Nonlinear Neumann problems with asymmetric nonsmooth potential

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Abstract

In this paper we study a scalar Neumann problem driven by the ordinary p-Lapacian and a nonsmooth potential. The nonlinearity exhibits an asymmetric behavior. Namely growth restriction is imposed in one direction only (either the positive direction or the negative direction). Using a variational approach based on the nonsmooth critical point theory for locally Lipschitz function, we prove the existence of a solution.

1 Introduction

In this paper, we study the following nonlinear Neumann problem with nonsmooth potential:

$$\begin{cases} -(|x'(t)|^{p-2}x'(t))' \in \partial j(t,x(t)) + g(t), \quad a.e. \text{ on } T=[0,b], \\ x'(0) = x'(b) = 0, \quad 1 < p < \infty, g \in L^q(T), \dfrac{1}{p} + \dfrac{1}{q} = 1. \end{cases} \tag{1.1}$$

The Neumann problem for ordinary differential equations, was studied by Dong [6] de Figueiredo-Ruf [7], Gupta [9], Iannacci-Nkashama [11], Mawhin-Ward-Willem [16], Villegas [17](semilinear problems, i.e., p=2) and Dang-Oppenheimer [3], Guo [8], Kourogenis-Papageorgiou [12] (nonlinear problems involving p-Laplacian type

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differential operators). With the exception of the works of de Figueiredo-Ruf [7], Villegas [17] and Dong [6], all the other papers proved existence results by imposing restrictions on the nonlinearity in both directions (i.e. as $x \to -\infty$ and as $x \to -\infty$). In the aforementioned works of de Figueiredo-Ruf, Villegas and Dong, the nonlinearity is restricted only as $x \to -\infty$ (we should point out that an analogous condition for Dirichlet problems, can be found in the work of Ma-Sanchez [14]). The goal of this paper is to extend the works of de Figueiredo-Ruf, Villegas and Dong, to equations driven by the ordinary p-Laplacian and having a nonsmooth potential. Our approach is variational based on the nonsmooth critical point theory of Chang [1] and Kourogenis-Papageorgiou [13]. From the three relevant semilinear smooth works of de Figueiredo-Ruf, Villegas and Dong, the first two are also based on variational methods, while the work of Dong is based on degree theoretic arguments.

In the next section, for the convenience of the reader we recall the basic definitions and facts from the nonsmooth critical point theory, which is based on the subdifferential for locally lipschitz functions, due to Clarke.

2 Mathematical Background

We start by recalling the basics from the subdifferential theory for locally Lipschitz functions. Our main references are the books of Clarke [2] and Denkowski-Migorski-Papageorgiou [4].

Let X be a Banach space and X^* its topological dual. By $< \cdot, \cdot >$ we denote the duality brackets for the pair (X, X^*). A function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$, we can find a neighborhood U of x and a constant $k_U > 0$ such that $|\varphi(z) - \varphi(y)| \leq k_U ||z - y||$ for all $z, y \in U$. From convex analysis we know that if $\phi : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is convex, lower semicontinuous and not identically $+\infty$, then ϕ is locally Lipschitz in the interior of its effective domain $\text{dom}\phi = \{x \in X : \phi(x) < +\infty\}$. Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, we define the generalized directional derivative at x in the direction $h \in X$, by

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

We can easily check that $\varphi^0(x; \cdot)$ is sublinear, continuous and so from the Hahn-Banach theorem it follows that $\varphi^0(x; \cdot)$ is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x) \subseteq X^*$, defined by

$$\partial\varphi(x) = \{x^* \in X^* :< x^*, h > \leq \varphi^0(x; h) \text{ for all } h \in X\}.$$

The set $\partial\varphi(x)$ is known as the generalized (or Clarke) subdifferential of φ at $x \in X$. If φ is also convex, then the generalized subdifferential, coincides with the subdifferential $\partial_c\varphi(x)$ in the sense of convex analysis, defined by

$$\partial_c\varphi(x) = \{x^* \in X^* :< x^*, h > \leq \varphi(x + h) - \varphi(x) \text{ for all } h \in X\}.$$

If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$. Moreover, if $\varphi, \psi : X \to$ are locally Lipschitz functions and $\lambda \in$, then

$$\partial(\varphi + \psi)(x) \subseteq \partial\varphi + \partial\psi \quad \text{and} \quad \partial(\lambda\varphi)(x) = \lambda\partial\varphi(x) \text{ for all } x \in X$$

Next let us say a few things about the critical point theory for locally Lipschitz not necessary smooth functions. So let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz function. A point $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. If $x \in X$ is a critical point of φ, then $c = \varphi(x)$ is the corresponding critical value. It is easy to check that x is a local extremum of φ (i.e. a local minimum or a local maximum), then x is a critical point of φ (i.e. $0 \in \partial\varphi(x)$).

As is the case with the smooth theory (i.e. $\varphi \in C^1(X)$), in the nonsmooth critical point theory, a basic tool is a compactness type condition, known as the "nonsmooth Palais-Smale condition" (nonsmooth PS-condition for short), which says the following:

A locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the "nonsmooth PS-condition" if any sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\sup_{n\geq 1} |\varphi(x_n)| < +\infty$ and $m(x_n) = \inf\{||x^*|| : x^* \in \partial\varphi(x_n)\} \to 0$ as $n \to \infty$, has a strongly convergent subsequence.

If $\varphi \in C^1(X)$, then as we already said $\partial\varphi(x) = \{\varphi'(x)\}$ and so we see that the nonsmooth PS-condition coincides with the smooth one (see for example Denkowski-Migorski-Papageorgiou [5]).

In our analysis of problem (1.1) we shall need the following basic geometric notion:

Definition 2.1. *Let Z be a Hausdorff topological space and E_1, D two nonempyt subsets of Z. We say that E_1 and D "link in Z" if*

(a) $E_1 \cap D = \emptyset$;

(b) there exists a closed set $E \supseteq E_1$ such that for any $\theta \in C(E, Z)$ with $\theta|_{E_1} = id_{E_1}$, we have $\theta(E) \cap D \neq \emptyset$.

Using this geometric notion, Kourogenis-Papageorgiou [13] proved the following abstract minimax principle. In fact the result of Kourogenis-Papageorgiou [13] is more general, but the version of the result which follows, suffices for our needs here.

Theorem 2.2. *If X is a reflexive Banach space, E_1 and D are two nonempty subsets of X with D closed, E_1 and D link in $X, \varphi : X \to \mathbb{R}$ is a locally Lipschitz function which satisfies the nonsmooth PS-condition, $\sup_{E_1} \varphi < \inf_D \varphi$ and $c = \inf_{\theta\in\Gamma} \sup_{v\in E} \varphi(\theta(v))$ with $\Gamma = \{\theta \in C(E, X) : \theta|_{E_1} = id_{E_1}\}$ and $E \supseteq E_1$ is as in the definition of linking sets, then $c \geq \inf_D \varphi$ and c is a critical value of φ, i.e., we can find a critical point $x_0 \in X$ such that $\varphi(x_0) = c$. Moreover, if $c = \inf_D \varphi$, then $x_0 \in D$.*

Remark 2.3. *With appropriate choices of linking sets E_1 and D, we can have nonsmooth versions of the Mountain Pass Theorem, the Saddle Point Theorem and the Generalized Mountain Pass Theorem. For details we refer to Kouragenis-Papageorgiou [13].*

3 Some Auxiliary Results

In this section, by solving an optimization problem, we introduce a quantity which will be used in our hypotheses on the nonsmooth potential $j(t,x)$. As we show this quantity is in fact the first eigenvalue of the negative scalar p-Laplacian with mixed boundary conditions.

So we introduce the set

$$C = \{x \in W^{1,p}(0,b) : \min_T x = 0\}$$

and we consider the following minimization problem

$$\gamma = \inf\{\frac{||x'||_p^p}{||x||_p^p} : x \in C, x \neq 0\}. \tag{3.1}$$

PROPOSITION 3.1. *There exists $x \in C \setminus \{0\}$ such that $\gamma = \frac{||x'||_p^p}{||x||_p^p}$ and $x(t) > 0$ for all $t \in (0,b)$.*

Proof. Let $\{x_n\}_{n\geq 1} \subseteq C$ with $||x_n||_p = 1$ be a minimizing sequence, i.e., $||x_n'||_p^p \downarrow \gamma$ as $n \to \infty$. Evidently, $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(T)$ is bounded and so by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(0,b)$ and $x_n \to x$ in $C(T)$ (recall that $W^{1,p}(0,b)$ is embedded compactly in $C(T)$). So we have $||x||_p = 1$, hence $x \neq 0$. Also if $t_n \in T$ is such that $\min_T x_n = x_n(t_n), n \geq 1$, then we may assume that $t_n \to t$ and so $x_n(t_n) \to x(t) = 0$, which means that $\min_T x \leq 0$. If the inequality is strict, we can find $s \in T$ such that $x(s) = \min_T x < 0$ and then for $n \geq 1$ large enough, we shall have $x_n(s) < 0$, which contradicts the fact that $x_n \in C$ for all $n \geq 1$. Therefore $\min_T x = 0$ and so $x \in C \setminus \{0\}$. Moreover, from the weak lower semicontinuity of the norm functional in a Banach space, we have $||x'||_p^p \leq \gamma$, hence $||x'||_p^p = \gamma$ (since $x \in C \setminus \{0\}$).

Next suppose that for some $t \in (0,b)$, we have $x(t) = 0$. We introduce the following two sets:

$$C_0 = \{y \in W^{1,p}(0,t) : \min_{[0,t]} y = 0\} \quad \text{and} \quad C_1 = \{y \in W^{1,p}(t,b) : \min_{[t,b]} y = 0\}.$$

To these two sets we correspond the following two quantities:

$$\gamma_0 = \inf\{\frac{\int_0^t |y'(s)|^p ds}{\int_0^t |y(s)|^p ds} : y \in C_0, y \neq 0\} \text{ and } \gamma_1 = \inf\{\frac{\int_t^b |y'(s)|^p ds}{\int_t^b |y(s)|^p ds} : y \in C_1, y \neq 0\}.$$

From the first part of the proof, we know that both infima are attained. Remark that the maps $x \to \hat{x}(s) = x(\frac{s}{t}b)$ and $x \to \bar{x}(s) = x(\frac{s-t}{b-t}b)$ are bijections of C onto C_0 and C_1 respectively. Therefore it follows that

$$\gamma_0 = (\frac{b}{t})^p \gamma \quad \text{and} \quad \gamma_1 = (\frac{b}{b-t})^p \gamma.$$

We have

$$\begin{aligned}\gamma = \frac{||x'||_p^p}{||x||_p^p} &= \frac{\int_0^t |x'(s)|^p ds + \int_t^b |x'(s)|^p ds}{||x||_p^p} \\ &\geq \frac{\gamma_0 \int_0^t |x(s)|^p ds + \gamma_1 \int_t^b |x(s)|^p ds}{||x||_p^p} \\ &\geq \frac{\min\{\gamma_0, \gamma_1\}\|x\|_p^p}{\|x\|_p^p} \\ &= \min\{\gamma_0, \gamma_1\} > \gamma,\end{aligned}$$

a contradiction. So we must have that $x(t) > 0$ for all $t \in (0, b)$. ∎

In the next Proposition, we identify γ as the first eigenvalue of the negative scalar p-Laplacian with mixed boundary conditions.

PROPOSITION 3.2. *γ is the first eigenvalue of $-(|x'(\cdot)|^{p-2}x'(\cdot))'$ with boundary conditions $x(0) = x'(b) = 0$; so*

$$\gamma = \frac{p-1}{b^p}\left(\int_0^1 \frac{ds}{(1-s^p)^{\frac{1}{p}}}\right)^p.$$

Proof. Let $x \in C \setminus \{0\}$ be the minimizer of (3.1) obtained in Proposition 3.1. Then at least $x(0) = 0$ or $x(b) = 0$. We may assume without any loss of generality that $x(0) = 0$ (otherwise consider the function $\overline{x}(t) = x(b-t)$). Then let $V = \{y \in W^{1,p}(0,b) : y(0) = 0\}$ and $\lambda_1 = \inf\left[\frac{\|y'\|_p^p}{\|y\|_p^p} : y \in V, y \neq 0\right]$. As before we can check that this minimization problem has a solution $u \in V$ and by virtue of the Lagrange multiplier rule we can check that $u \in V$ is a solution of the following nonlinear eigenvalue problem:

$$\left\{\begin{array}{l} -(|y'(t)|^{p-2}y'(t))' = \lambda_1|y(t)|^{p-2}y(t) \ a.e \ \text{ on } T \\ y(0) = y'(b) = 0 \end{array}\right\}$$

So we have that $\lambda_1 = \frac{p-1}{b^p}(\int_0^1 \frac{ds}{(1-s^p)^{1/p}})^p$ and $u(t) \neq 0$ for all $t \in (0, b)$ (see Mawhin [15]). We have

$$\begin{aligned}\gamma &= \min\{\frac{||y'||_p^p}{||y||_p^p} : y \in C, y \neq 0\} \\ &= \min\{\frac{||y'||_p^p}{||y||_p^p} : y \in C \cap V, y \neq 0\} \text{(see Proposition 3.1)} \\ &= \min\{\frac{||y'||_p^p}{||y||_p^p} : y \in V, y \neq 0\} = \lambda_1.\end{aligned}$$

∎

So $\gamma > 0$ has a natural intrinsic characterization in terms of the p-Lpalacian differential operator of the problem. We shall use γ in the hypotheses for the nonsmooth potential $j(t, x)$.

4 The Neumann Problem

In this section, using a variational approach based on the nonsmooth critical point theory, we prove an existence theorem for problem (1.1) by imposing unilateral asymptotic conditions on the nonsmooth potential $j(t,x)$. More precisely, our hypotheses on $j(t,x)$ are the following:

$H(j)$: $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(\cdot,0) \in L^1(T)$ and

(i) for all $x \in, t \to j(t,x)$ is measurable;

(ii) for almost all $t \in T, x \to j(t,x)$ is locally Lipschitz;

(iii) for every $r > 0$, there exists $a_r \in L^q(T)_+ (\frac{1}{p} + \frac{1}{q} = 1)$ such that for almost all $t \in T$, all $|x| \leq r$ and all $u \in \partial j(t,x)$, we have $|u| \leq a_r(t)$.

(iv) $\limsup_{x \to +\infty} \frac{u}{x^{p-1}} \leq h(t)$ uniformly for almost all $t \in T$ and all $u \in \partial j(t,x)$ with $h \in L^1(T)_+$ such that $h(t) \leq \gamma$ a.e. on T and this inequality is strict on a set of positive Lebesgue measure;

(v) $\limsup_{x \to -\infty} \{\max_{u \in \partial j(t,x)} u\} < -g(t) < \liminf_{x \to +\infty} \{\min_{u \in \partial j(t,x)} u\}$ uniformly for almost all $t \in T$.

Remark 4.1. *Hypothesis $H(j)$ (iv) imposes a unilateral growth restriction on $j(t,\cdot)$ (only in the positive direction). We have no growth restriction in the negative direction. So the nonsmooth potential in general exhibits an asymmetric behavior.*

Let $\varphi : W^{1,p}(0,b) \to \mathbb{R}$ be the functional defined

$$\varphi(x) = \frac{1}{p}||x'||_p^p - \int_0^b j(t,x(t))dt - \int_0^b g(t)x(t)dt, \quad x \in W^{1,p}(0,b).$$

We know that φ is locally Lipschitz (see Clarke [2], p.80 or Denkowski-Migorski-Papageorgiou [4], p.616).

Proposition 4.2. *If hypotheses $H(j)$ hold, then φ satisfies the nonsmooth PS-condition.*

Proof. Let $\{x_n\}_{n \geq 1} \subseteq W^{1,p}(0,b)$ be a sequence such that

$$|\varphi(x_n)| \leq M, \quad \text{for some } M_1 > 0 \text{ and all } n \geq 1 \quad \text{and } m(x_n) \to \infty \text{ as } n \to \infty.$$

Because $\partial\varphi(x_n) \subseteq W^{1,p}(0,b)^*$ is weakly compact and the norm functional in a Banach space is weakly lower semicontinuous, by the Weierstrass theorem, we can find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = ||x_n^*||, n \geq 1$. Let $A : W^{1,p}(0,b) \to W^{1,p}(0,b)^*$ be the nonlinear operator defined by

$$< A(x), y >= \int_0^b |x'(t)|^{p-2}x'(t)y'(t)dt \quad \text{for all } x, y \in W^{1,p}(0,b).$$

(hereafter by $< \cdot, \cdot >$ we denote the duality brackets for the pair $(W^{1,p}(0,b)^*, W^{1,p}(0,b))$). It is easy to see that A is monotone, demicontinuous, thus it is maximal monotone (see Hu-Papageorgiou [10], p.309). We know that

$$x_n^* = A(x_n) - u_n - g, \quad n \geq 1$$

with $u_n \in L^q(T), u_n(t) \in \partial j(t, x_n(t))$ a.e. on T (see Clarke [2], p.83 or Denkowski-Migorski-Papageorgiou [4], p.617).

We claim that the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(0,b)$ is bounded. Suppose that this is not true. Then by passing to a suitable subsequence if necessary, we may assume that $||x_n|| \to \infty$. Set $y_n = \frac{x_n}{||x_n||}, n \geq 1$. We may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(0,b) \quad \text{and } y_n \to y \text{ in } C(T) \text{ as } n \to \infty.$$

From the choice of the sequence $\{x_n\}_{n\geq 1}$, we have that

$$| < x_n^*, z > | \leq \varepsilon_n ||z|| \quad \text{for all } z \in W^{1,p}(0,b) \text{ with } \varepsilon_n \downarrow 0.$$

Choose $z \equiv 1 \in W^{1,p}(0,b)$. We obtain

$$|\int_0^b u_n(t)dt| \leq \varepsilon_n' + |\int_0^b g(t)dt| \text{ with } \varepsilon_n' \downarrow 0.$$

Dividing with $||x_n||^{p-1}$, we get

$$|\int_0^b \frac{u_n(t)}{||x_n||^{p-1}}dt| \leq \frac{\varepsilon_n'}{||x_n||^{p-1}} + \frac{||g||_1}{||x_n||^{p-1}} \to 0 \quad \text{as } n \to \infty. \tag{4.1}$$

By virtue of hypothesis $H(j)$ (iv), we know that given $\varepsilon > 0$, we can find $M_2 = M_2(\varepsilon) > 0$ such that for almost all $t \in T$, all $x \geq M_2 > 0$ and all $u \in \partial j(t,x)$, we have

$$u \leq (h(t) + \varepsilon)x^{p-1}. \tag{4.2}$$

Moreover, from hypothesis $H(j)$ (v), we see that we can find $M_3 > 0$ such that for almost all $t \in T$, all $x \geq M_3$ and all $u \in \partial j(t,x)$, we have

$$0 \leq u + g(t). \tag{4.3}$$

Therefore finally, we can say that for almost all $t \in T$, all $x \geq M_4 = \max\{M_2, M_3\}$ and all $u \in \partial j(t,x)$, we have

$$\begin{aligned} & |u + g(t)| = u + g(t) \leq (h(t) + \varepsilon)x^{p-1} + g(t) \quad \text{(see (4.2) and (4.3))}, \\ \Longrightarrow \quad & |u| \leq (h(t) + \varepsilon)|x|^{p-1} + 2|g(t)|. \end{aligned} \tag{4.4}$$

On the other hand, from hypotheses $H(j)$ (iii) and (v), we can find $M_5 \geq M_4$ such that for almost all $t \in T$, all $x < M_5$ and all $u \in \partial j(t,x)$, we have

$$|u| \leq a(t) - u \quad \text{for some } a \in L^q(T)_+ \tag{4.5}$$

we can take $a(t) = a_{M_5}(t) + |g(t)|$. Then we have

$$\int_0^b \frac{|u_n(t)|}{||x_n||^{p-1}}dt = \int_{\{x_n \geq M_5\}} \frac{|u_n(t)|}{||x_n||^{p-1}}dt + \int_{\{x_n < M_5\}} \frac{|u_n(t)|}{||x_n||^{p-1}}dt. \tag{4.6}$$

Using (4.4), we have

$$\begin{aligned} \int_{\{x_n \geq M_5\}} \frac{|u_n(t)|}{||x_n||^{p-1}}dt &\leq \int_{\{x_n \geq M_5\}} (h(t) + \varepsilon)|y_n(t)|^{p-1}dt + \int_{\{x_n \geq M_5\}} \frac{2|g(t)|}{||x_n||^{p-1}}dt \\ &\leq \int_0^b (h(t) + \varepsilon)|y_n^+(t)|^{p-1}dt + \int_0^b \frac{2|g(t)|}{||x_n||^{p-1}}dt \\ &\qquad \text{(here } y_n^+ = \max\{y_n, 0\}). \end{aligned} \tag{4.7}$$

Also using (4.5), we have

$$\begin{aligned}\int_{\{x_n<M_5\}} \frac{|u_n(t)|}{||x_n||^{p-1}}dt &\leq \frac{||a||_1}{||x_n||^{p-1}} - \int_{\{x_n<M_5\}} \frac{u_n(t)}{||x_n||^{p-1}}dt \\ &= \frac{||a||_1}{||x_n||^{p-1}} - \int_0^b \frac{u_n(t)}{||x_n||^{p-1}}dt + \int_{\{x_n\geq M_5\}} \frac{u_n(t)}{||x_n||^{p-1}}dt \\ &\leq \frac{||a||_1}{||x_n||^{p-1}} - \int_0^b \frac{u_n(t)}{||x_n||^{p-1}}dt \\ &\quad + \int_0^b (h(t)+\varepsilon)|y_n^+(t)|^{p-1}dt + \varepsilon_n'' \text{ with } \varepsilon_n'' \downarrow 0 \\ &\quad \text{(see 4.2).} \end{aligned} \tag{4.8}$$

Returning to (4.6), using (4.7) and (4.8) and recalling also (4.1), we obtain that

$$\begin{aligned}&\int_0^b \frac{|u_n(t)|}{||x_n||^{p-1}}dt \leq M_6 \quad (\text{for some } \ M_6>0 \ \text{ and } \ \text{all } n\geq 1), \\ \Longrightarrow \ &\Big\{\frac{u_n(\cdot)}{||x_n||^{p-1}}\Big\}_{n\geq 1} \subseteq L^1(T) \quad \text{is bounded.}\end{aligned} \tag{4.9}$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(0,b)$, we have

$$\begin{aligned}&| < x_n^*, y_n - y > | \leq \varepsilon_n ||y_n - y|| \text{ with } \varepsilon_n \downarrow 0, \\ \Longrightarrow &| < \frac{x_n^*}{||x_n||^{p-1}}, y_n - y > | \leq \frac{\varepsilon_n}{||x_n||^{p-1}}||y_n - y||, \\ \Longrightarrow &| < A(y_n), y_n - y > - \int_0^b \frac{u_n(t)}{||x_n||^{p-1}}(y_n - y)(t)dt - \int_0^b \frac{g(t)}{||x_n||^{p-1}}(y_n - y)(t)dt| \\ &\qquad \leq \frac{\varepsilon_n}{||x_n||^{p-1}}||y_n - y||. \end{aligned} \tag{4.10}$$

Because of (4.9) and since $y_n \to y$ in $C(T)$, it follows that

$$\int_0^b \frac{u_n(t)}{||x_n||^{p-1}}(y_n - y)(t)dt \to 0 \quad \text{as } n \to \infty.$$

Clearly we also have that

$$\int_0^b \frac{g(t)}{||x_n||^{p-1}}(y_n - y)(t)dt \to 0 \quad \text{as } n \to \infty.$$

Therefore from (4.10), we obtain

$$< A(y_n), y_n - y > \to 0 \quad \text{as } n \to \infty.$$

But A being maximal monotone, it is generalized pseudomonotone (see Hu-Papageorgiou [10], p.365) and so we have

$$\begin{aligned}&< A(y_n), y_n > \to < A(y_n), y >, \\ &\Longrightarrow ||y_n'||_p \to ||y'||_p \quad \text{as } n \to \infty.\end{aligned}$$

Since $y_n' \overset{w}{\to} y'$ in $L^p(T)$ and the latter space is uniformly convex, from the Kadec-Klee property we have that $y_n' \to y'$ in $L^p(T)$. Therefore we infer that $y_n \to y$ in $W^{1,p}(T)$. Since $||y_n|| = 1$ for all $n \geq 1$, it follows that $||y|| = 1$ and so $y \neq 0$.

Recall that for all $z \in W^{1,p}(0,b)$, we have

$$| < x_n^*, z > | \leq \varepsilon_n ||z|| \quad \text{with } \varepsilon_n \to 0.$$

Taking as a test function $z \equiv 1 \in W^{1,p}(0,b)$, we obtain

$$\begin{aligned} & | \int_0^b u_n(t)dt + \int_0^b g(t)dt| \leq \varepsilon_n, \\ \Longrightarrow \quad & | \int_0^b u_n(t)dt \to - \int_0^b g(t)dt \quad \text{as } n \to \infty. \end{aligned} \tag{4.11}$$

We shall use (4.11) to establish that y has roots on T. We proceed by contradiction. So suppose that y has no roots on T. We may assume that $y(t) > 0$ for all $t \in T$ (the analysis is similar if we assume that $y(t) < 0$ for all $t \in T$). Then $x_n(t) \to +\infty$ for all $t \in T$ as $n \to \infty$. We claim that this convergence is uniform in $t \in T$. To this end choose $\delta > 0$ such that $0 < \delta < \min_T y$. Because $y_n \to y$ in $C(T)$, we can find $n_0 = n_0(\delta) \geq 1$ such that

$$\begin{aligned} & |y_n(t) - y(t)| < \delta \quad \text{for all} \quad n \geq n_0 \quad \text{and all } t \in T, \\ \Longrightarrow \quad & y_n(t) \geq y(t) - \delta = \delta_1 > 0 \quad \text{for all } n \geq n_0 \text{ and all } t \in T. \end{aligned}$$

Since $\|x_n\| \to \infty$, given $\beta > 0$, we can find $n_1 = n_1(\beta) \geq 1$ such that

$$\|x_n\| \geq \beta > 0 \quad \text{for all } n \geq n_1.$$

Then for all $n \geq n_2 = \max\{n_0, n_1\}$ and all $t \in T$, we have

$$\begin{aligned} & \frac{x_n(t)}{\beta} \geq \frac{x_n(t)}{\|x_n\|} = y_n(t) \geq \delta_1 > 0 \\ & \qquad \text{(recall that } x_n(t) > 0 \text{ for all } n \geq n_0 \text{ and all } t \in T), \\ \Longrightarrow \quad & x_n(t) \geq \beta\delta_1 > 0. \end{aligned}$$

Since $\beta > 0$ is arbitrary, we conclude that $\min_T x_n \to +\infty$ as $n \to \infty$. Using this fact in conjunction with hypothesis H(j)(v) and Fatou's Lemma, we infer that

$$\int_0^b g(t)dt < \liminf_{n \to \infty} \int_0^b u_n(t)dt. \tag{4.12}$$

Comparing (4.11) and (4.12), we reach a contradiction. This proves that y has roots on T.

Next let $y^+ = \max\{y, 0\} \in W^{1,p}(0,b)$ (see Denkowski-Migorski-Papageorgiou [4]). We consider two cases:

Case I: $y^+ \equiv 0$

Then $\max_T y = 0$, since y has roots on T. Also from (4.7) and (4.8) and recalling that (4.1) holds and that $y_n^+ \to y^+$ in $C(T)$, we obtain

$$\frac{u_n}{\|x_n\|^{p-1}} \to 0 \text{ in } L^1(T) \text{ as } n \to \infty. \tag{4.13}$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(0,b)$, we have

$$\begin{aligned} & \left| < A(x_n), y > - \int_0^b u_n(t)y(t)dt - \int_0^b g(t)y(t)dt \right| \leq \varepsilon_n \|y\| \quad \text{with} \quad \varepsilon_n \downarrow 0, \\ \Longrightarrow \quad & \left| < A(y_n), y > - \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}} y(t)dt - \int_0^b \frac{g(t)}{\|x_n\|^{p-1}} y(t)dt \right| \\ & \leq \frac{\varepsilon_n}{\|x_n\|^{p-1}} \|y\| \end{aligned} \tag{4.14}$$

Remark that $A(y_n) \overset{w}{\to} A(y)$ in $W^{1,p}(0,b)^*$(since $y_n \to y$ in $W^{1,p}(0,b)$ and A is demicontinuous), $\int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}} y(t)dt \to 0$ (see (4.13)) and $\int_0^b \frac{g(t)}{\|x_n\|^{p-1}} y(t)dt \to 0$. So by passing to the limit as $n \to \infty$ in (4.14), we obtain

$$\begin{aligned} & < A(y), y >= \|y'\|_p^p = 0, \\ \Longrightarrow \quad & y \equiv \xi \in \mathbb{R} \end{aligned}$$

Because $\max_T y = 0$, we have that $y = \xi = 0$, a contradiction to the fact that $\|y\| = 1$.

Case II: $y^+ \neq 0$.

Because y has roots on T, we must have $\min_T y^+ = 0$ and so $y^+ \in C \setminus \{0\}$. Recall that

$$| < x_n^*, z > | \leq \varepsilon_n \|z\| \quad \text{for all } z \in W^{1,p}(0,b) \text{ and with } \varepsilon_n \downarrow 0.$$

Use as a test function $z = y_n^+ \in W^{1,p}(0,b)$ and divide with $\|x_n\|^{p-1}$. We obtain

$$\begin{aligned} & \left| \|(y_n^+)'\|_p^p - \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}} y_n^+(t)dt - \int_0^b \frac{g(t)}{\|x_n\|^{p-1}} y_n^+(t)dt \right| \leq \varepsilon_n \|y_n^+\| \\ \Longrightarrow \quad & \|(y_n^+)'\|_p^p \leq \varepsilon_n \|y_n^+\| + \int_0^b \frac{u_n(t)}{\|x_n\|^{p-1}} \chi_{\{y_n>0\}}(t) y_n(t)dt \\ & \qquad + \int_0^b \frac{g(t)}{\|x_n\|^{p-1}} y_n^+(t)dt. \end{aligned} \tag{4.15}$$

For every $n \geq 1$, we have

$$\begin{aligned} & u_n(t) \in \partial j(t, x_n(t)) \ a.e \text{ on } T, \\ \Longrightarrow \quad & \chi_{\{y_n>0\}}(t) u_n(t) \in \chi_{\{y_n>0\}}(t) \partial j(t, x_n(t)) \ a.e \text{ on } T. \end{aligned}$$

Because $x_n^+(t) \to +\infty$ uniformly in $t \in T$, by virtue of hypothesis H(j)(iv)(see also (4.2)), given $\varepsilon > 0$, we can find $n_0 = n_0(\varepsilon) \geq 1$ such that for all $n \geq n_0$ and almost all $t \in T$, we have

$$\chi_{\{y_n>0\}}(t) \frac{u_n(t)}{\|x_n\|^{p-1}} \leq \chi_{\{y_n>0\}}(t)(h(t)+\varepsilon)|y_n(t)|^{p-1} = (h(t)+\varepsilon)|y_n^+(t)|^{p-1}.$$

Using this in (4.15), we obtain

$$\begin{aligned} \|(y_n^+)'\|_p^p \leq \varepsilon_n \|y_n^+\| & + \int_0^b (h(t)+\varepsilon)|y_n^+(t)|^{p-1}dt \\ & + \int_0^b \frac{g(t)}{\|x_n\|^{p-1}} y_n^+(t)dt \quad \text{for all } n \geq n_0. \end{aligned} \tag{4.16}$$

Note that $\int_0^b \frac{g(t)}{\|x_n\|^{p-1}} y_n^+(t)dt \to 0$ as $n \to \infty$. So if we pass to the limit as $n \to \infty$ in (4.16) and eventually let $\varepsilon \downarrow 0$, we obtain

$$\begin{aligned} & \|(y_n^+)'\|_p^p \leq \int_0^b h(t)|y_n^+(t)|^p dt \leq \gamma \|y^+(t)\|_p^p \quad \text{(see hypothesis H(j)(iv))}, \\ \Longrightarrow \quad & \|(y^+)'\|_p^p = \gamma \|y^+(t)\|_p^p \ (\text{ since } y^+ \in C \setminus \{0\}). \end{aligned}$$

Then from Proposition 3.1 we know that $y^+(t) > 0$ for all $t \in (0, b)$. So returning to (4.17) we have

$$\|(y^+)'\|_p^p < \gamma \|y^+(t)\|_p^p,$$

a contradiction to the fact that $y^+ \in C \setminus \{0\}$.

So the analysis of Cases I and II implies that $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(0, b)$ is bounded. Hence by passing to a suitable subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \ \text{ in } \ W^{1,p}(0, b) \ \text{ and } \ x_n \to x \ \text{ in } \ C(T).$$

Recall that $\int_0^b u_n(t)(x_n - x)(t)dt \to 0$(see hypothesis H(j)(iii)) and $\int_0^b g(t)(x_n - x)(t)dt \to 0$. So it follows that

$$\lim < A(x_n), x_n - x >= 0.$$

Because A is maximal monotone, it is generalized pseudomonotone and so

$$\begin{aligned} & < A(x_n), x_n > \to < A(x), x > \\ \Longrightarrow \quad & \|x_n'\|_p \ \to \ \|x'\|_p. \end{aligned}$$

As before via the Kadec-Klee property of $L^p(T)$, we conclude that $x_n \to x$ in $W^{1,p}(0, b)$. ∎

Next we want to show that $\varphi\Big|_C$ is coercive, thus bounded below. This requires the following Lemma

LEMMA 4.3. *There exists $\beta_0 > 0$ such that $\|x'\|_p^p - \int_0^b h(t)x(t)^p dt \geq \beta_0 \|x'\|_p^p$ for all $x \in C$.*

Proof. Let $\psi(x) \equiv \|x'\|_p^p - \int_0^b h(t)x(t)^p dt$, $x \in C$. From hypothesis $H(j)(iv)$ and Proposition 1, we see that $\psi \geq 0$. Suppose that the claim of the Lemma is not true. Then exploiting the (p-1)-homogeneity of of ψ, we can find $\{x_n\}_{n\geq 1} \subseteq C$ with $\|x_n'\|_p = 1$ such that $\psi(x_n) \downarrow 0$. From Proposition 1 we know that $\|x_n\|_p \leq \frac{1}{\gamma}\|x_n'\|_p$ and so $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(0, b)$ is bounded. Therefore we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(0, b)$ and $x_n \to x$ in $C(T)$, with $x \in C$. We have

$$\begin{aligned} & \lim \psi(x_n) = 0 \geq \|x'\|_p^p - \int_0^b h(t)x(t)^p dt, \\ \Longrightarrow \quad & \|x'\|_p^p \leq \int_0^b h(t)x(t)^p dt \leq \gamma \|x\|_p^p \ \text{(see hypothesis H(j)(iv))}, \qquad (4.17) \\ \Longrightarrow \quad & \|x'\|_p^p = \gamma \|x\|_p^p \ (\text{ since } x \in C). \end{aligned}$$

If $x \equiv 0$, then $x_n' \to 0$ in $L^p(T)$, a contradiction to the fact that $\|x_n'\|_p = 1$. So $x \neq 0$ and from Proposition 1 we have that $x(t) > 0$ for all $t \in (0, b)$. Using this fact in (4.17), we obtain

$$\|x'\|_p^p < \gamma \|x\|_p^p,$$

a contradiction to the fact that $x \in C \setminus \{0\}$. ∎

Using this Lemma, we can now prove that $\varphi\Big|_C$ is coercive, thus bounded below.

PROPOSITION 4.4. *If hypotheses H(j) hold, then $\varphi\Big|_C$ is coercive, thus bounded below.*

Proof. In what follows by $|\cdot|_1$ we denote the Lebesgue measure on $\mathbb{R}$. By hypothesis H(j)(ii) for all $t \in T \setminus N$, $|N|_1 = 0$, the function $x \to j(t,x)$ is locally Lipschitz. So for all $t \in T \setminus N$, the function $x \to j(t,x)$ is differentiable at every $x \in \mathbb{R} \setminus D(t)$, $|D(t)|_1 = 0$ and for all $x \geq 0$, we have

$$\begin{aligned} j(t,x) - j(t,0) &= \int_0^x j'_r(t,r)dr \\ &\leq \int_0^x (h(t)+\varepsilon)r^{p-1}dr + 2|g(t)|x \\ &\quad (\text{ recall that } j'_r(t,r) \in \partial j(t,r) \text{ and see 4.4}) \\ &\leq \frac{1}{p}(h(t)+\varepsilon)x^p + a_\varepsilon(t) + \frac{\varepsilon}{\rho}x^p \qquad (4.18) \\ &\quad \text{with } \varepsilon > 0 \text{ and } a_\varepsilon(t) \in L^1(T)_+ \text{ (by Young's inequality)}. \end{aligned}$$

So if $x \in C$, we have

$$\begin{aligned} \varphi(x) &= \frac{1}{p}\|x'\|_p^p - \int_0^b j(t,x(t))dt - \int_0^b g(t)x(t)dt, \\ &\geq \frac{1}{p}\|x'\|_p^p - \frac{1}{p}\int_0^b h(t)x(t)^p dt - \frac{2\varepsilon}{p}\|x\|_p^p - \|a_\varepsilon\|_1 - \int_0^b g(t)x(t)dt \\ &\quad \text{(see 4.17)}, \\ &\geq \frac{1}{p}\|x'\|_p^p - \frac{1}{p}\int_0^b h(t)x(t)^p dt - \frac{3\varepsilon}{p}\|x\|_p^p - \xi_\varepsilon \text{ for some } \xi_\varepsilon > 0 \\ &\quad \text{(by Young's inequality)}, \\ &\geq \frac{\beta_0}{p}\|x'\|_p^p - \frac{3\varepsilon}{p}\|x\|_p^p - \xi_\varepsilon \text{ (see Lemma 4.3)}, \\ &\geq \frac{\beta_0}{p}\|x'\|_p^p - \frac{3\varepsilon}{\gamma p}\|x'\|_p^p - \xi_\varepsilon \text{ (recall the definition of } \gamma). \end{aligned}$$

Choose $\varepsilon < \frac{\beta_0\gamma}{3}$. Then

$$\varphi(x) \geq \beta_1\|x'\|_p^p - \xi_\varepsilon \quad \text{for some } \beta_1 > 0 \text{ and all } x \in C.$$

Because $\|x\|_p^p \leq \frac{1}{\gamma}\|x'\|_p^p$, it follows that

$$\begin{aligned} &\varphi(x) \geq \frac{\beta\gamma}{1+\gamma}\|x\|^p - \xi_\varepsilon \quad \text{for all } x \in C, \\ \Longrightarrow\ &\varphi\Big|_C \text{ is coercive, thus bounded below.} \end{aligned}$$

■

PROPOSITION 4.5. *If hypotheses H(j) hold, then* $\varphi\Big|_{\mathbb{R}}$ *is anticoercive, i.e* $\varphi(\xi) \to -\infty$ *as* $|\xi| \to \infty$, $\xi \in \mathbb{R}$.

Proof. From the mean value theorem for locally Lipschitz functions (see Clarke [2], p.41 or Denkowski-Migorski-Papageorgiou[4], p.609), we know that for all $t \in T \setminus N$, $|N|_1 = 0$ and all $x < y < 0$, we have

$$j(t,x) - j(t,y) = u(x-y) \quad \text{with } u \in \partial j(t, \lambda x + (1-\lambda x)), \lambda \in (0,1)$$

$(u, \lambda$ depending on $t \in T \setminus N)$.

Also by virtue of hypothesis H(j)(v), we can find $\delta > 0$ and $M_7 > 0$ such that if $y \leq -M_7 < 0$, then

$$u \leq -g(t) - \delta \quad \text{for all } t \in T \setminus N, |N|_1 = 0.$$

Hence for $t \in T \setminus N$, $|N|_1 = 0$ and $x < y \leq -M_7 < 0$, we have

$$\begin{aligned} & j(t,x) - j(t,y) = u(x-y) \geq \delta|x-y| - g(t)x + g(t)y \\ \Longrightarrow \quad & \int_0^b j(t,x)dt + \int_0^b g(t)xdt \geq \delta|x-y| + \int_0^b g(t)ydt, \\ \Longrightarrow \quad & \lim_{x \to -\infty} \left[\int_0^b j(t,x)dt + \int_0^b g(t)xdt \right] = +\infty. \end{aligned}$$

Similarly, we show that

$$\lim_{x \to +\infty} \left[\int_0^b j(t,x)dt + \int_0^b g(t)xdt \right] = +\infty.$$

Therefore finally we conclude that

$$\varphi(\xi) = -\int_0^b j(t,\xi)dt - \int_0^b g(t)\xi dt \to -\infty \quad \text{as } |\xi| \to \infty, \xi \in \mathbb{R}.$$

■

Now we are ready for an existence theorem for problem (1.1).

THEOREM 4.6. *If hypotheses H(j) hold and* $g \in L^q(T)(\frac{1}{p} + \frac{1}{q} = 1)$*, then problem (1.1) has a solution* $x \in C^1(T)$ *with* $|x'|^{p-2}x' \in W^{1,q}(0,b)$.

Proof. By virtue of Proposition 4.4 and 4.5, we can find $\xi \in \mathbb{R}_+ \setminus \{0\}$, such that

$$\varphi(\pm\xi) < \inf_C \varphi.$$

Let $E_1 = \{\pm\xi\}$, $E = \{y \in W^{1,p}(0,b) : -\xi \leq y(t) \leq \xi \text{ for all } t \in T\}$ and $\Gamma = \{\theta \in C(E, W^{1,p}(0,b)) : \theta\Big|_{E_1} = id_{E_1}\}$. For $\theta \in \Gamma$ we have $\theta(-\xi) = -\xi < 0 < \xi = \theta(\xi)$. Also the function $y \to \min_T \theta(y)$ is continuous from $E \subseteq W^{1,p}(0,b)$ into $W^{1,p}(0,b)$. To see this let $y_n \to y$ in E. Then $\theta(y_n) \to \theta(y)$ in $W^{1,p}(0,b)$ and so $\theta(y_n) \to \theta(y)$ in $C(T)$. Let $t_n \in T$ be such that $\theta(y_n)(t_n) = \min_T \theta(y_n)$, $n \geq 1$. We may assume that $t_n \to \hat{t} \in T$. Then $\theta(y_n)(t_n) \to \theta(y)(\hat{t})$. For every $n \geq 1$ and every $t \in T$ we have $\theta(y_n)(t_n) \leq \theta(y_n)(t)$. So passing to the limit as $n \to \infty$ we obtain

$\theta(y)(\hat{t}) \leq \theta(y)(t)$ for all $t \in T$ and so $\theta(y)(\hat{t}) = \min_T \theta(y)$. This proves the continuity of $y \to \min_T \theta(y)$ from E into $\mathbb{R}$. Since $\min_T \theta(-\xi) = -\xi$ and $\min_T \theta(\xi) = \xi$, from the intermediate value theorem we can find $y \in E$ such that $\min_T \theta(y) = 0$. Hence $\theta(E) \cap C = \emptyset$ while $E_1 \cap C = \emptyset$. So the sets E_1 and C link in $W^{1,p}(0,b)$. Because of Proposition 3.2, we can apply Theorem 2.2 and obtain $x \in W^{1,p}(0,b)$ such that $0 \in \partial\varphi(x)$ and $\inf_C \varphi \leq \varphi(x)$. From the inclusion $0 \in \partial\varphi(x)$, we obtain

$$A(x) = u + g \quad \text{with } u \in L^q(T), u(t) \in \partial j(t, x(t)) a.e \text{ on } T \tag{4.19}$$

$$\Longrightarrow \quad < A(x), \eta >= \int_0^b (u(t) + g(t))\eta(t)dt \quad \text{for all } \eta \in C_c^1(0,b). \tag{4.20}$$

We know that $|x'|^{p-2}x' \in W^{-1,q}(0,b) = W_0^{1,p}(0,b)^*$ (see Denkowski-Migorski-Papageorgiou[4], p.363). Thus if by $< \cdot, \cdot >_0$ we denote the duality brackets for the pair $(W^{-1,q}(0,b), W^{1,p}(0,b))$, from (4.20), we have

$$< -(|x'|^{p-2}x', \eta) >_0 = \int_0^b (u(t) + g(t))\eta(t)dt \quad \text{for all } \eta \in C_c^1(0,b). \tag{4.21}$$

But $C_c^1(0,b)$ is dense in $W_0^{1,p}(0,b)$. So from (4.21) it follows that

$$-(|x'(t)|^{p-2}x'(t)) = u(t) + g(t) \ a.e \ \text{ on } T. \tag{4.22}$$

From (4.22) it follows that $x \in C^1(T)$. Also from Green's identity, for every $\zeta \in W^{1,p}(0,b)$ we have

$$\begin{aligned} -\int_0^b (|x'(t)|^{p-2}x'(t))'\zeta(t)dt &= -|x'(b)|^{p-2}x'(b)\zeta(b) + |x'(0)|^{p-2}x'(0)\zeta(0) \\ &\quad + \int_0^b |x'(t)|^{p-2}x'(t)\zeta'(t)dt, \\ \Longrightarrow \quad \int_0^b (u(t) + g(t))\zeta(t) &= -|x'(b)|^{p-2}x'(b)\zeta(b) + |x'(0)|^{p-2}x'(0)\zeta(0) \\ &\quad + \int_0^b |x'(t)|^{p-2}x'(t)\zeta'(t)dt \ (\text{ see (4.22)}), \\ \Longrightarrow \quad |x'(0)|^{p-2}x'(0)\zeta(0) &= |x'(b)|^{p-2}x'(b)\zeta(b) \quad (\text{ see (4.19)}). \end{aligned}$$

Since $\zeta \in W^{1,p}(0,b)$ is arbitrary, it follows that

$$\begin{aligned} & |x'(0)|^{p-2}x'(0) = |x'(b)|^{p-2}x'(b) = 0, \\ \Longrightarrow \quad & x'(0) = x'(b) = 0. \end{aligned}$$

So we conclude that $x \in C^1(T)$ with $|x'|^{p-2}x' \in W^{1,q}(0,b)$, is a solution of (1.1). ∎

Following the same reasoning (with the obvious modifications), we can have on existence result when the growth restriction on $j(t,\cdot)$ is imposed in the negative direction. So our hypothesis on $j(t,x)$ are the following:
$\underline{H(j)'}$: $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(\cdot, 0) \in L^1(T)$, hypothesis H(j)(i), (ii), (iii) and (v) hold an (iv) $\limsup_{x \to -\infty} \frac{u}{|x|^{p-2}x} \leq h(t)$ uniformly for almost all $t \in T$ and all $u \in \partial j(t,x)$, with $h \in L^1(T)_+$ such that $h(t) \leq \gamma \ a.e$ on T and this inequality is strict on a set of positive Lebesgue measure.

Under these hypotheses we can prove the following existence theorem, which extends and improves the work of Villegas[17].

THEOREM 4.7. *If hypotheses $H(j)'$ hold and $g \in L^q(T)$ $(\frac{1}{p}+\frac{1}{q}=1)$, then the problem (1.1) has a solution $x \in C^1(T)$ with $|x'|^{p-2}x' \in W^{1,q}(0,b)$.*

We derive three corollaries which extend and improve the work of de Figueiredo-Ruf [7].

COROLLARY 4.8. *If hypotheses H(j)(i)→ (iv) or $H(j)'$(i)→ (iv) hold and*

$$\lim_{x\to-\infty}\left[\max_{u\in\partial j(t,x)} u\right] = -\infty \quad \textit{and} \quad \lim_{x\to+\infty}\left[\min_{u\in\partial j(t,x)} u\right] = +\infty \quad \textit{uniformly for a.a } t \in T,$$

then problem (1.1) has a solution $x \in C^1(T)$ with $|x'|^{p-2}x' \in W^{1,q}(0,b)$ for every $g \in L^q(T)$.

COROLLARY 4.9. *If hypotheses H(j)(i),(ii),(iii) hold and*
(iv) $0 < \liminf_{x\to+\infty} \frac{u}{x^{p-1}} \le \limsup_{x\to+\infty} \frac{u}{x^{p-1}} \le h(t)$ *uniformly for almost all $t \in T$ and all $u \in \partial j(t,x)$ with $h \in L^1(T)_+$ such that $h(t) \le \gamma$ a.e on T with strict inequality on a set of positive Lebesgue measure and* $\limsup_{x\to-\infty}\left[\max_{u\in\partial j(t,x)} u\right] = -\infty$ *uniformly for a.a $t \in T$,or (iv)'* $0 < \liminf_{x\to-\infty} \frac{u}{x^{p-1}} \le \limsup_{x\to-\infty} \frac{u}{x^{p-1}} \le h(t)$ *uniformly for almost all $t \in T$ and all $u \in \partial j(t,x)$ with $h \in L^1(T)_+$ such that $h(t) \le \gamma$ a.e on T with strict inequality on a set of positive Lebesgue measure and* $\liminf_{x\to+\infty}\left[\min_{u\in\partial j(t,x)} u\right] = +\infty$ *uniformly for a.a $t \in T$,then problem (1.1) has a solution $x \in C^1(T)$ with $|x'|^{p-2}x' \in W^{1,q}(0,b)$ for every $g \in L^q(T)$.*

COROLLARY 4.10. *If hypothesis H(j)(i),(ii),(iii) hold and* $\lim_{x\to-\infty}\left[\max_{u\in\partial j(t,x)} u\right] = -\infty$, $\lim_{x\to+\infty}\left[\min_{u\in\partial j(t,x)} u\right] = \lim_{x\to+\infty}\left[\max_{u\in\partial j(t,x)} u\right] = 0$ *uniformly for a.a $t \in T$, or* $\lim_{x\to-\infty}\left[\max_{u\in\partial j(t,x)} u\right] = 0$, $\lim_{x\to+\infty}\left[\min_{u\in\partial j(t,x)} u\right] = +\infty$ *uniformly for a.a $t \in T$, then problem (1.1) has a solution $x \in C^1(T)$ with $|x'|^{p-2}x' \in W^{1,q}(0,b)$ for every $g \in L^q(T)$ with $g(t) < 0$ a.e on T or $g(t) > 0$ a.e on T respectively.*

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