Nonlinear eigenvalue problems for some degenerate elliptic operators on $\mathbb{R}^N$

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Abstract

We study two nonlinear degenerate eigenvalue problems on $\mathbb{R}^N$. For the first problem we prove the existence of a positive eigenvalue while for the second we show the existence of a continuous family of eigenvalues. Our approach is based on standard tools in the critical point theory combined with adequate variational methods. We also apply an idea developed recently by Szulkin and Willem.

1 Introduction

In this paper we shall deal with the nonlinear eigenvalue problem

$$-\mathrm{div}(\mathcal{A}(x)\nabla u) = \lambda f(x,u), \qquad x \in \mathbb{R}^N. \tag{1}$$

We work under general conditions $N \geq 3$, and $\mathcal{A} : \mathbb{R}^N \to \mathbb{R}$ is a function which vanishes in at least one point on $\mathbb{R}^N$. Thus equation (1) becomes a degenerate elliptic equation.

The presence of the singular potential $\mathcal{A}(x)$ in the divergence operator represents the main point of interest in our investigation. At our best knowledge, the study of degenerate elliptic equations began around the 1800's with Legendre's famous equation

$$-\frac{d}{dx}\left\{(1-x^2)\frac{df}{dx}\right\} = \lambda f, \qquad x \in [-1,1]. \tag{2}$$

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For more information and connections on equation (2) the reader may consult Kevorkian [15] (p. 124-125) or Davies [11] (Example 1.2.4, p. 10). Between 1953 and 1954 the study of degenerate elliptic equations was carried on by Mikhlin's work [16], [17] who also pointed out the relevance of these problems in Mathematical Physics. This study continued in the 70's with a careful analysis of several linear degenerate elliptic problems (see, e.g., Murthy-Stampacchia [19] and Trudinger [23]). We also refer to more recent papers by Baouendi-Goulaouic [3], Edmunds-Peletier [13], Passaseo [18], Stredulinsky [22] for the treatment of some nonlinear classes of degenerate elliptic problems. In the non-singular case when $\mathcal{A}(x)$ is a function that does not vanish on $\mathbb{R}^N$ we can mention several studies devoted to the investigation of related problems. P. Drábek, for instance, proved in [12] that the problem

$$\begin{cases} -\mathrm{div}(a(x)\nabla u) = \lambda f(x,u), & x \in \mathbb{R}^N, \quad \lambda > 0 \\ u > 0, & x \in \mathbb{R}^N, \quad \lim\limits_{|x|\to\infty} u = 0 \end{cases}$$

has a solution in the weak sense assuming that $a \in L^\infty(\mathbb{R}^N)$ is a positive function and f is a Caratheodory function having sub-critical growth. The above formulation of the problem is not quite exact the result being proved in a more general case which involves the operator $-\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u)$ with $1 < p < N$.

There are more extensive results recorded in the particular case $f(x,u) = g(x)u$. We start by celebrating the result given by Brown-Cosner-Fleckinger [5] on a problem of the type

$$-\Delta u = \lambda g(x) u, \quad x \in \mathbb{R}^N. \tag{3}$$

They showed that (3) has a principal eigenvalue λ_1 if g is sufficiently smooth and satisfies an appropriate condition at infinity. In the case g is bounded and $g^+ \in L^{N/2}(\mathbb{R}^N)$ Allegretto proved in [2] the existence of infinitely many eigenvalues such that $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...$ and $\lambda_n \to \infty$ as $n \to \infty$. Recently, A. Szulkin and M. Willem in [21] improved the above results on equations (3) showing the existence of a sequence of eigenvalues $\lambda_n \to \infty$ when g satisfies the assumption

$g \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, $g^+ = g_1 + g_2 \neq 0$, $g_1 \in L^{N/2}(\mathbb{R}^N)$, $\lim\limits_{x\to y} |x-y|^2 g_2(x) = 0$, $\forall y \in \mathbb{R}^N$, $\lim\limits_{|x|\to\infty} |x|^2 g_2(x) = 0$.

The result of Szulkin-Willem is even more general, the existence of eigenvalues being proved for a general open set $\Omega \subset \mathbb{R}^N$.

Finally, we refer to Jin [14] who proved under the hypotheses that g is a locally Hölder continuous function on $\mathbb{R}^N$, the existence of a continuous family of eigenvalues for (3) when g is dominated by a non-negative function g_1 having the property that the problem

$$-\Delta u = \mu g_1(x) u, \quad x \in \mathbb{R}^N$$

has a principal eigenvalue $\mu > 0$.

In this paper we study two different eigenvalue problems. We are concerned only with the weak solutions for problems of type (1). Each time will be specified the space in which we are seeking solutions. The respective spaces will be weighted Sobolev spaces defined as the closure of $C_0^\infty(\mathbb{R}^N)$ under different norms.

First, we consider the equation

$$-\mathrm{div}(a(x)\nabla u) = \lambda\left(f(x,u) + \theta(x)|u|^{\gamma-1}u \cdot \int_{\mathbb{R}^N} (2F(x,u) - f(x,u)u)\,dx\right), \qquad x \in \mathbb{R}^N. \tag{4}$$

We denote by $2^\star$ the critical Sobolev exponent, i.e. $2^\star = 2N/(N-2)$ and let $p \in (2, 2^\star)$ be fixed. Suppose that the functions $a(x)$, $f(x,t)$ and $\theta(x)$ satisfy the hypotheses:

(A) $a \in C(\mathbb{R}^N)$, $a(x) \geq 0$ a.e. $x \in \mathbb{R}^N$ and there exists $q > Np/(2N + 2p - Np)$ such that $1/a \in L^q(\mathbb{R}^N)$;

(F1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}_+$ is a Carathéodory function, i.e. f is measurable and for $x \in \mathbb{R}^N$, the function $t \to f(x,t)$ is continuous on $\mathbb{R}$;

(F2) There exists an open set $\Omega \subset \mathbb{R}^N$ $(\Omega \neq \emptyset)$ and there exist two constants $0 \leq \delta_1 < \delta_2 \leq \infty$ such that $f(x,t) > 0$ on $\Omega \times (\delta_1, \delta_2)$;

(F3) There exists a non-negative function $\theta = \theta(x)$ such that

$$0 \leq f(x,t) \leq \theta(x)|t|^\gamma, \quad \forall x \in \mathbb{R}^N, \quad \forall t \in \mathbb{R}$$

where $0 < \gamma < 1$ and $\theta \in L^m(\mathbb{R}^N)$ with $m = \beta^\star/(\beta^\star - (\gamma+1))$. We denoted by β and $\beta^\star$ the real numbers $\beta = 2q/(q+1)$ and $\beta^\star = N\beta/(N-\beta)$ with q given by condition (A).

Condition (A) is inspired by condition (A1) in Cîrstea-Rădulescu [10] (see also Chabrowski [9] or Murthy-Stampacchia [19]) while conditions (F1)-(F3) are inspired by conditions (f1)-(f3) in Drábek [12]. Similar conditions may be founded in Rădulescu-Smets [20].

We point out that there exist functions $a : \mathbb{R}^N \to \mathbb{R}$ which satisfy the condition (A) and which vanish on $\mathbb{R}^N$. An example in that sense is the function $a_1(x) = |x|^{\nu_1}$ if $|x| \leq 1$ and $a_1(x) = |x|^{\nu_2}$ if $|x| \geq 1$ with $0 < \nu_1 < N/q < \nu_2$.

We define now $F(x,u) = \int_0^u f(x,t)\,dt$. By (F2) we obtain that $F(x,u) > 0$ on $\Omega \times (\delta_1, \delta_2)$ while by (F3) we deduce

$$0 \leq F(x,u) \leq \frac{1}{\gamma+1}\theta(x)|t|^{\gamma+1}, \qquad \text{on } \mathbb{R}^N \times \mathbb{R}. \tag{5}$$

Let us consider the weighted Sobolev space $\mathcal{D}_a(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_a = \left(\int_{\mathbb{R}^N} a(x)|\nabla u|^2\,dx\right)^{1/2}.$$

Clearly, $\mathcal{D}_a(\mathbb{R}^N)$ is a Hilbert space with respect to the scalar product

$$\langle u, v\rangle_a = \int_{\mathbb{R}^N} a(x)\nabla u \nabla v\,dx.$$

Definition 1. *We say that* $\lambda \in \mathbb{R}$ *is an eigenvalue of (4) if there exists* $u \in \mathcal{D}_a(\mathbb{R}^N) \setminus \{0\}$ *such that*

$$\int_{\mathbb{R}^N} a(x)\nabla u\nabla\varphi\,dx - \lambda \cdot \int_{\mathbb{R}^N} f(x,u)\varphi\,dx - \lambda \cdot \int_{\mathbb{R}^N} (2F(x,u) - f(x,u)u)\,dx \cdot \int_{\mathbb{R}^N} \theta(x)|u|^{\gamma-1}u\varphi\,dx = 0 \tag{6}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$*. Moreover, a function* $u \in \mathcal{D}_a(\mathbb{R}^N)$ *which verifies the above relation for a fixed* $\lambda \in \mathbb{R}$ *is called a weak solution of (4).*

The main result on problem (4) is given by the following theorem:

Theorem 1. *Assuming (A), (F1)-(F3) are fulfilled then there exists* $\lambda_0 > 0$*, an eigenvalue of (4). Moreover, the corresponding week solution,* $u_0 \in \mathcal{D}_a(\mathbb{R}^N) \setminus \{0\}$*, satisfies* $u_0 \geq 0$ *in* $\mathbb{R}^N$*.*

Second, we study problem (1) in the particular case $\mathcal{A}(x) = |x|^\alpha$. Thus equation (1) becomes

$$-\text{div}(|x|^\alpha \nabla u) = \lambda f(x,u), \qquad x \in \mathbb{R}^N. \tag{7}$$

We assume that $\alpha \in (0,2)$ and $f(x,t)$ is a function of the type $f(x,t) = g(x)t + r(x)|t|^{q-2}t$ with $2 < q < 2_\alpha^\star$. We denoted by $2_\alpha^\star$ the real number $2_\alpha^\star = 2N/(N-2+\alpha)$. Suppose that r is a positive function on $\mathbb{R}^N$ which satisfies the property $r \in L^s(\mathbb{R}^N)$ for s chosen such that $1/s + q/2_\alpha^\star = 1$. We show that imposing some conditions on g problem (7) has a continuous family of positive eigenvalues.

Let us consider the weighted Sobolev space $\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^2 \, dx \right)^{1/2}.$$

It is clear that $\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$ is a Hilbert space with respect to the scalar product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} |x|^\alpha \nabla u \nabla v \, dx.$$

Our basic assumption on g is that it is dominated by a function g_1 which satisfies the property:

(G) *A non-negative function* g_1 *on* $\mathbb{R}^N$ *is said to have the property* **(G)** *if* $g_1(x) > 0$ *a.e.* $x \in \mathbb{R}^N$ *and there are* $\mu > 0$ *and* $u \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N) \setminus \{0\}$ *such that*

$$\mu = \frac{\displaystyle\int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^2 \, dx}{\displaystyle\int_{\mathbb{R}^N} g_1(x) u^2 \, dx} = \inf_{\varphi \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |x|^\alpha |\nabla \varphi|^2 \, dx}{\displaystyle\int_{\mathbb{R}^N} g_1(x) \varphi^2 \, dx}.$$

Definition 2. *We say that* $\lambda \in \mathbb{R}$ *is an eigenvalue of (7) if there exists* $u \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N) \setminus \{0\}$ *such that*

$$\int_{\mathbb{R}^N} |x|^\alpha \nabla u \nabla \varphi \, dx - \lambda \cdot \int_{\mathbb{R}^N} f(x,u) \varphi \, dx = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$*. Moreover, a function* $u \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$ *which verifies the above relation for a fixed* $\lambda \in \mathbb{R}$ *is called a weak solution of (7).*

Our first result on problem (7) is given by the following theorem

Theorem 2. *Assuming* g_1 *is a non-negative function which satisfies property* **(G)** *and* $g \in L^\infty_{\rm loc}(\mathbb{R}^N)$ *is a function such that*

$$|g(x)| \le g_1(x) \qquad \text{a.e. } x \in \mathbb{R}^N$$

then any $\lambda \in (0, \mu)$ *is an eigenvalue for problem (7), where* μ *is a number defined in property* **(G)**.

We point out that there are a lot of functions having property **(G)**. Indeed, we can replace property **(G)** on g_1 by the property:

(G1) *A non-negative function* g_1 *is said to have property* **(G1)** *if* $g_1(x) > 0$ *a.e.* $x \in \mathbb{R}^N$ *and the application*

$$\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N) \ni u \longrightarrow \int_{\mathbb{R}^N} g_1(x) u^2 \, dx$$

is weakly continuous.

We establish the following result.

Theorem 3. *Assuming* g_1 *is a non-negative function which has property* **(G1)** *then* g_1 *has property* **(G)**. *Furthermore, if* g *is a function such that*

$$|g(x)| \le g_1(x) \qquad \text{a.e. } x \in \mathbb{R}^N$$

then $|g|$ *has property* **(G)**.

We remark that there exist functions which verify property **(G1)**. A class of such functions is offered by the following theorem.

Theorem 4. *Assuming* $g_1 > 0$ *a.e.* $x \in \mathbb{R}^N$ *is a function such that* $g_1 \in L^{N/(2-\alpha)}(\mathbb{R}^N)$ *then the application*

$$\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N) \ni u \longrightarrow \int_{\mathbb{R}^N} g_1(x) u^2 \, dx$$

is weakly continuous.

As an application of Theorems 2-4 we have:

Theorem 5. *Assuming* g *is a function such that* $|g| \in L^{N/(2-\alpha)}(\mathbb{R}^N)$ *then any number* $\lambda \in (0, \lambda_1)$ *is an eigenvalue for (7), where* $\lambda_1 > 0$ *reaches the minimum in the expression*

$$\lambda_1 = \inf_{\varphi \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |x|^\alpha |\nabla \varphi|^2 \, dx}{\displaystyle\int_{\mathbb{R}^N} |g(x)| \varphi^2 \, dx}.$$

2 Proof of Theorem 1

For each $\lambda \ge 0$ we associate to problem (4) the equation

$$-\text{div}(a(x)\nabla u) - \lambda f(x, u) = M(\lambda)\theta(x)|u|^{\gamma-1}u, \qquad x \in \mathbb{R}^N. \tag{8}$$

Clearly, λ_0 and u_0 solve Theorem 1 if and only if $M(\lambda) = \int_{\mathbb{R}^N}(2\lambda_0 F(x,u_0) - \lambda_0 f(x,u_0)u_0)\,dx$ is an eigenvalue of (8). To find the principal eigenvalue of (8) we solve the minimization problem

$$\text{minimize} \int_{\mathbb{R}^N} a(x)|\nabla u|^2\,dx - 2\lambda \cdot \int_{\mathbb{R}^N} F(x,u)\,dx,$$
$$\text{when } u \in \mathcal{D}_a(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} \theta(x)|u|^{\gamma+1}\,dx = 1. \quad (9)$$

Let

$$S_\lambda(u) = \int_{\mathbb{R}^N} a(x)|\nabla u|^2\,dx - 2\lambda \cdot \int_{\mathbb{R}^N} F(x,u)\,dx.$$

We show that problem (9) has sense, that is for every $\lambda \geq 0$ fixed $S_\lambda(u)$ is bounded from below for all $u \in \mathcal{D}_a(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}\theta(x)|u|^{\gamma+1}\,dx = 1$.

Indeed, using (5) we obtain that for any fixed $\lambda \geq 0$ holds the inequality

$$S_\lambda(u) \geq \|u\|_a^2 - \frac{2\lambda}{\gamma+1} \cdot \int_{\mathbb{R}^N} \theta(x)|u|^{\gamma+1}\,dx.$$

Thus for all $u \in \mathcal{D}_a(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}\theta(x)|u|^{\gamma+1}\,dx = 1$ we have

$$S_\lambda(u) \geq -\frac{2\lambda}{\gamma+1} > -\infty.$$

Let

$$\mu(\lambda) = \inf_{u \in \mathcal{D}_a(\mathbb{R}^N),\ \int_{\mathbb{R}^N}\theta(x)|u|^{\gamma+1}\,dx = 1} S_\lambda(u).$$

Lemma 1. *The function space $\mathcal{D}_a(\mathbb{R}^N)$ is continuously embedded in $W^{1,\beta}(\mathbb{R}^N)$ and in $L^{\beta^\star}(\mathbb{R}^N)$ where $\beta = 2q/(q+1)$ and $\beta^\star = N\beta/(N-\beta)$.*

Proof. Since $\beta = 2q/q+1$ it follows that $1 < \beta < 2 < N$. Using the Sobolev-Galiardo-Nirenberg inequality (see Theorem IX.9, p. 162 in [4]) combined with the Hölder inequality we get

$$\begin{aligned}
\left(\int_{\mathbb{R}^N} |u|^{\beta^\star}\,dx\right)^{1/\beta^\star} &\leq C_1 \cdot \left(\int_{\mathbb{R}^N} |\nabla u|^\beta\,dx\right)^{1/\beta} \\
&= C_1 \cdot \left(\int_{\mathbb{R}^N} \frac{1}{a(x)^{q/q+1}}\,|\nabla u|^\beta\,a(x)^{q/q+1}\,dx\right)^{1/\beta} \\
&\leq C_1 \cdot \left(\int_{\mathbb{R}^N} \frac{1}{a(x)^q}\,dx\right)^{1/2q} \cdot \left(\int_{\mathbb{R}^N} a(x)|\nabla u|^2\,dx\right)^{1/2} \\
&\leq C_2 \cdot \left(\int_{\mathbb{R}^N} a(x)|\nabla u|^2\,dx\right)^{1/2}
\end{aligned}$$

where C_1, C_2 are positive constants. By the above inequalities Lemma 1 follows. ■

Lemma 2. *For all $u \in \mathcal{D}_a(\mathbb{R}^N)$ there exists $C > 0$ such that*

$$\int_{\mathbb{R}^N} F(x,u)\,dx \leq C \cdot \|u\|_a^{\gamma+1}.$$

Proof. Applying Hölder's inequality and Lemma 1 we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,u)\,dx &\le \frac{1}{\gamma+1}\cdot\int_{\mathbb{R}^N}\theta(x)|u|^{\gamma+1}\,dx\\ &\le \frac{1}{\gamma+1}\cdot\left(\int_{\mathbb{R}^N}|\theta(x)|^m\,dx\right)^{1/m}\cdot\left(\int_{\mathbb{R}^N}|u|^{\beta^\star}\,dx\right)^{\gamma+1/\beta^\star}\\ &\le \frac{1}{\gamma+1}\cdot\left(\int_{\mathbb{R}^N}|\theta(x)|^m\,dx\right)^{1/m}\cdot(c\,\|u\|_a)^{\gamma+1}.\end{aligned}$$

∎

Lemma 3. *Assume conditions (A), (F1)-(F3) are fulfilled. Then problem (9) has a solution* $e_1 \ge 0$*. Moreover,* e_1 *is a weak solution of (8) corresponding to the eigenvalue*

$$M(\lambda) = \|e_1\|_a^2 - \lambda\cdot\int_{\mathbb{R}^N} f(x,e_1)e_1\,dx.$$

Proof. Let $\{u_n\}$ be a minimizing sequence for (9), i.e.

$$\int_{\mathbb{R}^N} a(x)|\nabla u_n|^2\,dx - 2\lambda\cdot\int_{\mathbb{R}^N} F(x,u_n)\,dx \to \mu(\lambda)$$

and $\int_{\mathbb{R}^N}\theta(x)|u_n|^{\gamma+1}\,dx = 1$, for all n. Then by Lemma 2 it follows that $\{u_n\}$ is bounded in $\mathcal{D}_a(\mathbb{R}^N)$. Since $\mathcal{D}_a(\mathbb{R}^N)$ is a Hilbert space we deduce that there exists $u \in \mathcal{D}_a(\mathbb{R}^N)$ such that passing eventually to a subsequence $\{u_n\}$ converges weakly to u in $\mathcal{D}_a(\mathbb{R}^N)$. By Proposition III.5 (iii), p. 35 in [4] we obtain

$$\int_{\mathbb{R}^N} a(x)|\nabla u|^2\,dx \le \liminf_{n\to\infty}\int_{\mathbb{R}^N} a(x)|\nabla u_n|^2\,dx. \tag{10}$$

On the other hand since $\{u_n\}$ is bounded in $\mathcal{D}_a(\mathbb{R}^N)$ by Lemma 1 we have that $\{u_n\}$ is bounded in $L^{\beta^\star}(\mathbb{R}^N)$. Moreover, we get $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$. Then by Theorem 10.36, p. 220 in [24] it follows that $\{u_n\}$ converges weakly to u in $L^{\beta^\star}(\mathbb{R}^N)$ or $\{|u_n|^{\gamma+1}\}$ converges weakly to $|u|^{\gamma+1}$ in $L^{\beta^\star/\gamma+1}(\mathbb{R}^N)$.

We define the operator $\Lambda : L^{\beta^\star/\gamma+1}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\langle\Lambda,\varphi\rangle = \int_{\mathbb{R}^N}\theta(x)\varphi\,dx.$$

Since $\theta \in L^m(\mathbb{R}^N)$ and $1/m = 1-(\gamma+1)/\beta^\star$ it is clear that Λ is linear and continuous. The above remarks imply that $\langle\Lambda,|u_n|^{\gamma+1}\rangle \to \langle\Lambda,|u|^{\gamma+1}\rangle$ or

$$\int_{\mathbb{R}^N}\theta(x)|u_n|^{\gamma+1}\,dx \to \int_{\mathbb{R}^N}\theta(x)|u|^{\gamma+1}\,dx.$$

Since $\theta(x)|u_n|^{\gamma+1}$ converges to $\theta(x)|u|^{\gamma+1}$ in $L^1(\mathbb{R}^N)$ it follows by Theorem IV.9, p. 58 in [4] that passing eventually to a subsequence, there exists $h_0 \in L^1(\mathbb{R}^N)$ such that

$$\frac{1}{\gamma+1}\,|\theta(x)|\,|u_n|^{\gamma+1} \le h_0(x) \qquad \text{a.e. } x\in\mathbb{R}^N.$$

Taking into account that $F(x,u_n(x)) \to F(x,u(x))$ a.e. $x \in \mathbb{R}^N$ and $|F(x,u_n)| \le \frac{1}{\gamma+1}\,\theta(x)\,|u_n|^{\gamma+1}$ a.e. $x \in \mathbb{R}^N$ we obtain via Lebesgues' Theorem (Theorem IV.2, p. 54 in [4]) that

$$\int_{\mathbb{R}^N} F(x,u_n)\,dx \to \int_{\mathbb{R}^N} F(x,u)\,dx. \tag{11}$$

Finally, we remark that

$$\int_{\mathbb{R}^N} \theta(x)|u|^{\gamma+1}\, dx = 1. \tag{12}$$

By (10), (11) and (12) we conclude that u is a solution of problem (9). Furthermore, since $|u|$ is also a solution we may assume that $u \geq 0$.

Let now $v \in \mathcal{D}_a(\mathbb{R}^N)$ be fixed. Then the application

$$h(\epsilon) = S_\lambda\left(\frac{u+\epsilon v}{(\int_{\mathbb{R}^N} \theta(x)|u+\epsilon v|^{\gamma+1}\, dx)^{1/(\gamma+1)}}\right)$$

is well defined in a suitable neighborhood of the origin and it possesses a minimum in $\epsilon = 0$. Then we obtain that $h^{'}(0) = 0$ which yields

$$\int_{\mathbb{R}^N} a(x)\nabla u \nabla v\, dx - \lambda \cdot \int_{\mathbb{R}^N} f(x,u)v\, dx = M(\lambda) \cdot \int_{\mathbb{R}^N} \theta(x)|u|^{\gamma-1}uv\, dx$$

with

$$M(\lambda) = \|u\|_a^2 - \lambda \cdot \int_{\mathbb{R}^N} f(x,u)u\, dx.$$

It follows that u is a weak solution of (8) corresponding to the eigenvalue $M(\lambda)$ specified above. ■

Proof of Theorem 1. We remark that for fixed $\psi \in \mathcal{D}_a(\mathbb{R}^N)$, the application

$$\lambda \longrightarrow \|\psi\|_a^2 - 2\lambda \cdot \int_{\mathbb{R}^N} F(x,\psi)\, dx$$

is an affine function and thus a concave function. As the infimum of any collection of concave functions is a concave function it follows that $\lambda \to \mu(\lambda)$ is a concave function on $[0,\infty)$ and thus a continuous function.

On the one hand it is clear that $\mu(0) > 0$. Considering $\omega \subset \Omega$ a nonempty bounded domain and fixing $\delta \in (\delta_1, \delta_2)$ there exists $\psi_0 \in C_0^\infty(\mathbb{R}^N)$ such that $\psi_0 = \delta$ on ω and $|\psi_0| \leq 2\delta$ on $\mathbb{R}^N$. By (F2) we obtain that

$$\int_{\mathbb{R}^N} F(x,\psi_0)\, dx \geq \int_\omega F(x,\psi_0)\, dx = \int_\omega F(x,\delta)\, dx > 0.$$

The above inequality yields $\lim\limits_{\lambda\to\infty} \mu(\lambda) = -\infty$. Thus $\lambda \to \mu(\lambda)$ is a continuous function with $\mu(0) > 0$ and $\lim\limits_{\lambda\to\infty} \mu(\lambda) = -\infty$. Then, clearly, there exists $\lambda_0 > 0$ such that $\mu(\lambda_0) = 0$. On the other hand we point out the fact that since u is the solution of (8) given by Lemma 3 we have $M(\lambda) = \mu(\lambda) + \int_{\mathbb{R}^N}(2\lambda F(x,u) - \lambda f(x,u)u)\, dx$. We conclude Theorem 1 holds true. ■

3 Proof of Theorem 2

We fix a λ in $(0,\mu)$. Let us define the energetic functional $J_\lambda : \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ by

$$J_\lambda(u) = \frac{1}{2} \cdot \int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^2\, dx - \frac{\lambda}{2} \cdot \int_{\mathbb{R}^N} g(x)u^2\, dx - \frac{\lambda}{q} \cdot \int_{\mathbb{R}^N} r(x)|u|^q\, dx.$$

It is clear that J_λ is well-defined on $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ and $J_\lambda \in C^1(\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N), \mathbb{R})$ with

$$\langle J_\lambda^{'}(u), v\rangle = \int_{\mathbb{R}^N} |x|^\alpha \nabla u \nabla v \, dx - \lambda \cdot \int_{\mathbb{R}^N} g(x) uv \, dx - \lambda \cdot \int_{\mathbb{R}^N} r(x)|u|^{q-2} uv \, dx,$$

for all $u, v \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$. Moreover, we observe that u is a solution of (7) if and only if u is a critical point of J_λ. To obtain the nontrivial critical points for J_λ we shall apply the Mountain Pass Theorem.

We establish some auxiliary results. We start by recalling the following key result, known as the Caffarelli-Kohn-Nirenberg inequality (see [6]).

Lemma 4. *Let $N \geq 2$, $\alpha \in (0, 2)$ and denote $2^\star_\alpha = \frac{2N}{N-2+\alpha}$. Then there exists $C_\alpha > 0$ such that*

$$\left(\int_{\mathbb{R}^N} |\varphi|^{2^\star_\alpha} dx \right)^{2/2^\star_\alpha} \leq C_\alpha \cdot \int_{\mathbb{R}^N} |x|^\alpha |\nabla \varphi|^2 dx$$

for every $\varphi \in C_0^\infty(\mathbb{R}^N)$.

As an immediate consequence, we get that $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ is continuously embedded in $L^{2^\star_\alpha}(\mathbb{R}^N)$. Let us illustrate some other useful remarks.

Remark 1. *For all Ω bounded domains in $\mathbb{R}^N$, with $0 \notin \overline{\Omega}$, the norm $\|\cdot\|$ and the $H^1_0(\Omega)$ norm, i.e.*

$$\|u\|^2_{H^1_0(\Omega)} = \int_\Omega |\nabla u|^2 \, dx$$

are equivalent. Using that fact and the Rellich-Kondrachov embedding theorem for Sobolev spaces (see [1], p. 144) we can deduce a result presented as Example 1.3 by Caldiroli-Musina in [7]: if Ω is a bounded domain in $\mathbb{R}^N$ with $0 \notin \overline{\Omega}$ then the embedding $D^{1,2}_\alpha(\Omega) \subset L^{2^\star_\alpha}(\Omega)$ is compact for $\alpha \in (0, 2)$, where $2^\star_\alpha = 2N/(N-2+\alpha)$. Moreover, we deduce that $\mathcal{D}^{1,2}_\alpha(\Omega)$ is compactly embedded in $L^i(\Omega)$ for all $i \in [1, 2^\star_\alpha]$.

Remark 2. *By Lemma 2.1 in Catrina-Wang [8] we deduce that*

$$D^{1,2}_\alpha(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N \setminus \{0\})}^{\|\cdot\|}.$$

Lemma 5. *There exist $a > 0$, $\rho > 0$ such that $J_\lambda(u) \geq a > 0$ for all $u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ with $\|u\| = \rho$.*

Proof. Using Hölder's inequality and Lemma 4 we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2} \cdot \|u\|^2 - \frac{\lambda}{2} \cdot \int_{\mathbb{R}^N} g(x) u^2 \, dx - \frac{\lambda}{q} \cdot \int_{\mathbb{R}^N} r(x)|u|^q \, dx \\ &\geq \frac{1}{2} \cdot \|u\|^2 - \frac{\mu - \sigma}{2\mu} \cdot \mu \cdot \int_{\mathbb{R}^N} g_1(x) u^2 \, dx - \frac{\lambda}{q} \cdot \|r\|_{L^s(\mathbb{R}^N)} \cdot \|u\|^q_{L^{2^\star_\alpha}(\mathbb{R}^N)} \\ &\geq \frac{\sigma}{2\mu} \cdot \|u\|^2 - \frac{\lambda}{q} \cdot \|r\|_{L^s(\mathbb{R}^N)} \cdot C_\alpha \cdot \|u\|^q \\ &\geq \left[\frac{\sigma}{2\mu} - \frac{\mu}{q} \cdot \|r\|_{L^s(\mathbb{R}^N)} \cdot C_\alpha \cdot \|u\|^{q-2} \right] \cdot \|u\|^2 \end{aligned}$$

where σ is a positive constant which lies in the interval $(0, \mu)$. The above inequalities show that the conclusion of Lemma 5 holds. ∎

Lemma 6. *There exists $e \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$ with $\|e\| > \rho$ (ρ given by Lemma 5) such that $J_\lambda(e) < 0$.*

Proof. Let $e_1 \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N) \setminus \{0\}$ be a function given in property **(G)** and $t > 0$. Then we have

$$J_\lambda(t \cdot e_1) = \frac{t^2}{2} \cdot \|e_1\|^2 - \frac{\lambda \cdot t^2}{2} \cdot \int_{\mathbb{R}^N} g(x) e_1^2 \, dx - \frac{\lambda \cdot t^q}{q} \cdot \int_{\mathbb{R}^N} r(x) |e_1|^q \, dx.$$

Passing to the limit as $n \to \infty$ we deduce that the above expression tends to $-\infty$, since $q > 2$. It follows that fixing a $t_0 > 0$ large enough and letting $e = t_0 \cdot e_1$ we obtain $J_\lambda(e) < 0$ and the Lemma is proved. ■

Lemma 7. *Assume that the hypotheses of Lemmas 5 and 6 are fulfilled. If*

$$\Gamma = \{\gamma \in C([0,1], D_\alpha^{1,2}(\mathbb{R}^N));\ \gamma(0) = 0,\ \gamma(1) = e\}$$

where e is given by Lemma 6, and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t))$$

then $c > 0$.

Proof. It is obvious that $c \geq 0$ because $c \geq \inf_{\gamma \in \Gamma} \max_{t \in \{0,1\}} J_\lambda(\gamma(t))$ and

$$\gamma(0) = 0 \Rightarrow J_\lambda(\gamma(0)) = J_\lambda(0) = 0,$$
$$\gamma(1) = e \Rightarrow J_\lambda(\gamma(1)) = J_\lambda(e) < 0.$$

We suppose that $c = 0$. Then $0 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t))$. It follows that

1) $\max_{t \in [0,1]} J_\lambda(\gamma(t)) \geq 0,\ \forall \gamma \in \Gamma$;

2) for all $\epsilon > 0$ there is $\gamma_\epsilon \in \Gamma$ such that, $\max_{t \in [0,1]} J_\lambda(\gamma_\epsilon(t)) < \epsilon$. Using a given by Lemma 5 we fix ϵ such that $0 < \epsilon < a$.

We have $\gamma_\epsilon(0) = 0$, $\gamma_\epsilon(1) = e$. It follows, $\|\gamma_\epsilon(0)\| = 0$, $\|\gamma_\epsilon(1)\| = \|e\| > \rho$ (where ρ is given by Lemma 5). But the application $t \to \|\gamma_\epsilon(t)\|$, is continuous and thus we conclude that there exists $t_\epsilon \in [0,1]$ such that $\|\gamma_\epsilon(t_\epsilon)\| = \rho$. Then $J_\lambda(\gamma_\epsilon(t_\epsilon)) \geq a > \epsilon$ and we have obtained a contradiction with 2).

We conclude that $c > 0$ and then Lemma 7 follows. ■

Proof of Theorem 2. Applying Lemma 7 and the Mountain Pass Theorem we deduce that there exists a sequence $\{u_n\} \in D_\alpha^{1,2}(\mathbb{R}^N)$ such that

$$J_\lambda(u_n) \to c, \quad J_\lambda'(u_n) \to 0 \text{ in } (D_\alpha^{1,2}(\mathbb{R}^N))^\star. \tag{13}$$

We show that $\{u_n\}$ is bounded in $\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$. Indeed, if we assume the contrary then we may suppose that passing eventually to a subsequence $\|u_n\| \to \infty$. Then for n large enough we have

$$\begin{aligned} c + 1 + \|u_n\| &\geq J_\lambda(u_n) - \frac{1}{q} \cdot \langle J_\lambda'(u_n), u_n \rangle \\ &= \left(\frac{1}{2} - \frac{1}{q}\right) \cdot \left[\|u_n\|^2 - \lambda \cdot \int_{\mathbb{R}^N} g(x) u_n^2 \, dx\right] \\ &\geq \left(\frac{1}{2} - \frac{1}{q}\right) \cdot \left(1 - \frac{\lambda}{\mu}\right) \cdot \|u_n\|^2. \end{aligned}$$

Now dividing the above inequality by $\|u_n\|$ and passing to the limit we obtain a contradiction. Hence $\{u_n\}$ is bounded in $D^{1,2}_\alpha(\mathbb{R}^N)$.

Since $\{u_n\}$ is bounded in $D^{1,2}_\alpha(\mathbb{R}^N)$ it follows that there exists $u \in D^{1,2}_\alpha(\mathbb{R}^N)$ and a sub-sequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that $\{u_n\}$ converges weakly to u in $D^{1,2}_\alpha(\mathbb{R}^N)$. By Remark 1 $\{u_n\}$ converges strongly to u in $L^i(\Omega)$, for all $\Omega \subset \mathbb{R}^N$ bounded domains with $0 \notin \overline{\Omega}$ and for all $i \in [1, 2^\star_\alpha]$.

We prove now that u is a weak solution of problem (7). By Remark 2 it is enough to verify that

$$\langle J'_\lambda(u_n), \varphi\rangle \to \langle J'_\lambda(u), \varphi\rangle, \ \ \forall\, \varphi \in C^\infty_0(\mathbb{R}^N \setminus \{0\}).$$

Let $\varphi \in C^\infty_0(\mathbb{R}^N \setminus \{0\})$ be fixed. We set $\Omega = \mathrm{supp}(\varphi)$ $(0 \notin \overline{\Omega})$. Since $u_n \rightharpoonup u$ in $D^{1,2}_\alpha(\mathbb{R}^N)$ we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^\alpha \nabla u_n \nabla\varphi \, dx = \int_{\mathbb{R}^N} |x|^\alpha \nabla u \nabla\varphi \, dx.$$

Furthermore, we have

$$\left|\int_\Omega r(x)[|u_n|^{q-2}u_n - |u|^{q-2}u]\varphi \, dx\right| \leq \|\varphi\|_{L^\infty(\Omega)} \cdot \|r\|_{L^s(\Omega)} \cdot \||u_n|^{q-2}u_n - |u|^{q-2}u\|_{L^{2^\star_\alpha/q}(\Omega)}.$$

Taking into account that $u_n \to u$ in $L^i(\Omega)$ for all $i \in [1, 2^\star_\alpha]$ and since $2^\star_\alpha/q \in [1, 2^\star_\alpha]$ we deduce by Theorem A.2 in [25] that the right-hand side of the above inequality converges to 0, as $n \to \infty$.

With the same arguments we have

$$\left|\int_\Omega g(x)[u_n - u]\varphi \, dx\right| \leq \|\varphi\|_{L^\infty(\Omega)} \cdot \|g\|_{L^\infty(\Omega)} \cdot \|u_n - u\|_{L^1(\Omega)} \to 0 \quad \text{as } \ n \to \infty.$$

We conclude that

$$\langle J'_\lambda(u_n), \varphi\rangle \to \langle J'_\lambda(u), \varphi\rangle$$

for all $\varphi \in C^\infty_0(\mathbb{R}^N \setminus \{0\})$, and thus u is a weak solution of equation (7).

Finally we remark that u can not be trivial. Indeed by (13) we deduce that for n large enough we have

$$\begin{aligned} \frac{c}{2} &\leq J_\lambda(u_n) - \frac{1}{2} \cdot \langle J'_\lambda(u_n), u_n\rangle \\ &\leq \lambda \cdot \left(\frac{1}{2} - \frac{1}{q}\right) \cdot \int_{\mathbb{R}^N} r(x)|u_n|^q \, dx. \end{aligned}$$

Since $c > 0$ it is enough to prove that $\int_{\mathbb{R}^N} r(x)|u_n|^q \, dx \to \int_{\mathbb{R}^N} r(x)|u|^q \, dx$ as $n \to \infty$. Indeed, by $u_n \rightharpoonup u$ in $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ we deduce that $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$ and $\|u_n\|$ is bounded.

Using Lemma 4 it follows that $\|u_n\|_{L^{2^\star_\alpha}(\mathbb{R}^N)}$ is bounded. Then the hypotheses of Theorem 10.36, p. 220 in [24] are fulfilled and we obtain that $\{|u_n|^q\}$ converges weakly to $|u|^q$ in $L^{2^\star_\alpha/q}(\mathbb{R}^N)$. On the other hand the application $T : L^{2^\star_\alpha/q}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\langle T, v\rangle = \int_{\mathbb{R}^N} r(x) v \, dx$$

is linear and continuous provided $r \in L^s(\mathbb{R}^N)$ and $1/s + q/2^\star_\alpha = 1$. Thus we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} r(x)|u_n|^q \, dx = \int_{\mathbb{R}^N} r(x)|u|^q \, dx.$$

We conclude that u is not trivial and the proof of Theorem 2 is now complete. ■

4 Proof of Theorem 3

The proof of Theorem 3 uses some ideas from Szulkin-Willem [21].

In order to prove that g_1 satisfies property **(G)** provided that it verifies property **(G1)** we consider the minimization problem

(P) minimize $\int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^2\, dx;\quad u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N),\quad \int_{\mathbb{R}^N} g_1(x)u^2\, dx = 1.$

We show that if g_1 satisfies property **(G1)** then there exists $e_1 \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)\setminus\{0\}$, $e_1 \geq 0$ solution for problem **(P)**.

We let $\{u_n\} \subset \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ be a minimizing sequence for **(P)**, i.e.

$$\int_{\mathbb{R}^N} |x|^\alpha |\nabla u_n|^2\, dx \to c := \inf(\mathbf{P})$$

and $\int_{\mathbb{R}^N} g_1(x)u_n^2\, dx = 1$, for all n. Then it is clear that $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ and thus there exists $u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ such that u_n converges weakly to u in $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$. It follows that

$$\int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^2\, dx \leq \liminf_{n\to n} \int_{\mathbb{R}^N} |x|^\alpha |\nabla u_n|^2\, dx$$

and $\int_{\mathbb{R}^N} g_1(x)u^2\, dx = \lim_{n\to\infty} g_1(x)u_n^2\, dx = 1$. We conclude that u is a solution of **(P)**. Moreover, since $|u|$ is also a solution we may assume $u \geq 0$. Furthermore, we obtain that g_1 satisfies property **(G)**.

For the second part of Theorem 3 we consider

$$0 \leq |g(x)| \leq g_1(x) \qquad \text{a.e. } x \in \mathbb{R}^N.$$

We let $\{u_n\}$ be a sequence in $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ which converges weakly to $u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$. Then we have

$$\int_{\mathbb{R}^N} g_1(x)u_n^2\, dx \to \int_{\mathbb{R}^N} g_1(x)u^2\, dx$$

or $g_1(x)u_n^2$ converges to $g_1(x)u^2$ in $L^1(\mathbb{R}^N)$. By Theorem IV.9, p. 58 in [4] we deduce that passing eventually to a sub-sequence $g_1(x)u_n^2 \to g_1(x)u^2$ a.e. $x \in \mathbb{R}^N$ and there exists $h_0 \in L^1(\mathbb{R}^N)$ such that $g_1(x)u_n^2 \leq h_0(x)$ a.e. $x \in \mathbb{R}^N$ and for all n. It follows that

$$|g(x)|u_n^2 \to |g(x)|u^2 \qquad \text{a.e. } x \in \mathbb{R}^N$$

and

$$|g(x)|u_n^2 \leq h_0(x) \qquad \text{a.e. } x \in \mathbb{R}^N$$

for all n. Using Lebesgues' Theorem (Theorem IV.4, p. 54 in [4]) we deduce that

$$\int_{\mathbb{R}^N} |g(x)|u_n^2\, dx \to \int_{\mathbb{R}^N} |g(x)|u^2\, dx$$

as $n \to \infty$, i.e. $|g|$ has the property **(G)**. ■

5 Proof of Theorem 4

The proof of Theorem 4 follows the lines of the final part in the proof of Theorem 2. We present it in detail for reader's convenience.

Let $\{u_n\} \subset \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ be a sequence which converges weakly to u in $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$. Then $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$ and by Lemma 4 we deduce that $\{u_n\}$ is bounded in $L^{2^\star_\alpha}(\mathbb{R}^N)$. Then we can apply Theorem 10.36, p. 220 in [24] to obtain that u_n^2 converges weakly to u^2 in $L^{2^\star_\alpha/2}(\mathbb{R}^N)$. Defining the operator $E : L^{2^\star_\alpha/2}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\langle E, w\rangle = \int_{\mathbb{R}^N} g_1(x) w \, dx$$

we remark that it is linear and continuous provided that $g_1 \in L^{N/2-\alpha}(\mathbb{R}^N)$. We get

$$\int_{\mathbb{R}^N} g_1(x) u_n^2 \, dx \to \int_{\mathbb{R}^N} g_1(x) u^2 \, dx$$

as $n \to \infty$ and the conclusion of Theorem 4 follows. ■

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