Small maximal partial t-spreads

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Abstract

Lower and upper bounds for the size of the smallest maximal partial t-spreads in $\mathrm{PG}(n, q)$ are presented. In some cases these bounds are sharp.

1 Introduction

Let $\mathrm{PG}(n, q)$ denote the n-dimensional projective space over the finite field of order q. A *partial t-spread* of $\mathrm{PG}(n, q)$ is a set of mutually disjoint t-spaces in $\mathrm{PG}(n, q)$. It is called *maximal* if no t-space can be added to obtain a larger partial t-spread. A *t-spread* of $\mathrm{PG}(n, q)$ is a set of t-spaces in $\mathrm{PG}(n, q)$ that partitions the point set of $\mathrm{PG}(n, q)$. $\mathrm{PG}(n, q)$ has a t-spread if and only if $t + 1$ divides $n + 1$, see e.g. [5, p. 29]. If n is odd and $t = (n - 1)/2$, then a (partial) t-spread of $\mathrm{PG}(n, q)$ is simply called a *(partial) spread* of $\mathrm{PG}(n, q)$.

In this paper we will be interested in the following question: "What is the size of the smallest maximal partial t-spreads of $\mathrm{PG}(n, q)$?" We will obtain upper and lower bounds on this size, see Theorem 2.7, which in a few cases are sharp and in many cases reasonably close to one another. Under some extra assumptions (which might never be fulfilled), Theorem 3.2 improves upon Theorem 2.7 and increases the number of cases in which the bounds are sharp significantly.

In order to obtain the upper bound for the size of the smallest maximal partial t-spreads, we will use information on the size of the largest partial t-spreads in $\mathrm{PG}(n, q)$. Recall that if $t+1$ divides $n+1$, then $\mathrm{PG}(n, q)$ has t-spreads. If $t+1$ does not divide $n + 1$, then $\mathrm{PG}(n, q)$ has no t-spread and the following upper bounds on the size of a partial t-spread are known.

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Theorem 1.1 *Let $n+1=k(t+1)+r$, $1 \leq r \leq t$. Suppose $\mathcal{S}$ is a partial t-spread of* PG(n,q) *of size $q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1}-s$.*

1. **(Beutelspacher [1])** *If $r=1$, then $s \geq q-1$.*
2. **(Drake and Freeman [6])** *If $r>1$, then $s \geq \lfloor\theta\rfloor+1$, where $2\theta = \sqrt{1+4q^{t+1}(q^{t+1}-q^r)}-(2q^{t+1}-2q^r+1)$.*

To get a clearer view on the upper bound of Drake and Freeman, the value of θ can be approximated.

Corollary 1.2 *Let $n+1=k(t+1)+r$, $1 \leq r \leq t$. Suppose $\mathcal{S}$ is a partial t-spread of* PG(n,q) *of size $q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1}-s$.*

1. *If $r=1$, then $s \geq q^r-1$.*
2. *If $r>1$ and $t+1 \geq 2r$, then $s \geq \frac{q^r}{2}-1$.*
3. *If $r>1$ and $t+1<2r$, then $s \geq \frac{q^r}{2}-\frac{q^{2r-t-1}}{2}+1$.*

In Example 1.4, we will recall the construction of the largest known maximal partial t-spreads in PG(n,q). It will be useful later on. We will need the following lemma.

Lemma 1.3 (Beutelspacher [1]) *If π_a is an a-space in* PG$(a+b+1,q)$, *$a \geq b$, then it is possible to partition the points of* PG$(a+b+1,q) \setminus \pi_a$ *by a set of q^{a+1} b-spaces.*

Proof Embed PG$(a+b+1,q)$ in PG$(2a+1,q)$ and take a spread $\mathcal{S}$ in PG$(2a+1,q)$ containing π_a. The elements of $\mathcal{S} \setminus \{\pi_a\}$ intersect PG$(a+b+1,q)$ in a b-spread of PG$(a+b+1,q) \setminus \pi_a$. ■

Example 1.4 In [1], Beutelspacher gives a construction for partial t-spreads in PG(n,q), $n+1=k(t+1)+r$, $k \geq 1$, $0 \leq r \leq t$, of size $q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1}-q^r+1$. Let

$$\pi_{t+r} \subseteq \pi_{2(t+1)+r-1} \subseteq \ldots \subseteq \pi_{k(t+1)+r-1} = \mathrm{PG}(n,q)$$

be a chain of subspaces in PG(n,q), $\dim(\pi_{i(t+1)+r-1}) = i(t+1)+r-1$, $i=1,2,\ldots,k$. Using the proof of Lemma 1.3, take a partition $\mathcal{S}_j$ by t-spaces of $\pi_{(j+1)(t+1)+r-1} \setminus \pi_{j(t+1)+r-1}$ for each $j \in \{1,\ldots,k-1\}$. Let π_t be a t-space in π_{t+r}. Then

$$\mathcal{S} = \cup_{1 \leq j \leq k-1}\mathcal{S}_j \cup \{\pi_t\}$$

is a maximal partial t-spread of size $q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1}-q^r+1$ in PG(n,q). This is the largest known example.

Hence in Theorem 1.1, for $r=1$, Beutelspacher's bound is sharp, and for $r>1$, Drake and Freeman's bound is approximately halfway in between the trivial upper bound and the largest known example.

2 Small maximal partial t-spreads

A *blocking set with respect to t-spaces in* $\mathrm{PG}(n,q)$ is a set of points that has nonempty intersection with every t-space of $\mathrm{PG}(n,q)$. By this definition, a partial t-spread $\mathcal{S}$ in $\mathrm{PG}(n,q)$ is maximal if and only if the set of points covered by $\mathcal{S}$ is a blocking set with respect to t-spaces. Hence, in order to construct a small maximal partial t-spread, it makes sense to start from a small blocking set with respect to t-spaces and to try to find a partial t-spread which covers all its points and as little extra points as possible. This is exactly what we will do in Subsection 2.1.

Theorem 2.1 (Bose and Burton [3]) *If B is a blocking set with respect to t-spaces in* $\mathrm{PG}(n,q)$*, then $|B| \geq |\mathrm{PG}(n-t,q)|$. Equality holds if and only if B is an $(n-t)$-space.*

A blocking set with respect to t-spaces that contains an $(n-t)$-space is called *trivial*. In Theorem 2.2, for $q > 2$ the smallest nontrivial blocking sets with respect to t-spaces in $\mathrm{PG}(n,q)$ are characterized, while for $q = 2$ a lower bound on their size is recalled. In its statement *blocking sets in projective planes* are mentioned. These are blocking sets with respect to lines. It is not hard to see that if $q = 2$, then every blocking set in $\mathrm{PG}(2,q)$ is trivial and it is known that if $q > 2$, then every nontrivial blocking set in $\mathrm{PG}(2,q)$ has size at least $q + \sqrt{q} + 1$, see [4].

Theorem 2.2

1. **(Beutelspacher [2])** *In* $\mathrm{PG}(n,2)$*, $n \geq 3$, the size of a nontrivial blocking set with respect to t-spaces is greater than $2^{n-t+1} + \sqrt{2} \times 2^{n-t-1} - 1$.*

2. **(Beutelspacher [2], Heim [10])** *In* $\mathrm{PG}(n,q)$*, $q > 2$, the smallest nontrivial blocking sets with respect to t-spaces are cones with vertex an $(n-t-2)$-space π_{n-t-2} and base a nontrivial blocking set of minimal cardinality in a plane skew to π_{n-t-2}.*

2.1 A construction

The following lemma gives some motivation for the way in which we will construct maximal partial t-spreads.

Lemma 2.3 (Beutelspacher [2]) *If $\mathcal{U}$ is a set of subspaces of* $\mathrm{PG}(m,q)$*, $m \geq 1$, that partitions the point set of* $\mathrm{PG}(m,q)$*, then either*

1. $\mathcal{U} = \{\mathrm{PG}(m,q)\}$*, or*

2. $|\mathcal{U}| \geq q^{\beta+1} + 1$*, where $\beta = \lceil \frac{m-1}{2} \rceil$.*

If equality is reached in the second case, then $\mathcal{U}$ consists of one β-space and $q^{\beta+1}$ $(m-1-\beta)$-spaces.

Remark 2.4 The lower bound in the second case of Lemma 2.3 can be reached by applying Lemma 1.3 with $a = \beta$ and $b = m - 1 - \beta$. The same tactic can be used to construct partitions of PG(m, q) consisting of one α-space and $q^{\alpha+1}$ $(m-1-\alpha)$-spaces for every $\beta \leq \alpha < m$.

We will now construct partial t-spreads in PG(n, q), $n \geq 3t + 1$; for some comments on this lower bound on n, see Remark 2.6. Since their elements will cover the point set of an $(n - t)$-space, they will be maximal. The dimension n can be written in a unique way as $n = k(t + 1) + t - 1 + r$, with $0 \leq r \leq t$. Assuming that $n \geq 3t + 1$, we assume that $k \geq 2$. Let B be an $(n - t)$-space in PG(n, q) and write $n - t + 1 = k(t + 1) + r$. As in Example 1.4, it is possible to take a t-spread $\mathcal{S}'$ of $B \setminus \pi_{t+r}$ where π_{t+r} is a $(t + r)$-space in B.

If $r = 0$, then π_{t+r} can be added to $\mathcal{S}'$ to obtain a maximal partial t-spread $\mathcal{S}$ of PG(n, q) of size

$$\frac{q^{k(t+1)} - 1}{q^{t+1} - 1},$$

which is in fact a t-spread of B.

If $r > 0$, then we can use the construction from Remark 2.4. As shown there, it is possible to partition the set of points of π_{t+r} by a set $\mathcal{U}$ consisting of one β-space and $q^{\beta+1}$ $(t + r - \beta - 1)$-spaces, where $\beta = \lceil (t + r - 1)/2 \rceil$. By Lemma 2.3, it is not possible to partition the points of π_{t+r} using a smaller number of subspaces. We will now construct a set $\mathcal{S}''$ of $q^{\beta+1} + 1$ mutually skew t-spaces that intersect B exactly in the elements of $\mathcal{U}$.

Let $\gamma = \lfloor (t+r-1)/2 \rfloor$, $\beta^* = \lfloor (t-r-1)/2 \rfloor$ and $\gamma^* = \lceil (t-r-1)/2 \rceil$. Let π_β and π^i_γ, $i = 1, 2, \ldots, q^{\beta+1}$ be the β-space and the $q^{\beta+1}$ $(t+r-\beta-1)$-spaces that partition π_{t+1}. If $t+r$ is even, then take a $(t+1)$-space π^*_{t+1} that intersects B in a line l skew to π_{t+r} and let π^*_β be a β-space in π^*_{t+1} containing l. Now construct a partition of π^*_{t+1} consisting of π^*_β and $q^{\beta+1}$ γ^*-spaces $\pi^{*,i}_{\gamma^*}$, $i = 1, 2, \ldots, q^{\beta+1}$. Let $\pi^*_{\beta^*}$ be a β^*-space in π^*_β skew to l. Then let $\mathcal{S}'' = \{\langle \pi_\beta, \pi^*_{\beta^*} \rangle\} \cup \{\langle \pi^i_\gamma, \pi^{*,i}_{\gamma^*} \rangle : i = 1, 2, \ldots, q^{\beta+1}\}$. In a similar way, if $t + r$ is odd, then take a t-space π^*_t that intersects B in a point P skew to π_{t+r} and let π^*_β be a β-space in π^*_t containing P. Now construct a partition of π^*_t consisting of π^*_β and $q^{\beta+1}$ γ^*-spaces $\pi^{*,i}_{\gamma^*}$, $i = 1, 2, \ldots, q^{\beta+1}$. Let $\pi^*_{\beta^*}$ be a β^*-space in π^*_β skew to l. Then let $\mathcal{S}'' = \{\langle \pi_\beta, \pi^*_{\beta^*} \rangle\} \cup \{\langle \pi^i_\gamma, \pi^{*,i}_{\gamma^*} \rangle : i = 1, 2, \ldots, q^{\beta+1}\}$.

In this way, considering the set $\mathcal{S} = \mathcal{S}' \cup \mathcal{S}''$, we obtain a maximal partial t-spread of size

$$q^r \frac{q^{k(t+1)} - 1}{q^{t+1} - 1} + q^{\beta+1} - q^r + 1.$$

Theorem 2.5 *In* PG(n, q), $n = k(t+1)+t-1+r$, *with* $k \geq 2$, *there exist maximal partial* t*-spreads of size*

$$q^r \frac{q^{k(t+1)} - 1}{q^{t+1} - 1} + q^{\beta+1} - q^r + 1, \text{ where } \begin{cases} \beta = -\infty & \text{if } r = 0, \\ \beta = \lceil (t + r - 1)/2 \rceil & \text{otherwise.} \end{cases} \quad (1)$$

Remark 2.6 The condition that $n \geq 3t + 1$ was imposed to ensure that—using the notation from above—$B \setminus \pi_{r+t}$ is nonempty, such that the spaces π^*_{t+1} and π^*_t can be chosen skew to π_{r+t}. It seems plausible that when $2t+1 \leq n \leq 3t$ a slightly modified construction will yield maximal partial t-spreads of size (1). Note that for $r=0$, or equivalently $n=2t+1$, this is immediately clear by taking a t-spread of $\mathrm{PG}(2t+1,q)$. In the cases where $2t+1 \leq n \leq 3t$ the spaces π^*_{t+1} and π^*_t would necessarily intersect $\pi_{t+r} = B$ nontrivially, such that the spaces π_β, π^i_γ, $\pi^*_{\beta^*}$, $\pi^{*,i}_{\gamma^*}$ would need to be chosen carefully to make sure that the resulting t-spaces of $\mathcal{S}''$ are skew. We do not pursue this line of thought here, since in Theorem 2.7 we will need to assume that $k > 2$ anyway. Also, it seems unlikely that if $2t+1 \leq n \leq 3t$, then partial t-spreads of size (1) can be called small. Again, for $r=0$, it is immediately clear that they are not.

2.2 Lower bounds

Let $\mathcal{S}$ be a maximal partial t-spread in $\mathrm{PG}(n,q)$, $n \geq t$. The dimension n can be written in a unique way as $n = k(t+1)+t-1+r$, with $0 \leq r \leq t$. If $n < 2t+1$, then $|\mathcal{S}| = 1$. So, from now on, assume that $k \geq 1$ and $(k,r) \neq (1,0)$. Let B denote the smallest blocking set with respect to t-spaces contained in $\cup_{\pi_t \in \mathcal{S}} \pi_t$. We will distinguish two cases.

Case 1. The set B is an $(n-t)$-space

The smallest possibility for B is an $(n-t)$-space $\pi_{n'}$. We can write $n' = n-t$ as $n'+1 = k(t+1)+r$. Let $\mathcal{S}_1$ denote the set of elements of $\mathcal{S}$ that are contained in π'_n and let $\mathcal{S}_2 = \mathcal{S} \setminus \mathcal{S}_1$. Write $v = q^r(q^{k(t+1)}-1)/(q^{t+1}-1)$ and $|\mathcal{S}_1| = v-s$. By Corollary 1.2 (i) if $r=0$, then $s \geq 0$, (ii) if $r=1$, then $s \geq q^r-1$, (iii) if $r>1$ and $t+1 \geq 2r$, then $s \geq q^r/2-1$, and (iv) if $r>1$ and $t+1 < 2r$, then $s \geq q^r/2 - q^{2r-t-1}/2+1$.

The points of $A := \pi'_n \setminus \cup_{\pi_t \in \mathcal{S}_1} \pi_t$ must be covered by the elements of $\mathcal{S}_2$, which means that the elements of $\mathcal{S}_2$ must intersect π'_n in a partition of A by subspaces of dimension at most $t-1$. Note that $|A| = \theta_{r-1} + s\theta_t$, where $\theta_i = |\mathrm{PG}(i,q)| = (q^{i+1}-1)/(q-1)$.

To find a lower bound on the size of $\mathcal{S}_2$, we can take a lower bound on the size of a partition of A by subspaces of dimension at most $t-1$. The size of such a partition is bounded from below by a lower bound for the size of a partition $\mathcal{P}$ of a set of size $|A|$ by subsets whose sizes are elements of $\{\theta_{t-1}, \theta_{t-2}, \ldots, \theta_0 = 1\}$. Let $a_{t-1} = \lfloor |A|/\theta_{t-1} \rfloor$ and define recursively $a_i = \lfloor (|A| - \sum_{j=t-1}^{i+1} a_j\theta_j)/\theta_i \rfloor$ for $i = t-2, t-3, \ldots, 0$; then $|\mathcal{P}| \geq \sum_{i=0}^{t-1} a_i$. This lower bound can be attained by applying a greedy algorithm, that is, by taking as many disjoint subsets of size θ_{t-1} as possible, then taking as many disjoint subsets of size θ_{t-2} that are disjoint from the previously chosen subsets, etcetera.

Let us for instance discuss the case $r=1$. In this case, $|A| = 1+s\theta_t$ and $s \geq q-1$. If $s = q-1$, then $|A| = q^{t+1}$. The best possible way to partition a set of q^{t+1} points by subspaces of dimension at most $t-1$ is by taking q^2-q $(t-1)$-spaces and q points, from which the lower bound $|\mathcal{S}_2| \geq q^2$ is obtained. Therefore $|\mathcal{S}| \geq v-(q-1)+q^2$ such that $|\mathcal{S}| \geq v+q^2-q+1$. If $s > q-1$, then

$|\mathcal{S}| \geq v - s + \lceil (1 + s\theta_t)/\theta_{t-1} \rceil = v + s(q-1) + \lceil (1+s)/\theta_{t-1} \rceil$, such that as before $|\mathcal{S}| \geq v + q^2 - q + 1$.

The other cases are handled similarly. We will only look at the situation where s attains the lower bound from Corollary 1.2. As above, if s is larger, then the same lower bound for $|\mathcal{S}|$ is immediately obtained.

If $r > 1$, $t + 1 \geq 2r$ and $s = q^r/2 - 1$, then $|A| = (q^{t+r} + q^{t+r-1} + \ldots + q^{t+1} - q^t - q^{t-1} - \ldots - q^r)/2$. The best possible way to partition a set consisting of this number of points by subspaces of dimension at most $t-1$ is by taking $q^{r+1}/2 - q$ $(t-1)$-spaces, $q/2$ $(r-1)$-spaces and $q/2$ $(r-2)$-spaces, from which the lower bound $|\mathcal{S}| \geq v + (q^{r+1} - q^r)/2 + 1$ follows.

If $r > 1$, $t + 1 < 2r$ and $s = (q^r - q^{2r-t-1} + 2)/2$, then $|A| = (q^{t+r} + \ldots + q^{2r} + 2q^t + \ldots + 2q^{r+1} + 4q^r + 3q^{r-1} + \ldots + 3q^{2r-t-1} + 4q^{2r-t-2} + \ldots + 4)/2$. The best possible way to partition a set consisting of this number of points by subspaces of dimension at most $t-1$ is by taking $(q^{r+1}+q)/2$ $(t-1)$-spaces, one r-space, $(q-1)/2$ $(r-1)$-spaces, $q/2$ $(2r-t-3)$-spaces and one point, from which the lower bound $|\mathcal{S}| \geq v + (q^{r+1} - q^r + q^{2r-t-1} + 3q + 1)/2$ follows.

Summarizing, the following lower bounds were obtained for the size of a maximal partial t-spread $\mathcal{S}$ containing $\pi_{n'}$.

1. If $r = 0$, then $|\mathcal{S}| \geq \frac{q^{k(t+1)}-1}{q^{t+1}-1}$.

2. If $r = 1$, then $|\mathcal{S}| \geq q\frac{q^{k(t+1)}-1}{q^{t+1}-1} + q^2 - q + 1$.

3. If $r > 1$ and $t + 1 \geq 2r$, then $|\mathcal{S}| \geq q^r \frac{q^{k(t+1)}-1}{q^{t+1}-1} + (q^{r+1} - q^r)/2 + 1$.

4. If $r > 1$ and $t+1 < 2r$, then $|\mathcal{S}| \geq q^r \frac{q^{k(t+1)}-1}{q^{t+1}-1} + (q^{r+1} - q^r + q^{2r-t-1} + 3q + 1)/2$.

Case 2. The set B is not an $(n-t)$-space

Now suppose that B is not an $(n-t)$-space. If $q > 2$, then by Theorem 2.2 the size of B is at least the size of a cone with vertex an $(n-t-2)$-space π_{n-t-2} and base the smallest nontrivial blocking set in a plane skew to π_{n-t-2}. So, in this case

$$|B| \geq \frac{q^{n-t-1}-1}{q-1} + (q + \sqrt{q} + 1)q^{n-t-1}. \tag{2}$$

If $q = 2$, then by Theorem 2.2

$$|B| > 2^{n-t+1} + \sqrt{2} \times 2^{n-t-1} - 1. \tag{3}$$

Noting that $|\mathcal{S}| \geq |B|/\theta_t$, some calculations show that if $k \geq 2$, then a set of t-spaces covering at least (2) points when $q > 2$ or at least (3) points when $q = 2$ has size greater than (1).

2.3 Conclusions

Theorem 2.7 *Let $s(t,n,q)$ denote the size of the smallest maximal partial t-spreads in $\mathrm{PG}(n,q)$ and write $n = k(t+1)+t-1+r$, $0 \le r \le t$. Let $\beta = \lceil (t+r-1)/2 \rceil$. If $k \ge 2$, then the following hold.*

1. *If $r=0$, then $s(t,n,q) = \frac{q^{k(t+1)}-1}{q^{t+1}-1}$.*

2. *If $r=1$, then $q\frac{q^{k(t+1)}-1}{q^{t+1}-1} + q^2 - q + 1 \le s(t,n,q) \le q\frac{q^{k(t+1)}-1}{q^{t+1}-1} + q^{\beta+1} - q + 1$.*

3. *If $r > 1$ and $t+1 \ge 2r$, then $q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1} + (q^{r+1}-q^r)/2 + 1 \le s(t,n,q) \le q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1} + q^{\beta+1} - q^r + 1$.*

4. *If $r > 1$ and $t+1 < 2r$, then $q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1} + (q^{r+1}-q^r+q^{2r-t-1}+3q+1)/2 \le s(t,n,q) \le q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1} + q^{\beta+1} - q^r + 1$.*

If $\mathcal{S}$ is a maximal partial t-spread in $\mathrm{PG}(n,q)$ whose size lies in the corresponding interval above, then $\cup_{\pi_t \in \mathcal{S}} \pi_t$ contains an $(n-t)$-space in $\mathrm{PG}(n,q)$.

Proof Theorem 2.5 immediately supplies the upper bounds. The second part of Subsection 2.2 shows that a maximal partial t-spread $\mathcal{S}$ in $\mathrm{PG}(n,q)$ whose size is at most the appropriate upper bound covers an $(n-t)$-space. The lower bounds are provided by the first part of Subsection 2.2. ■

Corollary 2.8

1. *In $\mathrm{PG}(2k+1,q)$, $k \ge 2$, the smallest maximal partial line-spreads have size $q\frac{q^{2k}-1}{q^2-1} + q^2 - q + 1$.*

2. *In $\mathrm{PG}(3k+2,q)$, $k \ge 2$, the smallest maximal partial planespreads have size $q\frac{q^{3k}-1}{q^3-1} + q^2 - q + 1$.*

Proof In these cases, the lower and upper bound from Theorem 2.7 coincide. ■

Corollary 2.9 *If $n \ne 3$, then the size of the smallest maximal partial linespreads in $\mathrm{PG}(n,q)$ is known.*

Remarks 2.10

1. The case where $r = 0$ in Theorem 2.7 was already known, see Beutelspacher [1].

2. In the difficult cases where $2t+1 \le n \le 3t$, we don't obtain new results. In these cases, the gap between the best known lower bounds and the smallest known examples of maximal partial t-spreads is quite large. For example, for maximal partial linespreads in $\mathrm{PG}(3,q)$ this lower bound is proved by Glynn [9] and equals $2q$, while the examples are by Gács and Szőnyi [8] and have size $(c\log q + 1)q + 1$, where $c = 2$ if q is odd and $c \le 6.1$ if $q > q_0$ is even.

3. If $t \le n \le 2t$, then $s(n,t,q) = 1$.

3 Wishful thinking

In [7], it is conjectured that the size of Beutelspacher's maximal partial t-spreads in $\mathrm{PG}(n,q)$ from Example 1.4 is maximal, i.e., that there exist no larger maximal partial t-spreads in $\mathrm{PG}(n,q)$. Although I am unable to prove this, I am inclined to believe that it is true. In this section, we will have a look at what implications it would have if the conjecture were correct.

Conjecture 3.1 (Eisfeld and Storme [7]) If $n+1=k(t+1)+r$, $1\leq r\leq t$, then the largest maximal partial t-spreads in $\mathrm{PG}(n,q)$ have size $q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1}-q^r+1$.

Of course the assumption of the correctness of Conjecture 3.1 does not affect the upper bounds in Theorem 2.7, but it significantly increases the lower bounds. Indeed, it implies that, under the conditions of Theorem 2.7, a maximal partial t-spread of $\mathrm{PG}(n,q)$ with $r>0$ has size at least $q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1}+q^{r+1}-q^r+1$.

Theorem 3.2 *Let $s(t,n,q)$ denote the size of the smallest maximal partial t-spreads in $\mathrm{PG}(n,q)$ and write $n=k(t+1)+t-1+r$, $0\leq r\leq t$. Let $\beta=\lceil (t+r-1)/2\rceil$ and assume that Conjecture 3.1 is true. If $k\geq 2$, then the following hold.*

1. *If $r=0$, then $s(t,n,q)=\frac{q^{k(t+1)}-1}{q^{t+1}-1}$.*
2. *If $r>0$, then $q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1}+q^{r+1}-q^r+1\leq s(t,n,q)\leq q^r\frac{q^{k(t+1)}-1}{q^{t+1}-1}+q^{\beta+1}-q^r+1$.*

If $\mathcal{S}$ is a maximal partial t-spread in $\mathrm{PG}(n,q)$ whose size lies in the corresponding interval above, then $\cup_{\pi_t\in\mathcal{S}}\pi_t$ contains an $(n-t)$-space in $\mathrm{PG}(n,q)$.

Corollary 3.3 *Assume that Conjecture 3.1 is correct. Then the following hold.*

1. *In $\mathrm{PG}((k+1)(t+1)+t-2,q)$, $k\geq 2$, the smallest maximal partial t-spreads have size $q^t\frac{q^{k(t+1)}-1}{q^{t+1}-1}+q^{t+1}-q^t+1$.*
2. *In $\mathrm{PG}((k+1)(t+1)+t-3,q)$, $t>1$, $k\geq 2$, the smallest maximal partial t-spreads have size $q^{t-1}\frac{q^{k(t+1)}-1}{q^{t+1}-1}+q^t-q^{t-1}+1$.*

Proof In the respective cases $r=t$ and $r=t-1$, such that in both cases $\beta=r$, implying that in these cases the lower and upper bound from Theorem 3.2 coincide. ■

Corollary 3.4 *If Conjecture 3.1 is true, then for $n\notin\{5,6\}$ the size of the smallest maximal partial planespreads in $\mathrm{PG}(n,q)$ is known.*

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