Attainable lengths for circular binary words avoiding k powers

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Abstract

We show that binary circular words of length n avoiding $7/3^+$ powers exist for every sufficiently large n. This is not the case for binary circular words avoiding k^+ powers with $k < 7/3$.

1 Introduction

The word *banana* can be abbreviated as $b(an)^{5/2}$. By this, we mean that the suffix *anana* of *banana* consists of *an*, repeated two and a half times. In particular, *banana* contains the **square** $anan = (an)^2$. On the other hand, the word $onion = (oni)^{5/3}$ contains no squares. However, if we imagine the letters of *onion*, not as labels in sequence, but as labels on a necklace, *onion* is equivalent to *ononi*, which commences with the square $(on)^2$.

Let w be a word, $w = w_1w_2\dots w_n$ where the w_i are letters. We say that w is **periodic** if for some p we have $w_i = w_{i+p}$, $i = 1, 2, \dots, n-p$. We call p a **period** of w. Let k be a rational number. A k **power** is a word w of period $p = w/k$. A k^+ **power** is a word which is an r power for some $r > k$. A word is k^+ **power free** if none of its subwords is a k^+ power. Traditionally, a 2 power is called a **square**; a 2^+ power is called an **overlap**; a 3 power is a **cube**.

We denote the number of letters in w by $|w|$, and the number of times a specific letter a appears in w by $|w|_a$. When w is a binary word, that is, a word over $\{0, 1\}$, we use the notation $\bar{w}$ for the **binary complement** of w, obtained from w by replacing 0's with 1's, and vice versa.

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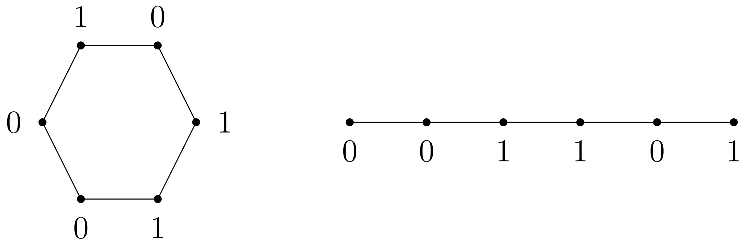

Figure 1: A 2^+ free circular word.

Word v is a **conjugate** of word w if there are words x and y such that $w = xy$ and $v = yx$. Let w be a word. The **circular word** w is the set consisting of w and all of its conjugates. We say that **circular word** w **is** k^+ **power free** if all of its elements are k^+ power free; that is, all the conjugates of the 'ordinary word' w are k^+ power free. The conjugates of w are the subwords of ww of length $|w|$. It follows that w is circular k power free if and only if ww contains no k powers of length at most $|w|$.

Example 1. The set of conjugates of word 001101 is

$$\{001101, 011010, 110100, 101001, 010011, 100110\}.$$

Each of these is 2^+ power free, so that 001101 is a circular 2^+ power free word. On the other hand, 0101101 is 2^+ power free, but its conjugate 1010101 is a 7/2 power. Thus 0101101 is not a circular 2^+ power free word.

At the turn of the last century, Axel Thue showed that there are infinite sequences over $\{a, b\}$ not containing any overlaps, and infinite sequences over $\{a, b, c\}$ not containing any squares [11]. In addition to studying ordinary words, Thue studied circular words, proving that overlap-free circular words of length m exist exactly when m is of the form 2^n or 3×2^n.

Say that x^k is **unavoidable on** n **letters** if any sufficiently long string on n letters contains a k power. Dejean generalized Thue's work to repetitions with fractional exponents. She conjectured [4] that

$$RT(n) = \begin{cases} 2, & n = 2 \\ 7/4, & n = 3 \\ 7/5, & n = 4 \\ n/(n-1), & n > 4 \end{cases}$$

where we define the **repetitive threshold function** RT by

$$RT(n) = \sup\{k : x^k \text{ is unavoidable on } n \text{ letters}\}.$$

It was recently shown [2] that there are ternary square-free circular words of length n for $n \geq 18$ (but not for $n = 17$). The authors have shown that there are binary $5/2^+$ power free circular words of every length [1]. This is optimal in the sense that no binary circular word of length 5 avoids both 5/2 powers and cubes.

On the other hand, one feels that 'accidental' problems with short lengths should perhaps be ignored.

Let $L(n,s)$ be the set of s power free circular words over $\{0,1,\ldots,n-1\}$. Let $\mathcal{L}(n,s)$ be the set of lengths of words in $L(n,s)$. For example, $L(2,2) = \{\epsilon, 0, 1, 01, 10, 010, 101\}$ and $\mathcal{L}(2,2) = \{0,1,2,3\}$. On the other hand, if $k > 5/2$, then $\mathcal{L}(2,k)$ is the set of non-negative integers. We wish to know for which k $\mathcal{L}(2,k)$ contains all integers greater than or equal to some N_0.

Define the **circular repetitive threshold function** by

$$CRT(n) = \inf\{s : \mathcal{L}(n,s) \supseteq \{N_s, N_s+1, N_s+2, \ldots\} \text{ for some integer } N_s.\}$$

We prove the following:

Main Theorem: $CRT(2) = 7/3$.

2 A few properties of the Thue-Morse substitution

The Thue-Morse word t is defined to be $t = \mu^\omega(0) = \lim_{n\to\infty} \mu^n(0)$, where $\mu : \{0,1\}^* \to \{0,1\}^*$ is the substitution generated by $\mu(0) = 01$, $\mu(1) = 10$. Thus

$$t = 0110100110010110100101100110 1001\cdots$$

The Thue-Morse word has been extensively studied. (See [5, 8, 9, 11] for example.) We use the following facts about t:

1. Word t is 2^+ power free.
2. If w is a subword of t then so is $\overline{w}$, the binary complement of w.
3. Neither 00100 nor 11011 is a subword of t.

The following lemma is proved in [1]:

Lemma 2. *Let $k \geq 6$ be a positive integer. Then t contains a subword of length $4k$ of the form* $01101001v10010110$.

If w is a binary word with period p, then $\mu(w)$ has period $2p$. This means that when w is a k power, so is $\mu(w)$. Again, if the circular word w contains a k power, so does the circular word $\mu(w)$. Here is a partial converse [10]:

Lemma 3. *Let $\alpha > 2$ be a rational number. Let w be a binary word, and suppose that $\mu(w)$ contains an α power z of period p, $|z| = \alpha p$. Then w contains a word u of period $p/2$, with $|u| \geq |z|/2$.*

Proof: Note that $\alpha > 2$ is necessary, since 01 is 2 power free, but $\mu(01)$ contains the square 11.

Write $z = (z_1 z_2 \cdots z_p)^n z_1 z_2 \cdots z_m$ where the z_i are letters, n, m are integers, $n \geq 2$ and $m < p$. Write $\mu(w) = xzy$. If $|x|$ is even, then for some $\underline{z}$ we can write the even length prefix $(z_1 z_2 \cdots z_p)^2$ of z as $\mu(\underline{z})$. We see that

$$\begin{aligned} p &= |\underline{z}| \\ &= |\mu(\underline{z})|_1 \\ &= |(z_1 z_2 \cdots z_p)^2|_1 \\ &= 2|(z_1 z_2 \cdots z_p)|_1 \end{aligned}$$

so that p is even. If x is odd, then $|xz_1|$ is even, and we can write $(z_2 \cdots z_p z_1)^2 = \mu(\underline{z})$ for some $\underline{z}$. Again we find that p is even.

Without loss of generality, assume that z is the longest subword of $\mu(w)$ having period p. We will show that $|x|$ is even. Suppose that $|x|$ is odd. Write $x = \mu(\underline{x})x_0$, where x_0 is a letter, $\underline{x}$ some word. Since p is even, write $xz_1z_2 \cdots z_p z_1$ as $\mu(\underline{x})x_0 z_1 \mu(\underline{z}) z_p z_1$ for some $\underline{z}$. It follows that $x_0 = \bar{z_1} = z_p$. Now, however, $x_0 z$ has period p, but is longer than z. This is a contradiction. We conclude that $|x|$ must be even. Symmetrically, $|y|$ must be even, so that $|z|$ is even also. This implies that m is even and $z = \mu(u)$ where $u = (z_1 z_3 \cdots z_{p-1})^n z_1 z_3 \cdots z_{m-1}$. We see that u has period $p/2$, while $|u| = |z|/2$.□

Corollary 4. *Let k be a rational number. Let w be a binary circular k^+ power free word. Then $\mu(w)$ is circular k^+ power free.*

Proof: Suppose that $\mu(w)$ is not circular k^+ power free. This means that $\mu(w)\mu(w) = \mu(ww)$ contains some α power z, $\alpha > k$, $|z| \leq |\mu(w)|$. Word z has period $p = |z|/\alpha$. By the previous lemma, ww contains a word u of period $p/2$, with $|u| = \lceil |z|/2 \rceil \leq |w|$. Moreover, u is a β power, where $\beta = |u|/(p/2) = \lceil |z|/2 \rceil/(p/2) \geq |z|/p = \alpha$.

Now ww contains a k^+ power u, with $|u| \leq |w|$. This means that w is not circular k^+ power free. ■

3 $CRT(2) \geq 7/3$

Certainly $CRT(2) \geq RT(2) = 2$. Karhumäki and Shallit prove the following theorem [7]:

Theorem 5. *Let x be a binary word avoiding α powers, with $2 < \alpha \leq 7/3$. Then there exist $u, v \in \{\epsilon, 0, 1, 00, 11\}$ and a binary word y avoiding α powers, such that $x = u\mu(y)v$.*

This allows the following result:

Lemma 6. *Suppose $2 < \alpha \leq 7/3$. Let x be a binary word, $|x| > 6$, such that every conjugate of x avoids α powers. Then there exists a binary word y such that $\mu(y)$ is a conjugate of x. In particular, $|x| = 2|y|$ and all conjugates of y avoid α powers.*

Proof: Suppose that there exists a binary word y such that $\mu(y)$ is a conjugate of x. If u is a conjugate of y containing an α power, then $\mu(u)$ is a conjugate of x containing an α power, which is impossible. It will thus suffice to show that there exists a binary word y such that $\mu(y)$ is a conjugate of x.

If no conjugate of x contains 00 or 11 as a subword, then x is $(01)^{|x|/2}$ or $(10)^{|x|/2}$. Since $|x|/2 \geq 3 > 7/3$, this is impossible.

Replacing x by its binary complement if necessary, suppose that a conjugate of x contains 11 as a subword. Since $|x| > 6$, and no conjugate of x can contain 111 as a subword, assume that a conjugate z of x begins with 011. Applying the previous theorem, write $z = u\mu(y')v$, some binary word y', and some $u, v \in \{\epsilon, 0, 1, 00, 11\}$. We see that $u = \epsilon$ is forced, and z in fact must begin with 0110. Write $z = \mu(01y'')v$. If we can show that $v = \epsilon$ we will be done.

Clearly $v \neq 00$; otherwise the conjugate $v\mu(y')$ of x commences 000. Since 000 is a cube, this is impossible.

Suppose $v = 11$. If $\mu(y')$ ends in 01, then $\mu(y')v$ ends in 0111, which is impossible. We therefore deduce that $\mu(y')$ ends in 10, and the conjugate $\mu(y'')v0110$ of x ends in the 7/3 power 0110110. This is impossible.

Suppose $v = 0$. This implies that 01 is a suffix of $\mu(y'')$; otherwise $10\mu(y'')v01$ ends in 10001, and a conjugate of x contains the cube 000. Since $\mu(y'')$ has 01 for a suffix, we deduce that $\mu(y'')$ ends in 0101 or 1001. If $\mu(y'')$ ends in 0101, then $\mu(y'')v$ ends in the 5/2 power 01010; if $\mu(y'')$ ends in 1001, then $\mu(y'')v01$ ends in the 7/3 power 1001001. We conclude that $v \neq 0$.

The last possibility to be avoided is that $v = 1$. Suppose this is the case. Either $\mu(y'')$ ends in 10, and $\mu(y'')v01$ ends in the 5/2 power 10101, or $\mu(y'')$ ends in 01, so that $\mu(y'')v0110$ ends in the 7/3 power 0110110. We conclude that $v \neq 1$.

This means that $v = \epsilon$, and $z = \mu(y')$.□

Theorem 7. *Suppose* $2 < \alpha \leq 7/3$ *and* m *is a positive integer. There is a circular binary word of length* m *avoiding* α *powers if and only if* m *is of the form* 2^n *or* 3×2^n.

Proof: The if direction follows from Thue's result on the lengths of overlap-free binary words. There is an overlap free binary circular word of each length 2^n or 3×2^n, and such an overlap free word must avoid α powers.

Now suppose that x is a circular binary word avoiding α powers. By induction on the previous theorem, $|x|$ has the form $r \times 2^n$, where $r \leq 6$, and there is a circular binary word avoiding α powers of length r. The only positive integer 6 or less not of the form 2^n or 3^n is 5. One finds that no circular binary word of length 5 avoids $5/2^+$ powers. Thus $r \neq 5$, and theorem is proved. ■

Corollary 8. $CRT(2) \geq 7/3$.

4 Circular $7/3^+$ power free words

Consider the words

- $f_0 = 00100$
- $f_1 = 11011$

Neither of the f_i appears in the Thue-Morse word t. Note that f_0 is the binary complement of f_1. Let the word $\mathcal{B}$ be a subword of the Thue-Morse word with $|\mathcal{B}| \geq 90$, of the following form:

$$\mathcal{B} = 1101001v1001011$$

Notice that f_1 and $\mathcal{B}$ have a common prefix of length 4. A candidate for the word $\mathcal{B}$ may be obtained from the word of Lemma 2 by deleting the first and last letters. We see then that word $\mathcal{B}$ may be taken to have any length $4k - 2$, $k \geq 23$.

Let w_1 be a circular word of the form $\mathcal{B}f_0f_1f_0$. Let w_3 be a circular word of the form $\mathcal{B}f_0$. We have $|w_i| \equiv i \pmod 4$, $i = 1, 3$.

Lemma 9. *No word of the form $a\mathcal{B}c$ with $|ac| \le 15$ is a k power for $k > 7/3$.*

Proof: Suppose $a\mathcal{B}c$ is a k power for $k > 7/3$, where $|ac| \le 15$. This means that $a\mathcal{B}c$ is periodic with some period p, $|a\mathcal{B}c| > 7p/3$. Its subword $\mathcal{B}$ must also then have period p. Since $\mathcal{B}$ is a subword of t, this means that $|\mathcal{B}| \le 2p$. In total then, $15 \ge |ac| = |a\mathcal{B}c| - |\mathcal{B}| > 7p/3 - 2p = p/3$, so that $45 > p$. However, then $90 \le |\mathcal{B}| \le 2p < 2 \times 45 = 90$. This is a contradiction. ∎

Lemma 10. *Suppose that a word of the form σb is a k power for $k > 7/3$, $|\sigma| \le 3$, b a subword of t. Let σb have period $p < 3|\sigma b|/7$. Then $p \le 8$.*

Proof: We have $|\sigma b| > 7p/3$, whence $|\sigma b| \ge \lfloor 7p/3 \rfloor + 1$. The word b has period p, but is a subword of t. Thus, $|b| \le 2p$. Now, $3 \ge |\sigma| = |\sigma b| - |b| \ge \lfloor 7p/3 \rfloor + 1 - 2p = \lfloor p/3 \rfloor + 1$. We conclude that $2 \ge \lfloor p/3 \rfloor$, or $p \le 8$.□

Lemma 11. *Consider a word of the form $s\beta$ where β is a prefix of $\mathcal{B}$, and s is a suffix of f_0, $|s| \le 4$. Then for $k > 7/3$, $s\beta$ is not a k power.*

Proof: Word s will be a suffix of 0100. Since $0\mathcal{B}$ is a subword of t, the result is true when $s = 0$. Let $\pi_1 = 1101001\ 0110$ and let $\pi_2 = 1101001\ 10010$. (The spaces are for clarity; they highlight the two possible prefixes of v in $\mathcal{B}$. The final 0 in π_2 reflects the fact that the overlap 100110011 cannot appear in t.)

By the construction of $\mathcal{B}$, one of π_1, π_2 is a prefix of $\mathcal{B}$. It follows that either β is a prefix of one of the π_k, or one of the π_k is a prefix of β.

To get a contradiction, suppose that $s\beta$ has period p, $|s\beta| > 7p/3$. Write $s = \sigma 0$. Then $b = 0\beta$ is a subword of t, so that by Lemma 10, $p \le 8$. If π_k is a prefix of β, then $s\pi_k$ has period p. On the other hand, if β is a prefix of π_k, then $s\pi_k$ has a prefix $s\beta$, $|s\beta| > 7p/3$. Let q be the maximal prefix of $s\pi_k$ with period p. For each choice $p = 1, 2, \ldots, 8$, and for each possibility $k = 1, 2$, we show two things:

1. Word q is a proper prefix of $s\pi_k$. This eliminates the case where π_k is a prefix of β.

2. We have $|q| \le 7p/3$. This eliminates the case where β is a prefix of π_k. We thus obtain a contradiction.

As an example, suppose $p = 6$. In $s\pi_1 = s\mathbf{1}101001\ \mathbf{0}110$, the letters in bold-face differ. This means that prefix q of period 6 is a prefix of $s1101001$, which has length $|s| + 7 \le 11 \le 7p/3 = 7 \times 6/3 = 14$. Again, in $s\pi_2 = s110\mathbf{1}001\ 10\mathbf{0}1$, the letters in bold-face differ. Any prefix of $s\pi_2$ of period 6 is thus a prefix of $s110100110$, which has length at most 14.

The following table bounds $|q|$ in the various cases. The pairs of bold-face letters certify the given values.

p	σ	$0\pi_i$		$\lvert q\rvert$		$\lvert q\rvert/p$
1	0	**01**101001 $\cdots$		2		2
	(0)**10**	01101001 $\cdots$	$\le$	2	$\le$	2
2	**0**	0**1**101001 $\cdots$		2		1
	(0**1**0	**0**1101001 $\cdots$	$\le$	3	$\le$	3/2
3	(01)**0**	01**1**01001 $\cdots$	$\le$	5	$\le$	5/3
4	(01)0	**0**110**1**001 $\cdots$	$\le$	7	$\le$	7/4
5	(01)**0**	0110**1**001 $\cdots$	$\le$	7	$\le$	7/5
6	(01)0	01**1**01001 **0**110	$\le$	11	$\le$	11/6
	(01)0	0110**1**001 10**0**10	$\le$	13	$\le$	13/6
7	(01)0	**0**110100**1** $\cdots$	$\le$	10	$\le$	10/7
8	(01)**0**	0110100**1** $\cdots$	$\le$	10	$\le$	5/4

The parentheses abbreviate rows of the table. For example, cases $\sigma = 10$ and $\sigma = 010$ are together in the second row of the table. The bold-faced pair will work whether $\sigma = 10$ or $\sigma = 010$. We have q a proper prefix of σ, whence $|q| \le 2$. Similarly, when $p = 5$, one pair works for all values of σ. Evidently, one could also verify this lemma via computer. ■

Lemma 12. *Consider a word of the form βr where β is a subword of t, and $|r| \le 4$. Then for $k > 7/3$, βr is not a k power.*

Proof: This assertion follows from the last by symmetry. ■

Corollary 13. *Let w_3 contain a k power z, some $k > 7/3$. Then z contains f_0 as a subword.*

Proof: Word z is an ordinary subword of some conjugate of w_3. The conjugates of w_3 have one of the forms $b''f_0b'$ or $f''\mathcal{B}f'$ where $f_0 = f'f''$ or $\mathcal{B} = b'b''$. We know that z cannot be a subword of $\mathcal{B}$, since t is 2^+ power free. If z does not contain f_0 therefore, then z has one of the forms $f''\mathcal{B}f'$, $f''\beta'$ or $\beta''f'$ with $|f'|$, $|f''| \le 4$, β' a prefix of $\mathcal{B}$, β'' a suffix of β''. These possibilities are ruled out by Lemmas 9, 11 and 12 respectively. ■

Lemma 14. *Fix $k > 2$. Suppose z has period $p < |z|/k$. Let u be a subword of z with $|u| \le \min(\lfloor (k-2)p \rfloor + 2, p)$. Then z contains a subword uvu for some v.*

Proof: Let au be a prefix of z with a as short as possible. Because z has period p, $|a| \le p - 1$. Write $z = aub$. Here

$$\begin{aligned}
|b| &= |z| - |au| \\
&\ge \lfloor kp \rfloor + 1 - |au| \\
&\ge \lfloor kp \rfloor + 1 - (p-1) - [\lfloor (k-2)p \rfloor + 2] \\
&= \lfloor kp \rfloor + 1 - (p-1) - [\lfloor kp \rfloor - 2p + 2] \\
&= p
\end{aligned}$$

Since $|u| \le p$ and $z = aub$ has period p, u is a subword of b. Pick v so that vu is a prefix of b. Then uvu is a subword of z. ■

Corollary 15. *Let z be a word of the form $b''f_0b'$ where b' and b'' are a prefix and suffix respectively of $\mathcal{B}$. Suppose that z is a k power, some $k > 7/3$. Then the longest period of z is at most 8.*

Proof: Suppose that $p > 8$. Then $\min(\lfloor (k-2)p \rfloor + 2, p) > \min(\lfloor 8/3 \rfloor + 2, 8) = 4$, so that $\min(\lfloor (k-2)p \rfloor + 2, p) \geq 5$. By Lemma 14 every subword of z of length 5 appears at least twice in z. However, $|f_0| = 5$, but f_0 only appears once in z. This is a contradiction. ∎

Lemma 16. *Let z be a word of the form $b''f_0b'$ where b' and b'' are a prefix and suffix respectively of $\mathcal{B}$. Then z is not a $7/3^+$ power.*

Proof: Suppose that z is a k power, some $k > 7/3$. By the last corollary, z has period $p \leq 8$. Word f_0 does not have period 1 or 2. Therefore, $p \geq 3$, and $7p/3 \geq 7$. We find then, that z must have a subword of length 8 containing f_0.This subword must have the form $b''f_0b'$ where b' and b'' are a prefix and suffix respectively of $\mathcal{B}$. The possible candidates are thus 01100100, 11001001, 10010011, 0010011 and 00100110. None of these have period 1, 2, 3, 4, 5 or 6. This implies that we must in fact have $7 \leq p \leq 8$. we find then that $|z| \geq \lfloor 7p/3 \rfloor + 1 \geq 17$. Certainly then, $|b'| \geq 4$ or $|b''| \geq 4$. This implies that z contains either 101100100 or 001001101 as a subword; however, neither of these words has period 7 or 8. This is a contradiction. ∎

Lemma 17. *Word w_3 is $7/3^+$ power free.*

Proof: Let w_3 contain a k power z, some $k > 7/3$. By Corollary 13, z contains f_0 as a subword, so that z has the form $b''f_0b'$ where b' and b'' are a prefix and suffix respectively of $\mathcal{B}$. This is impossible by Lemma 16. ∎

Lemma 18. *Let w_1 contain a k power z, some $k > 7/3$. Then z contains f_1.*

Proof: Word z cannot contain $\mathcal{B}$ as a subword. Otherwise, we could write $z = a\mathcal{B}c$, where $|ac| \leq |f_0f_1f_0| = 15$. This is impossible by Lemma 9. It follows that z is a subword of a conjugate of w_1 of the form $b''f_0f_1f_0b'$ where b' and b'' are a prefix and suffix respectively of $\mathcal{B}$. Suppose that z does not contain f_1. This means that z is a subword of either

- a word $b''f_0f_1'$, where f_1' is a prefix of f_1, $|f_1'| \leq 4$, or
- a word $f_1''f_0b'$ where f_1'' is a suffix of f_1, $|f_1''| \leq 4$.

Recall that every prefix (suffix) of f_1 of length at most 4 is also a prefix (suffix) of $\mathcal{B}$. We have thus returned to the case where k power z is a subword of a word $b''f_0b'$ where b' and b'' are a prefix and suffix respectively of $\mathcal{B}$. This is impossible, by Lemma 16. ∎

Lemma 19. *Word w_1 is $7/3^+$ power free.*

Proof: Let w_1 contain a k power z, some $k > 7/3$. Let z have period p, $|z|/p > 7/3$. By the last lemma, z contains f_1 as a subword. Since f_1 can appear in z only once, we find that $p \leq 8$. Arguing as in Lemma 16, we find that z has period 7 or 8, and contains either 010011011 or its reversal. (These are binary complements of 101100100 and 001001101 used in Lemma 16.) However, these words do not have period 7 or 8. This is a contradiction. ∎

Lemma 20. *There exist binary circular* $7/3^+$ *power free words of every odd length greater than or equal to* 105.

Proof: The words w_1 and w_3 give these lengths. ■

Theorem 21. *There exist binary circular* $7/3^+$ *power free words of every length greater than or equal to* 210.

Proof: This follows by combining the last lemma with Corollary 4. ■

Together with Theorem 7, this establishes our

Main Theorem: $CRT(2) = 7/3$.

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