the Higman-Sims graph related to* T	$X \not\leq T, X \cong 5^2 : 4$ *the eigenvalues of* $X/O_5(X)$ *on* $O_5(X)$ *are* $1, w; w, w$ *or* w, w^{-1} *with* $\langle w \rangle = GF(5)^*$
14	He	$\mathrm{Aut}(T)$	$\mathrm{Sp}_4(4) : 2$	*set of points of the rank five graph for* T	$7^{1+2} : \mathbb{Z}_6$

In particular, the rank 3 graphs for J_2 and HS, respectively, and the rank 5 graph for He are Cayley graphs.

3.3 The exceptional groups of Lie type.

If G is an exceptional group of Lie type, then there is no example:

Theorem 5. *[2, Theorem 10] Let (G, Ω) be a primitive permutation group with $\mathrm{soc}(G) = T$ an exceptional group of Lie type. Then there is no regular subgroup in G.*

3.4 The classical groups.

Let G be a classical group.

In Tables 1,3,4,5 which are taken from [2], see also [3], we present all the examples which are known to us. In the second column of the table the socle T of the almost simple group is given and in the third we present a group G with $T \leq G \leq \mathrm{Aut}(G)$ such that G acts primitively on Ω and such that G contains a regular subgroup. In almost all examples G is the smallest group satisfying these properties. The fourth column contains the structure of a stabilizer T_ω, for an $\omega \in \Omega$. In all the cases Ω can be considered as the set of cosets of $N_G(T_\omega)$ in G. If Ω is a nice object, then we list it in the fith column. By $V_1 \oplus V_3$ we mean the set of antiflags consisting of a 1 and a 3-dimensional subspace which intersect trivially. In the last column we present the structure of the regular subgroup of G.

Table 3. Examples with T a linear group.

No	T	G	T_ω	Ω	X
1	$L_2(11)$	T	A_5		$\mathbb{Z}_{11}$
2	$L_2(11)$	$PGL_2(11)$	A_4		$Frob(11:5)$
3	$L_2(23)$	T	S_4		$Frob(23:11)$
4	$L_2(29)$	T	A_5		$Frob(29:7)$
5	$L_2(59)$	T	A_5		$Frob(59:29)$
6	$L_3(3)$	T	$Frob(13:3)$		$3^2:2^{\cdot}D_8$
7	$L_4(2)$	$Aut(T)$	$L_3(2)$	$V_1 \oplus V_3$	$P\Gamma L_2(4)$
8	$L_4(3)$	$PGL_4(3)$	$(4 \times L_2(9)):2$		$3^3:(13:3\times 2)$
9	$L_4(4)$	$T:2$	$(5\times L_2(16)).2$		$2^6:(7:3^2):2$
10	$L_5(2)$	T	$Frob(31:5)$		$2^6:(S_3\times L_3(2))$
11	$L_5(2)$	T	$2^6:(S_3\times L_3(2))$	lns/pls	$Frob(31:5)$
12	$L_2(q)$ $q-1\equiv 2(4)$	$PGL_2(q), q=7$ $T, \quad q\neq 7$	D_{q+1}		$Frob(q:(q-1)/2)$
13	$L_n(q)$	$PGL_n(q)$	$q^{n-1}:GL_{n-1}(q)/(n,q-1)$	iso. pts	$\mathbb{Z}_{\frac{(q^n-1)}{(q-1)}}$, more

Table 4. Examples with T a symplectic group.

No	T	G	T_ω	Ω	X
1	$Sp_4(2)'$	$Sp_4(2)$	$L_2(4)$	$\{1,\ldots 6\}$	$\mathbb{Z}_6, S_3$
2	$Sp_4(4)$	$T:2$	$L_2(16):2$		$P\Gamma L_2(4)$
3	$Sp_6(2)$	T	$G_2(2)$		$P\Gamma L_2(4)$
4	$Sp_6(2)$	T	$L_2(8):3$		$2^4.L_2(4)$
5	$PSp_6(3)$	T	$L_2(27):3$		$3^{1+4}:2.PSp_4(3)$
6	$Sp_6(4)$	$T:2$	$G_2(4)$		$P\Gamma L_2(16)$
7	$Sp_8(2)$	T	$O_8^-(2)$		$P\Gamma L_2(4)$

Table 5. Examples with T an orthogonal group.

No	T	G	T_ω	Ω	X
1	$\Omega_8^+(2)$	T	$\mathrm{Sp}_6(2)$	non-sing. pts	$\mathrm{P\Gamma L}_2(4)$
2	$\Omega_8^+(2)$	T	A_9		$2^6 : \mathbb{Z}_{15}, 2^4.\mathrm{L}_2(4),$ $3 : (2^4 : (5 : 4))$
3	$\Omega_8^+(4)$	$T : 2$	$\mathrm{Sp}_6(4)$	non-sing. pts	$\mathrm{P\Gamma L}_2(16)$

My main result on classical groups is as follows:

Theorem 1. *[2, Theorem 13] Let (G, Ω) be a primitive permutation group with $T = \mathrm{soc}(G) \cong \mathrm{P}\Omega_8^+(q)$. Suppose there is a subgroup X of G which acts regularly on Ω. Then (G, Ω, X) is one of the examples in Table 4.*

To prove this theorem I also showed the next two theorems.

Theorem 2. *[2, Theorem 11] Let (G, Ω) be a primitive permutation group with $T = \mathrm{soc}(G) \cong \mathrm{P}\Omega_7(q)$, q even or odd. Suppose that $G_\omega \cap T \cong \mathrm{G}_2(q)$, for ω in Ω. Then G has a regular subgroup X if and only if $q \in \{2, 4\}$. If there is a regular subgroup X in G, then $X \cong \mathrm{P\Gamma L}_2(q^2)$.*

and

Theorem 3. *[2, Theorem 12] Let (G, Ω) be a primitive permutation group with $T = \mathrm{soc}(G) \cong \mathrm{P}\Omega_{2n+1}(q)$, q odd. Let V be the natural module for T and assume that Ω equals the set of totally isotropic subspaces of dimension i of V, for some i in $\{1, \ldots, n\}$. Then there is no regular subgroup in G.*

4 On certain prime divisors of classical groups.

Recall that Zsigmondy [20] has shown the following.

Lemma 4.1. *[20] Let p be a prime and $s \in \mathbb{N}$. Then one of the following holds:*

(a) there exists a prime r (called a Zsigmondy prime for $p^s - 1$*) which divides $p^s - 1$, but does not divide $p^i - 1$, for $i = 1, \ldots, s - 1$;*

(b) $s = 2$ and p is a Mersenne prime;

(c) $s = 6$ and $p = 2$.

Notice, if r is a Zsigmondy prime for $p^s - 1$, then $r \equiv 1 \mod s$.

Let $q = p^a$ for some prime p. If p^n is not as in statements (b) or (c) of Lemma 4.1, then we denote by q_n the largest Zsigmondy prime for $q^n - 1$. Notice that there is a paper by R. M. Guralnick, T. Pentilla, C. E. Praeger, J. Saxl where the authors determine all the maximal subgroups of the classical groups for which the orders are divisible by certain Zsigmondy primes [10].

Let $t = xp^i$ be a natural number with $(x, p) = 1$. We denote the p–part p^i of t by t_p. Frequently we will use the following fact.

In order to determine the regular subgroups of a unitary group in its action on the non-isotropic 1-spaces we show the following lemma. Let $G \cong \Gamma\mathrm{L}_m(q^2)$ and let V be the natural module for $E \leq G$ where $E \cong \mathrm{GL}_m(q^2)$.

Lemma 4.2. *Let $G \cong \Gamma\mathrm{L}_m(q^2)$ and let X be a subgroup of G such that*

(1) the order of X divides $q^{2m-1}(q^{2m}-1)$;

(2) $(q^{2m}-1)/(q+1)$ divides $|X|$.

Then one of the following holds.

(a) $m = 2$ and $\mathrm{SL}_2(q^2) \trianglelefteq X$;

(b) $m = 3$ and $q = 2$;

(c) X is a subgroup of a subgroup K of G and

(c.a) $K \cong \Gamma\mathrm{L}_1(q^{2m})$;

(c.b) K induces semilinear $GF(q^{2m})$-mappings on V.

Proof. Let X be a subgroup of G satsifying (1) and (2). Assume that X is not a subgroup in (a), (b) or (c). Let p be the prime such that $q = p^a$. If $m = 1$, then (c) holds with $K = G$. Therefore, by assumption $m > 1$ and $(m, q) \neq (3, 2)$. Notice that the Zsigmondy prime q_{2m} exists by Lemma 4.1. The orders of X and of $Y := X \cap E$ (recall $E \leq G$ with $E \cong \mathrm{GL}_m(q^2)$) are divisible by this prime. Hence Y is a group as described by the Main Theorem in [10], that is, Y is one of the groups listed in the family of examples (2.1) – (2.9). Conditions (1) and (2) and the fact that q^2 is not a prime implies immediately that Y is a group in (2.1) or in (2.6) – (2.9). Recall the Little Theorem of Fermat stating if r is a prime and u a natural number, then r divides $u^r - u$. In particular, if r does not divide u, then r divides $u^{r-1} - 1$. As $2m$ is the smallest natural number i such that q_{2m} divides $q^i - 1$, the Little Theorem of Fermat implies that $r = q_{2m}$ divides $q^{r-1} - 1$ and therefore

$$2m \mid q_{2m} - 1.$$

In particular, $q_{2m} = m + 1$ is not possible.
If Y is in (2.1), then Y has a normal subgroup isomorphic to $\mathrm{SL}_m(q^2), \mathrm{Sp}_m(q^2)$, $\mathrm{SU}_m(q)$ or $\Omega^{\epsilon}_m(q)$ with $\epsilon \in \{0, +, -\}$, see [10, p. 170]. If $\mathrm{SL}_m(q^2) \trianglelefteq Y$, then (1) implies $m = 2$ and (a) of the statement holds. In all other cases $|Y|_p > q^{2m-1}$ in contradiction with (1).

Hence Y is a member of the families (2.6) – (2.9). Then Y is nearly simple, that is, $S \leq Y/(Y \cap Z) \leq \mathrm{Aut}(S)$ for some non-abelian simple group S where $Z = Z(E)$. Let Y be of type (2.6). Then we can derive from [10, pp.173-174] that $S \cong \mathrm{A}_n$ and $n = 5$ or 6. If $n = 5$, then $m = 2$, $q = p$, $q_4 = 5$, and $p \equiv \pm 2(5)$ or $p = 2$, see [10, pp.173-174]. If $p = 2$, then (a) holds, so by assumption $p \equiv \pm 2(5)$ and $p \neq 2$. Then, as $|X : Y| \leq 2$ and $(p^2 + 1, p + 1) = 2$, (2) implies

$$(p^2 + 1)/(p^2 + 1, 4)$$

divides $|\mathrm{Aut}(S)| = 2^3 \cdot 3 \cdot 5^2$. This implies $p^2 + 1 \leq 2^5 \cdot 3 \cdot 5$, so $p \leq 21$ and therefore $p \in \{7, 13, 17\}$. If $p = 13, 17$, then $(p^2 + 1)$ is divisible by 17 and 29, respectively, which yields a contradiction. If $p = 7$, then by (2) $2^2 \cdot 3 \cdot 5 \mid |X|$, which is not possible, as $|Y \cap Z| \,|(7^2 - 1)$ and $Y/(Y \cap Z)$ is isomorphic to a subgroup of S_5. Suppose $n = 6$. Then $m = 2$, $q = 3$, $q_4 = 5$, see [10, pp.173-174], and (a) holds.
Now let Y be of type (2.7). Then it follows with [10, pp. 175-176] that $S \cong \mathrm{J}_3$, $m = 9, p = 2, q = 4$ and $q_{18} = 19$. By (2) the divisor 73 of $2^{18} - 1$ should divide the order of X and, as here $|Z| = 3$ and $|X : Y| \leq 2$, it should also divide the order of Y and S, but it does not, see for instance [5, p. 82].
Assume that Y is of type (2.8). Then it follows that $S \cong \mathrm{G}_2(q^2)$, $m = 6$ and $p = 2$. It holds $|\mathrm{G}_2(q^2)| = q^{12}(q^{12} - 1)(q^4 - 1)$. Thus the latter number has to divide $|Y|$ and $|X|$, which is a contradiction to (1).
Finally assume that Y is of type (2.9). Then we derive from [10, p. 175 - 177] that $S \cong L_2(s), s \geq 7$ for a prime s different from p, $m = (s-1)/2, q = p$ and $q_{2m} = s$. The order of Y divides $|\mathrm{Aut}(S)||Z| = s(s^2-1)(p^2-1)$. Hence by (2), as $|Y : X| \leq 2$, it follows that $(p^{s-1} - 1)/[(p-1, 2)(p+1)]$ divides $s(s^2 - 1)(p^2 - 1)$, which implies also

$$\star \qquad (p^{s-1} - 1)/[(p^{s-1} - 1, 2(p+1)(p^2 - 1))] \leq s(s^2 - 1).$$

If $s = 7$, then $(p^6-1)/[(p^6-1, 2(p+1)(p^2-1))]$ has to divide $7(7^2-1) = 2^4 \cdot 3 \cdot 7$, so $p_6 = 7$ and $p_3 = 3$, which is impossible. If $s = 11$, then $(p^{10}-1)/[(p^{10}-1, 2(p+1)(p^2-1))]$ has to divide $11(11^2 - 1) = 11 \cdot 2^3 \cdot 3 \cdot 5$ and therefore $p_{10} = 11$ and $p_5 = 5$, which is not possible. If $s = 13$, then $(p^{12} - 1)/[(p^{12} - 1, 2(p+1)(p^2 - 1))]$ divides $13(13^2 - 1) = 13 \cdot 2^3 \cdot 3 \cdot 7$. There are not enough prime divisors p_{12}, p_6, p_4. If $s = 17$ we obtain the same contradiction. Therefore $s \geq 19$ and $p^{s-5} > s^3$. But this contradicts with $\star$, as p^{s-5} is at most the left side of $\star$. This final contradiction proves the lemma. ∎

5 Simple facts about factorizations.

In this section we will present some simple facts about factorizations which we shall use throughout the text. In order to determine the primitive groups admitting a regular subgroup we shall quote the classification of the maximal factorizations of the almost simple groups by Liebeck, Praeger and Saxl [17] many times. In this section we also analyse for which primitive permutation groups we are able to quote [17].

Lemma 5.1. *Let G be a group and A and B two subgroups. The following statements are equivalent*

(a) $G = AB$.

(b) $G = AB^g$ *for all* $g \in G$.

(c) $G = AB^n$ *for all* $n \in N_{\mathrm{Aut}(G)}(A)$.

(d) $|G : A| = |B : A \cap B|$.

(e) $|G : B| = |A : A \cap B|$.

Proof. The equivalence of (a),(c), (d) and (e) is obvious. Assume (a). Then B acts transitively on the set of cosets of A in G, so B^g acts transitively on these cosets, as well. Thus (b) holds. Clearly, (b) implies (a). ∎

The next lemmas, in particular 5.2 will be applied in Section 5.

Lemma 5.2. *Let X be a complement to the subgroup A in G and let B be a maximal subgroup of G containing X. Then $G = AB$ and X is a complement to $A \cap B$ in B. In particular, if B_1 and B_2 are maximal subgroups of B containing $B \cap A$ and X respectively, then $B = B_1B_2$, moreover $|B : B_2|$ divides $|B : X| = |B \cap A|$ and $|B : B_1|$ divides $|B : A \cap B| = |X|$.*

Lemma 5.3. *Let the group $G = U \times N$ be a direct product of two of its subgroups and suppose that $G = AB$ for two subgroups A and B of G which are not contained in U and do not contain N. Then $N = \overline{AB}$ is a non-trivial factorization of N with $\overline{A}$ and $\overline{B}$ homomorphic images of A and B, respectively. In particular, if N is almost simple and* $\mathrm{soc}(N) \not\leq A, B$*, then we obtain a core-free factorization of N.*

Last we discuss some possible obstacles. Let G be an almost simple group with socle T. M. W. Liebeck, C. E. Praeger and J. Saxl determined all the factorizations $G = AB$ of the group G such that A and B are maximal core-free subgroups of G. Now suppose that G acts primitively on a set Ω and that X is a subgroup of G which acts regularly on Ω. Then $G = AX$ for $A = G_x$ the stabilizer in G of an element $x \in \Omega$. Let B be a subgroup of G containing X which is maximal with respect being core-free. Then B is not necessarily a maximal subgroup of G.

For a subgroup A of G, write $A \max^- G$ to mean that A is maximal among core-free subgroups of G. And write $A \max^+ G$ to mean that A is both core-free and maximal in G.

If $G = AB$ with $A, B \max^\epsilon G$ with $\epsilon \in \{+,-\}$, call the factorization a $\max^\epsilon$ factorization of G. This notion was introduced in [18] by Liebeck, Praeger and Saxl. We also call a $\max^+$ factorization of G simply a *maximal factorization* of G. In [18], the following has also been proven.

Lemma 5.4. *[18, Lemma 2] Suppose that $G = AB$ with core-free subgroups of G. Let $G^\star = AT \cap BT$, $A^\star = A \cap G^\star$, and $B^\star = B \cap G^\star$. Then*

(a) $G^\star = A^\star B^\star$ and $A^\star T = B^\star T = G^\star$;

(b) either $A^\star \max^+ G^\star$ or $A \cap T$ is non maximal in T; similarly for $B^\star$.

In [3] we showed:

Corollary 5.5. *Suppose that G as well as T act primitively on a set Ω and that there is a regular subgroup X in G which does not contain T. Let $x \in \Omega$, set $A = G_x$ and let B be a max$^-$ subgroup of G which contains X. Then $G^\star = A^\star B^\star$ is a max$^+$ factorization, where $G^\star = BT$ and $B = B^\star$.*

Let (G, Ω) be a primitive permutation group which has a regular subgroup X. Let $A = G_\omega$ with $\omega \in \Omega$ and let B be a max$^-$ subgroup of G which contains X. If T acts primitively on Ω, then we may assume that B is maximal in G by Corollary 5.5.

6 Proof of the Main Theorem.

In this section we prove Theorem 1.

Let V be a vector space of dimension $n, n \geq 3$ over the finite field $GF(q^2)$, and let $(,)$ be a non-degenerate sesquilinear form from $V \times V$ to $GF(q^2)$, that is $(,)$ is left linear, the field $GF(q^2)$ possesses an involutory automorphism τ and

$$(v, u) = (u, v)^\tau = (u, v)^q \text{ for all } u, v \in V.$$

Let $\hat{T}$ be the group of isometries of $(V, (,))$ such that $\hat{T} \cong \mathrm{SU}_n(q)$. In this section we assume that $T = \hat{T}/Z(\hat{T})$, so $T \cong \mathrm{U}_n(q), n \geq 3$, and we assume that G is a group with $T \leq G \leq \mathrm{Aut}(T)$ and that Ω is the set of non-isotropic 1-spaces of V. By the Lemma of Witt $T \cong \mathrm{U}_n(q)$ acts transitively on Ω. This action is also primitive, as one may easily check. Moreover, $\mathrm{Aut}(T)$ acts on Ω.

We first find a subgroup R of $\mathrm{Aut}(T)$ which acts semiregularly on Ω. Suppose that $n = 2m$, $m \geq 2$, is even. Let B be the stabilizer in $\mathrm{Aut}(T)$ of a maximal totally isotropic subspace W of V. Then $B = O_p(B) : L$ with $L \cong \Gamma\mathrm{L}_m(q^2)/Z, Z \leq Z(\mathrm{GL}_{2m}(q))$ of order $q+1$, and $O_p(B)$ is an irreducible $GF(q)E$-module of dimension m^2 where $E \leq L$ and $E \cong \mathrm{SL}_m(q^2)Z/Z$, see for instance [9, 3.2]. If $(m, q) \neq (3, 2)$, then there exists a Zsigmondy prime q_{2m}.

Lemma 6.1. *Let $(m, q) \neq (3, 2)$ and let B be the stabilizer in* $\mathrm{Aut}(T)$ *of a maximal totally isotropic subspace W of V. If $\overline{x} \in \overline{B} = B/O_p(B)$ is an element whose order is the Zsigmondy prime q_{2m}, then the centralizer R of $\overline{x}$ in $O_p(B)$ acts semiregularly on Ω.*

Proof. Assume that R does not act semiregularly on the set of non-isotropic 1-spaces. Then there is $r \in R$ with $C_V(r) > W$ (remind that $O_p(B) = C_W(B)$). This is not possible as $\overline{x}$ acts on $C_V(r)$, but W and V/W are irreducible modules for $\overline{x}$. ∎

Notice the following:

Lemma 6.2. *Let $(n, q) = (6, 2)$ and let s be an element of order* 3 *of T. Then s fixes an element in Ω.*

Proof. Let $\hat{T} \cong \mathrm{SU}_6(2)$ be the covering group of T and let x be a preimage of s in $\hat{T}$. Then either x is of order 3 or x^3 is a central element of $\hat{T}$ of order 3. According to the Atlas [5, p. 118] the latter case can not happen. Thus x is of order 3. By considering the Jordan Normal Form of x we see that this element is diagonisable. Clearly x has at least two different eigenvalues a, b on V. Let V_a, V_b be the eigenspaces of x to the eigenvalues a and b, respectively. We claim that V_a is non-degenerate. Let v and w be non-zero vectors in V_a and V_b, respectively. Then

$$(v, w) = (v^x, w^x) = (av, bw) = ab^2(v, w),$$

which yields $ab^2 = 1$, and then $a = b$, or $(v, w) = 0$. Thus V_a is perpendicular to the other eigenspaces and therefore, V_a is non-degenerate.

It follows in particular, that V_a contains a non-isotropic vector. This implies the assertion. ■

Proof of Theorem 1. Assume there is a subgroup X of G which acts regularly on Ω. In particular

$$|X| = |\Omega| = q^{n-1}(q^n - 1)/(q + 1).$$

Assume that $(n, q) = (6, 2)$. Then

$$|X| = 2^5 \cdot 3 \cdot 7.$$

Hence there is an element s of order 3 in X. Then s acts semiregularly on the set Ω and Lemma 6.2 implies that s is not in T. This and the fact that $\mathrm{Aut}(T)/T \cong \mathrm{S}_3$ implies that $|X \cap T| = 2^4 \cdot 7$ or $2^5 \cdot 7$. In particular, by the Theorem of Burnside, $X \cap T$ is soluble. As every element of order 7 of T is self-centralizing in T, see [5, p. 117], it follows that $|O_2(X \cap T)| = 8$ and $O_{2,7}(X \cap T) \cong 2^3 : 7$. This implies that $(X \cap T)/O_2(X \cap T) \cong D_{14}$ is isomorphic to a subgroup of $\mathrm{Aut}(O_2(X \cap T)) \cong \mathrm{GL}_3(2)$. The latter is not possible as $\mathrm{GL}_3(2)$ does not contain a subgroup isomorphic to D_{14}. Thus $(n, q) \neq (6, 2)$. Moreover, according to [3, (10.2)] $(n, q) \neq (4, 2)$.

Let A be the stabilizer of a non-isotropic 1-space $\langle v\rangle$ and B be a maximal subgroup of G containing X. As T as well as $\mathrm{Aut}(T)$ act primitively on Ω, there is a subgroup $T \le G^\star \le G$ which admits a maximal factorization $G^\star = AC$ such that $X \le C$ and $A = G^\star_x$ for some $x \in \Omega$, see Corollary 5.5. Thus we may assume that T is not a subgroup of B. Then $G = AB$ is a factorization as determined in [17] and therefore $n = 2m$ is even and B is isomorphic to one of the following groups, see [17, Theorem A].

(a) the stabilizer of a maximal totally isotropic subspace W of V;

(b) the stabilizer of a symplectic form on V, $B \cap T \cong \mathrm{PSp}_{2m}(q)$;

(c) the stabilizer of the direct sum $V = V_1 \oplus V_2$ of V in two maximal totally isotropic subspaces V_1 and V_2, $q = 2$ and $n \ge 8$ or $q = 4$ and $G \ge T.4$;

(d) $B \cap T \cong \mathrm{Suz}$ and $q = 2$, $n = 12$

We rule out one case after another.

(a) B is the stabilizer of a maximal totally isotropic subspace W of V. Thus $B = O_p(B) : L$ with

$$\mathrm{SL}_m(q^2)Z/Z \le L \le \Gamma\mathrm{L}_m(q^2)/Z,$$

where Z is the subgroup of $Z(\mathrm{GL}_{2m}(q))$ of order $q+1$ and $O_p(B)$ is an irreducible $GF(q)L'$-module of dimension m^2, where $L' = [L, L]$. Let $\overline{B} = B/O_p(B)$ and let $\hat{X}$ be a preimage of $\overline{X}$ in $\Gamma\mathrm{L}_m(q^2)$. Then $\hat{X}$ satisfies the assumptions of Lemma 4.2. Hence $\hat{X}$ is isomorphic to a subgroup of $\Gamma\mathrm{L}_1(q^{2m})$ or $m = 2$ and $\mathrm{SL}_2(q^2)$ is a normal subgroup of $\hat{X}$ by 4.2. In the latter case $(q^4-1)/(q-1,2)$ divides the order of $\overline{X}$, which is not possible. Thus $\overline{X}$ is isomorphic to a quotient of a subgroup of $\Gamma\mathrm{L}_1(q^{2m})$. In particular, $|\overline{X}|_p$ is a divisor of $2m$ and therefore $U := X \cap O_p(B) \neq 1$.

Let q_{2m} be a Zsigmondy prime (which exists by Lemma 4.1) and let $\overline{x} \in \overline{X}$ be an element of order q_{2m}. As $O_p(B)$ is an abelian group, $\overline{x}$ acts on U and, as $|U| \leq q^{2m-1}$, this action is trivial. Therefore U is a subgroup of the centralizer R of $\overline{x}$ in $O_p(B)$. By Lemma 6.1 R acts semiregularly on Ω, so in particular the order of R divides $|X|$ and $|R| \leq q^{2m-1}$.

Assume first $m = 2$ or 4. Then $m^2 = 4$ or 16 and U is a non-trivial p-group of order at most q^3 or q^7, respectively. On the other hand $O_p(B)$ is an m^2-dimensional $GF(q)L$-module and every irreducible $\langle\overline{x}\rangle$-submodule is of dimension at least 4 or 8, respectively. If $m = 2$ it follows $R = 1$ and if $m = 4$, then $|R| \leq q^7$ and therefore $q^9 \leq |O_p(B)/R| \leq q^{16}$ and it follows again that $R = 1$ in contradiction to $U \leq R$. Hence $m \neq 2, 4$.

As $m \neq 2, 4$, we get $|\overline{X}|_p = (2m)_p \leq p^{m-2} \leq q^{m-2}$. Hence

$$|U| \geq q^{2m-1-(m-2)} = q^{m+1}.$$

Again we are going to derive a contradiction to the fact that $U \leq R$.

As $|O_p(B)|$ and $|\overline{x}|$ are coprime, it follows $O_p(B) = [O_p(B), \overline{x}] \times R$, see for instance [1, (24.6)]. Notice, that the dimension of every non-trivial $GF(q)\langle\overline{x}\rangle$-submodule is divisible by $2m$, as the order of $\overline{x}$ is the Zsigmondy prime q_{2m}. Therefore, $|[O_p(B), \overline{x}]| = q^{r2m}$ for some natural number r and $|R| = q^{m^2-r2m} = q^{m(m-2r)}$. So $m(m-2r)$ is at most $2m-1$, which implies $m - 2r = 0$ or 1. We obtain the contradiction that R is of order 1 or q^m and the subgroup U of R of order at least q^{m+1}.

(b) B is the stabilizer of a symplectic form on V, so $B \cap T \cong \mathrm{PSp}_{2m}(q)$. As B is a maximal subgroup of G, we may assume that G is the normalizer of $B \cap T$ in G, so $G = T \cdot \langle d\rangle : \langle f\rangle$ with d a diagonal automorphism of order $(2, q-1)$ and f the Frobeniusautomorphism of order $2a$ where $q = p^a$, p prime, see for instance [14, (4.5.6)], and $B \cong 2 \times \mathrm{Aut}(\mathrm{PSp}_{2m}(q))$. According to [17, 3.3.7] $A \cap B$ fixes a $(2m-2)$-dimensional non-degenerate subspace W of the natural $GF(q)B$-module. Now Lemma 5.3 yields $B = B_W X$ and $B/Z(B) = (B_W/Z(B))(XZ(B)/Z(B))$ is one of the factorizations listed in Tables 1, 2 or 3 of [17]. It follows that $2m-2 = 2$ and that $q = 2, 4$. hence $m = 2$ and our assumption then implies $q = 4$ and

$$|X| = 4^3(4^4-1)/5 = 2^6 \cdot 3 \cdot 17.$$

We obtain a contradiction as there is no subgroup of order $2^6 \cdot 3 \cdot 17$ or $2^5 \cdot 3 \cdot 17$ in $\Gamma\mathrm{L}_2(16)$.

(c) B is the stabilizer of the direct sum $V = V_1 \oplus V_2$ of V in two maximal totally isotropic subspaces V_1 and V_2, $q = 2$ and $m \geq 3$ or $q = 4$ and $G \geq T.4$.

As $NT = \mathrm{Aut}(T)$ with N the normalizer of $T \cap B$ in $\mathrm{Aut}(T)$, we may assume $G = \mathrm{Aut}(T)$. Then

$$B \cong (q-1).\mathrm{P\Gamma L}_m(q^2).2,$$

(this can be derived from [14, (4.2.4)], for instance). Let $B_1 \le B$ be such that $B_1 \cong (q-1).\mathrm{P\Gamma L}_m(q^2)$ and let $X_1 = X \cap B_1$. Then $|X : X_1| \le 2$. Let $\hat{X}_1$ be the preimage of X_1 in $\Gamma\mathrm{L}_m(q^2)$, then, as $|Z(\Gamma\mathrm{L}_m(q^2)) : Z(B_1)| = (q^2-1)/(q-1) = q+1$, it follows that the order of $\hat{X}_1$ divides $|X_1|(q+1) = q^{2m-1}(q^{2m}-1)$. Now Lemma 4.2 gives $m = 2$ and $q = 4$ (we know $(m,q) \neq (2,2)$) or X induces semilinear mappings on V. Similiar as in (b) we see that the latter case is not possible. Hence $m = 2$ and $q = 4$, $B \cong 3.\mathrm{L}_2(16).4.2$, so that $B/Z(B) \cong \mathrm{Aut}(\mathrm{L}_2(16))$ and $|X| = 2^6 \cdot 3 \cdot 17$. Let L be a maximal subgroup of $\mathrm{L}_2(16).4$ which is divisible by 17 and 2 but not by 5. Then $L \cong 17 : 8$ and there is no subroup X_1 in B_1 of order $2^6 \cdot 3 \cdot 17$ or $2^5 \cdot 3 \cdot 17$. This contradiction shows that (c) is not possible.

(d) $B \cap T \cong \mathrm{Suz}$ and $q = 2$, $2m = 12$. Then, as B acts irreducibly on V, $Z(B) = 1$ and $B \le \mathrm{Aut}(\mathrm{Suz})$. According to Theorem 4 Aut(Suz) does not admit an exact factorization. This final contradiction proves the theorem. ∎

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