Classification of transitive deficiency one partial parallelisms*

Mauro Biliotti Vikram Jha Norman L. Johnson

Abstract

The transitive partial parallelisms of deficiency one in $PG(3,q)$ are classified under mild assumptions on the corresponding group.

1 Introduction.

In this article, we consider various coverings of the points of a projective space by lines. These coverings are called "spreads" and we further consider if there is a set of mutually disjoint spreads that cover the line set of the projective space. A covering of the set of lines of a projective space by spreads is called a "parallelism" and it is of fundamental importance to formulate construction techniques, examples and theory of this area of incidence geometry.

In particular, we consider parallelisms in $PG(3,q)$ where $q = p^r$, p a prime and r a positive integer. A parallelism then requires that there are $1+q+q^2$ spreads of the cover. The authors have previously considered "transitive" parallelisms, which are parallelisms admitting an automorphism group G as a subgroup of $P\Gamma L(4,q)$, acting transitively on the spreads of the parallelism. In this situation, the group G can be essentially completely determined and it is isomorphic to a solvable subgroup of $\Gamma L(1,q^3)$ or it is $PSL(2,7)$, $q = 2$ and the parallelism is one of the two parallelisms in $PG(3,2)$. Specifically, this result involves the analysis of p-primitive automorphisms

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(collineations of order dividing $q^3 - 1$, for $p^r = q$, but not dividing $p^t - 1$ for $t < 3r$), which always exist when there are transitive groups.

Theorem 1. *(Biliotti, Jha, Johnson [3]) Let $\mathcal{P}$ be a parallelism in $PG(3, q)$ and let G be a collineation group of $PG(3, q)$ which leaves $\mathcal{P}$ invariant and contains a collineation of order a p-primitive divisor u of $q^3 - 1$.*

Then, one of the two situations occurs:

(1) G is a subgroup of $\Gamma L(1, q^3)/Z$, where Z denotes the scalar group of order $q - 1$, and fixes a plane and a point.

(2) $u = 7$ and one of the following subcases occurs:

(a) G is reducible, fixes a plane or a point and either

(i) G is isomorphic to A_7 and $q = 5^2$, or

(ii) G is isomorphic to $PSL(2, 7)$ and $q = p$, for a prime $p \equiv 2, 4 \bmod 7$ or $q = p^2$, for a prime $p \equiv 3, 5 \bmod 7$. When $q = p = 2$, $\mathcal{P}$ is one of the two regular parallelisms in $PG(3, 2)$, or

(b) G is primitive, G is isomorphic to $PSL(2, 7)$ or A_7 and $q = p$, for an odd prime $p \equiv 2, 4 \bmod 7$ or $q = p^2$, for a prime $p \equiv 3, 5 \bmod 7$.

It might be noted that parallelisms in $PG(3, q)$, where all spreads are regular (the 'regular parallelisms') correspond to translation planes of order q^4 and kernel $GF(q)$ that contain a set of $1 + q + q^2$ derivable nets that mutually share a $GF(q)$-regulus (e.g. [13]). There are two parallelisms in $PG(3, 2)$, necessarily regular, whose corresponding translation planes are the Lorimer-Rahilly and Johnson-Walker planes of order $2^4 = 16$. These two planes, although non-isomorphic, may be obtained by transposition of their spreads and correspond to the two parallelisms linked by a polarity of the projective space. There are also regular parallelisms in $PG(3, 8)$, due to Denniston [6] and in $PG(3, 5)$, due to Prince [26]. Furthermore, Prince also completely determines all cyclic parallelisms in $PG(3, 5)$, of which exactly two are regular parallelisms. All of these regular parallelisms are in an infinite class of regular parallelisms constructed by Penttila and Williams [25]. This class of transitive (actually, cyclic) parallelisms actually produces two classes, as each regular parallelism is non-isomorphic to its dual parallelism. The corresponding pairs of translation planes of order q^4 are interrelated by transposition of their spread sets.

The two parallelisms in $PG(3, 2)$ are of particular interest in that these parallelisms are 'two-transitive'. The third author [12] has shown that any two-transitive parallelism is, in fact, one of these two parallelisms in $PG(3, 2)$.

Theorem 2. *(Johnson [12]) Let $\mathcal{P}$ be a two-transitive parallelism in $PG(3, q)$. Then $q = 2$ and $\mathcal{P}$ is one of the two parallelisms in $PG(3, 2)$.*

The proof given in [12] uses the classification theorem of finite simple groups and it might be pointed out that in the authors' work mentioned above on transitive parallelisms, the same result may be obtained directly from consideration of the subgroups of $P\Gamma L(4, q)$ and this is done in Biliotti, Jha and Johnson [3].

The consideration of group actions on parallelisms provides a crucial mechanism to understand their construction. In the work of the third author (see e.g. [13]), a construction of parallelisms is given using a central collineation group of an associated Desarguesian spread Σ. In particular, it is possible to determine parallelisms containing Σ admitting a central collineation group G of Σ such that G acts transitively on the remaining spreads of the parallelism. If the group G is the full central collineation group with a fixed line (component) then it will turn out that the remaining spreads are Hall spreads. We note that a Hall spread in this context is essentially defined as a spread obtained from a Desarguesian spread by the derivation process of replacement of a regulus (net). However, if the group G is not the full central collineation group, G can still act transitively on the remaining spreads of the parallelism but the spreads could be of another type not isomorphic to the Hall spreads. In the third author's work [14], there are a variety of examples of such parallelisms, where the 'other' spreads are all spreads derived from the so-called 'conical Knuth' spreads.

We recall that a flock of a quadratic cone is simply a covering of the non-vertex points of the cone by mutually disjoint conics and we assume that the cone lives in $PG(3, q)$. It is now well known (but see, for example, the survey article by Johnson and Payne [18]) that there are spreads, the 'conical spreads', that correspond to conical flocks. That such an equivalence exists between such spreads and conical flocks is due to Gevaert, Johnson and Thas [10]. The conical spreads are unions of q reguli that mutually share exactly one line and hence the associated translation planes are derivable in a variety of ways. Any such derived plane by one of the 'base reguli' is called a 'derived conical flock plane'. So as not to confuse the reader with notation,when q is odd, we note that there is the concept of 'derivation of conical flocks' that produces from one flock a set of $q + 1$ flocks and corresponds to a set of $q + 1$ points of the Klein quadric such that no three are incident with a line of the quadric, called BLT-sets in honor of L. Bader, G. Lunardon and J.A. Thas who introduced this concept (for details, e.g. see [18]). In this article, a 'derived conical flock plane' is a plane obtained from a conical flock plane by reversing a base regulus.

Hence, in the known cases, when a parallelism admits a group G that fixes one spread and acts transitively on the remaining spreads, the fixed spread, called the 'socle' (i.e. either 'socle spread' or 'socle plane') is Desarguesian and the remaining spreads are derived conical flock spreads. Hence, it is an open question whether this is always the case.

In this article, we show that if G is considered within $PGL(4, q)$ then we may completely determine and/or more-or-less classify the parallelisms that fix one spread and act transitively on the remaining. Actually, we note in the last section that the known examples satisfy a more general hypothesis, namely that the Sylow p-subgroups are in $PGL(4, q)$, so our theorem reflects this.

Our main theorems are as follows:

Theorem 3. *Let $q = p^r$, for p a prime . Let $\mathcal{P}$ be a parallelism in $PG(3, q)$ admitting an automorphism group G that fixes one spread (the socle) and acts transitively on the remaining spreads. Assume that the Sylow p-subgroups of G are in $PGL(4, q)$ or, if $q = 8$, that G itself is a subgroup of $PGL(4, q)$.*

Then

(1) the socle is Desarguesian,

(2) the associated group G contains an elation group of order q^2 acting on the socle and

(3) the remaining spreads of the parallelism are isomorphic derived conical flock spreads.

Theorem 4. *Let $\mathcal{P}$ be a parallelism in $PG(3, q)$, for $q \neq 8$, admitting an automorphism group G that fixes one spread (the socle) and acts transitively on the remaining spreads. If $q = p^r$, for p a prime and $(r, q) = 1$, then the following occurs:*

(1) the socle is Desarguesian,

(2) the associated group G contains an elation group of order q^2 acting on the socle and

(3) the remaining spreads of the parallelism are isomorphic derived conical flock spreads.

2 Background.

In this section, we provide the necessary background results necessary to read the proof of our main result.

We shall use the useful technical result proved in the authors' work [4], which we list for convenience.

Theorem 5. *(Biliotti, Jha, Johnson [4]) Let π denote a translation plane of order q^2. Assume that π admits a collineation group G that contains a normal subgroup N such that G/N is isomorphic to $PSL(2, q)$.*

Then π is one of the following planes:

(1) Desarguesian,

(2) Hall,

(3) Hering,

(4) Ott-Schaeffer

(5) one of three planes of Walker of order 25 or

(6) the Dempwolff plane of order 16.

Theorem 6. *(Johnson [15] Theorem 2.3) Let V be a vector space of dimension $2r$ over F isomorphic to $GF(p^t)$, p a prime, $q = p^t$. Let T be a linear transformation of V over F which fixes three mutually disjoint r-dimensional subspaces. Assume that $|T|$ divides $q^r - 1$ but does not divide $LCM(q^s - 1); s < r,\ s \mid r$. Then:*

(1) all T-invariant r-dimensional subspaces are mutually disjoint and the set of all such subspaces defines a Desarguesian spread;

(2) the normalizer of $\langle T \rangle$ in $GL(2r, q)$ is a collineation group of the Desarguesian plane Σ defined by the spread of (1);

(3) Σ may be thought of as a $2r$-dimensional vector space over F; that is, the field defining Σ is an extension of F.

Theorem 7. *(Gevaert and Johnson [9]) Let π be a translation plane of order q^2 with spread in $PG(3, q)$ that admits an affine elation group E of order q such that there is at least one orbit of components union the axis of E that is a regulus in $PG(3, q)$. Then π corresponds to a flock of a quadratic cone in $PG(3, q)$.*

In the case above, the elation group E is said to be 'regulus-inducing' as each orbit of a 2-dimensional $GF(q)$-vector space disjoint from the axis of E will produce a regulus.

Theorem 8. *(Johnson [16]) Let π be a translation plane of order q^2 with spread in $PG(3,q)$ admitting a Baer group B of order q. Then, the $q-1$ component orbits union $FixB$ are reguli in $PG(3,q)$. Furthermore, there is a corresponding partial flock of a quadratic cone with $q-1$ conics. The partial flock may be uniquely extended to a flock if and only if the net defined by $FixB$ is derivable.*

Theorem 9. *(Payne and Thas [24]) Every partial flock of a quadratic cone of $q-1$ conics in $PG(3,q)$ may be uniquely extended to a flock.*

Theorem 10. *(Hering [8], Ostrom [22], [23]) Let π be a translation plane of order p^r, p a prime, and let E denote the collineation group generated by all elations in the translation complement of π. Then one of the following situations apply:*
(i) E is elementary Abelian,
(ii) E has order $2k$, where k is odd, and $p = 2$,
(iii) E is isomorphic to $SL(2,p^t)$,
(iv) E is isomorphic to $SL(2,5)$ and $p = 3$,
(v) E is isomorphic to $S_z(2^{2s+1})$ and $p = 2$.

Theorem 11. *(Johnson and Ostrom [17] (3.4)) Let π be a translation plane with spread in $PG(3,q)$ of even order q^2. Let E denote a collineation group of the linear translation complement generated by affine elations. If E is solvable then either E is an elementary Abelian group of elations all with the same axis, or E is dihedral of order $2k$, k odd and there are exactly k elation axes.*

Theorem 12. *(Gleason [11]) Let G be a finite group operating on a set Ω and let p be a prime. If Ψ is a subset of Ω such that for every $\alpha \in \Psi$, there is a p-subgroup Π_α of G fixing α but no other point of Ω then Ψ is contained in an orbit.*

Theorem 13. *(André [1]) If Π is a finite projective plane that has two homologies with the same axis ℓ and different centers then the group generated by the homologies contains an elation with axis ℓ.*

Theorem 14. *(See Lüneburg [21] (49.4), (49.5)) Let τ be a linear mapping of order p of a vector space V of characteristic p and dimension 4 over $GF(p^r)$ and leaving invariant a spread.*

(1) Then the minimal polynomial of τ is $(x-1)^2$ or $(x-1)^4$. If the minimal polynomial is $(x-1)^4$ then $p \geq 5$.

(2) The minimal polynomial of τ is $(x-1)^2$ if and only if τ is an affine elation (shear) or a Baer p-collineation (fixes a Baer subplane pointwise).

Theorem 15. *(Foulser [7]) Let π be a translation plane of odd order q^2. Then Baer p-elements and elations cannot coexist.*

Theorem 16. *(Biliotti, Jha and Johnson [5]) Let π be a translation plane of order q^2, $q = p^r$, p odd, with spread in $PG(3,q)$. If π admits two mutually disjoint 'large' quartic p-groups (orders $> \sqrt{q}$) then the generated group is isomorphic to $SL(2,q)$ and the plane is Hering or the order is 25 and the plane is one of the Walker planes.*

3 Fundamentals.

In this section, it will be convenient to provide several fundamental lemmas, which will then be applied for subsequent arguments. In this section, $\mathcal{P}$ will be a parallelism containing the socle spread Σ and G is a collineation group of Σ that acts transitively on the $q(q+1)$ spreads of $\mathcal{P}-\{\Sigma\}$, where $p^r = q$, p a prime and r a positive integer. If q is not 8, we assume that any Sylow p-subgroup of G is in $PGL(4,q)$, acting on the spread or in $GL(4,q)$ acting on the associated translation plane π_Σ corresponding to the spread Σ. Let K denote the field isomorphic to $GF(q)$ such that all spreads of $\mathcal{P}$ are in $PG(3,K)$.

Lemma 1. *Any element σ of G that acts like an elation on Σ fixes some spread of $\mathcal{P}-\{\Sigma\}$.*

Proof. Choose the axis of σ be L and assume that M, N are components of Σ such that $M\sigma = N$. Choose a basis for the associated 4-dimensional K-subspace so that $x = (x, x_2)$, $y = (y_1, y_2)$ and L is $x = 0; x_1 = x_2 = 0$, M is $y = 0; y_1 = y_2 = 0$ and N is $y = x; x_1 = y_1$ and $x_2 = y_2$.

Then $\sigma : (x, y) \longmapsto (x, x + y)$. We note that $\{(0, x_2, 0, y_2); x_2 \text{ and } y_2 \text{ in } K\}$ is a 2-dimensional K-subspace which is left invariant by σ. But, this subspace π_o then is a line of $PG(3,K)$, which lies in a unique spread S_{π_o} of $\mathcal{P}-\{\Sigma\}$. Since σ fixes π_o, it follows that σ fixes S_{π_o}. ■

Lemma 2. *Any Sylow p-subgroup S_p of G that has order $p^a q$, contains an elation group of order p^a.*

Proof. S_p is a subgroup of $GL(4,q)$ acting on the translation plane π_Σ corresponding to Σ. Therefore, S_p fixes a 1-dimensional K-subspace X pointwise. Let L denote the unique component containing X. The maximal order subgroup that can fix X pointwise and be in $GL(4,q)$ and act faithfully on L has order q. Thus, it follows that there must be a subgroup of order p^a that fixes L pointwise. ■

Lemma 3. *Assume that the socle spread Σ is Desarguesian. Then any line of a non-socle spread that is fixed by an elation with axis L is a Baer subplane of a regulus of Σ that contains L.*

Proof. Any such line is a Baer subline of π_Σ and must non-trivially intersect L if it is fixed by an elation with axis L. Any Baer subplane defines a unique regulus net of Σ containing it as a subplane. ■

Lemma 4. *Assume that Σ is Desarguesian and assume that σ is a non-identity elation of Σ with axis L.*

(1) Then σ fixes exactly $q(q+1)$ Baer subplanes of π_Σ that share L as a component. Furthermore, if E is the full elation group of Σ with axis L then any element of E fixes the same set of $q(q+1)$ Baer subplanes.

(2) σ fixes exactly q non-socle spreads.

Proof. We have seen that σ must fix a Baer subplane of π_{Σ}, which is also a 2-dimensional K-subspace. Let π_o be a Baer subplane (and also a 2-dimensional K-subspace), fixed by an elation σ of Σ with elation axis. Then by Lemma 3, there is a regulus R of Σ containing π_o. Choose this regulus to be the standard regulus;

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in K.$$

Also, choose this representation so that $x = 0$ is L. Under this representation, Σ is forced to have the following form:

$$\begin{aligned} x &= 0, y = x \begin{bmatrix} u+\rho t & \gamma t \\ t & u \end{bmatrix} \forall u, t \in K, \\ &\text{for constants } \rho \text{ and } \gamma. \end{aligned}$$

Since Σ is Desarguesian, there are exactly q reguli on Σ that share L. We have seen that we may represent σ is the form $(x, y) \to (x, x+y)$. Furthermore, note that σ then must fix each of the q regulus nets R_t defined as follows:

$$\begin{aligned} x &= 0, y = x \begin{bmatrix} u+\rho t & \gamma t \\ t & u \end{bmatrix} \forall u \in K, \text{ for } t \text{ fixed in } K \\ &\text{for constants } \rho \text{ and } \gamma. \end{aligned}$$

Note that whenever σ fixes a Baer subplane π_1 sharing L, there is a regulus net R_{π_1} fixed by σ and containing π_1 as a Baer subplane. Furthermore σ fixes each Baer subplane of R_{π_1}. Thus, σ fixes at least $q(q+1)$ Baer subplanes of Σ. Assume that σ fixes a Baer subplane π_2 not in the previous set. Then π_2 shares L and lies on a regulus net containing L. There are exactly $q(q+1)$ regulus nets of π_{Σ} containing L. Since the regulus R_{π_2} defined by π_2 cannot be any of the q reguli defined by π_o within π_{Σ}, it follows that R_{π_2} shares exactly one component distinct from L in each of these q reguli. However, since σ fixes R_{π_2} and each of the reguli R_i defined above, for $i = 1, 2, ..., q$, it follows that σ fixes $R_{\pi_2} \cap R_i$, a contradiction. Hence, σ fixes exactly $q(q+1)$ Baer subplanes of π_{Σ}.

Consider an elation σ with axis L which fixes a non-socle spread Σ'. L now is a Baer subplane of $\pi_{\Sigma'}$ and there are exactly $q+1$ components of Σ' that non-trivially intersect L. Each of these is fixed by σ. Suppose there is a component M of Σ' that is fixed by σ. Then M is a Baer subplane of π_{Σ} that is fixed by a central collineation σ of π_{Σ}. However, this implies that M must non-trivially intersect the axis L of ρ. Hence, there are exactly $q+1$ Baer subplanes of π_{Σ} fixed by σ. Let X be any given 1-dimensional K-subspace of L. Then there are exactly $q+1$ Baer subplanes of Σ that are fixed by σ that lie as components in Σ'. Since there are exactly $q(q+1)$ Baer subplanes of π_{Σ} that share X, it follows that σ fixes exactly q non-socle spreads. ■

Lemma 5. *Assume that Σ is Desarguesian and G contains a unique non-trivial elation subgroup E. Then the order of E is q^2 and each non-socle spread is fixed by a subgroup of E of order q that acts as a Baer group on that non-socle spread. Hence, the non-socle spreads are derived conical flock spreads.*

Proof. Let Σ'' be a non-socle spread fixed by an elation subgroup $E_{\Sigma''}$ of E. Then $E_{\Sigma'}$ fixes exactly q non-socle spreads. If Σ'' is not fixed by $E_{\Sigma'}$, then it is not fixed by any element of $E_{\Sigma'}$. So if $E_{\Sigma''}$ is the elation subgroup of E that fixes Σ'' then, our previous arguments show that $E_{\Sigma'} \cap E_{\Sigma''} = \langle 1 \rangle$.

Since each such elation group fixes exactly q Baer subplanes among the common set of $q(q+1)$ Baer subplanes of π_Σ, it follows that E is partitioned into a set of exactly $q+1$ mutually disjoint subgroups E_i, where E_i are elation subgroups of order p^a that fix exactly q non-socle spreads. Since E of order p^z is partitioned by groups of order p^a, then $z = ka$ and

$$(p^a - 1)(q + 1) + 1 = p^{ka}$$

$$q + 1 = (p^{ka} - 1)/(p^a - 1) = 1 + p^a + p^{2a} + ... + p^{(k-1)a}.$$

Thus,

$$q = p^a + p^{2a} + ... + p^{(k-1)a}.$$

Assume that p^a is not q. Then,

$$qp^{-a} = 1 + p^a + ... + p^{(k-2)a}.$$

Thus, it follows that $(k - 2)a = 0$ or $k = 2$, but then $q = p^a$.

Thus, the order of E is q^2 and the order of the $E_i' s$ is q. Thus, there is a Baer group of order q fixing any non-socle spread. Thus, apply Theorems 8 and 9 to conclude that the non-socle planes are isomorphic derived conical flock planes. This completes the proof of the lemma. ∎

Lemma 6. *If Σ is Desarguesian, there is an elation group of order q^2 and the non-socle planes are isomorphic derived conical flock planes.*

Proof. If Σ is Desarguesian then the group G is a subgroup of $\Gamma L(2, q^2)$ and the Sylow p-subgroups S_p are in $GL(4, q)$ and have order at least q. Assume that $q = 2$. If S_2 has order 2 and is not an elation group, then S_2 is a Baer collineation and this Baer subplane is a line of a non-socle spread. Hence, S_2 has order ≥ 2.

In general, $S_p \cap GL(2, q^2)$ is an elation subgroup, which must be non-trivial. Since the only possible non-linear mapping involves the automorphism $x \to x^q$, it follows that S_p is an elation group if p is odd and contains an elation subgroup of index at most 2, when $p = 2$. We know from Lemma 1 that any elation will fix a Baer subplane and this Baer subplane will be line of a unique non-socle spread. Let π_o be a Baer subplane (and also a 2-dimensional K-subspace), fixed by an elation σ of Σ with elation axis L. Then by Lemma 3, there is a regulus R of Σ containing π_o. Choose this regulus to be the standard regulus;

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \forall u \in K.$$

Also, choose this representation so that $x = 0$ is L. Let E_{π_o} denote the full elation subgroup of S_p that fixes π_o, and let E_{π_o} have order p^a. Under this representation, Σ is forced to have the following form:

$$x = 0, y = x \begin{bmatrix} u + \rho t & \gamma t \\ t & u \end{bmatrix} \forall u, t \in K,$$

for constants ρ and γ.

Since G is transitive on $q(q+1)$ spreads, there is an elation subgroup of order p^a that fixes a non-socle spread. By transitivity, any non-socle spread is fixed by an elation group of order p^a. An elation group acting on a non-socle spread is a Baer group. Hence, every non-socle spread is fixed by a Baer group of order p^a. A Baer group will fix exactly $q+1$ components of a fixed non-socle spread. Hence, the associated elation group will fix $q+1$ Baer subplanes. Since we may assume that one of these Baer subplanes belongs to the standard regulus containing the axis $x=0$ of E, acting on Σ and since there is an elation group of order p^a fixing each Baer subplane of the standard regulus, we represent the elation group E_{π_o} on Σ as follows:

$$\left\langle \begin{bmatrix} 1 & 0 & u+t\rho & t\gamma \\ 0 & 1 & t & u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; t \in \kappa, u \in \lambda \right\rangle,$$

where λ is an additive subgroup of order p^a and κ is an additive subgroup of order at least $q/(2,q)$.

There are $q(q+1)$ Baer subplanes of Σ non-trivially intersecting $x=0$ in a given 1-dimensional subspace X. There is one of these in each non-socle spread.

Now look at the partial spread R_t :

$$x=0, y=x\begin{bmatrix} t\rho & t\gamma \\ t & 0 \end{bmatrix}; t \in GF(q).$$

We know that γ is non-zero. Change bases by $\begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}$, where

$$C=\begin{bmatrix} 0 & 1 \\ 1/\gamma & -\rho/\gamma \end{bmatrix}.$$

Then the image of the partial spread above is:

$$x=0, y=x\begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}; t \in GF(q),$$

clearly a regulus. Thus, R_t is a regulus of Σ. Note that the elation subgroup

$$E_t=\left\langle \begin{bmatrix} 1 & 0 & t\rho & t\gamma \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; t \in \kappa \right\rangle$$

of order at least $q/(2,q)$ will fix all Baer subplanes of R_t. Each of these Baer subplanes lies in a unique non-socle spread, so E_t fixes a non-socle spread. By transitivity, this implies that E_{π_o} of order p^a is of order q or $q/(2,q)$. Hence, when p is odd, we have an elation group of order $q \cdot q = q^2$ and when $p=2$, we have an elation group of order at least $q/2 \cdot q/2 = q^2/4$.

Since we have transitivity, each non-socle plane is fixed by some group of order $q/2$ that acts like a Baer group on the non-socle plane and is an elation group of Σ. We know that there is an elation group of order at least $q^2/4$. By the previous lemma, if all of these elation groups are subgroups of the same super elation group E of π_Σ, we are finished.

Hence, we may assume that two of these elation groups of order $q/2$, each of which acts like a Baer group on a non-socle plane have distinct axes when acting as elation groups of Σ. If $q/2 > 2$, then the group generated by elations in Σ is a group isomorphic to $SL(2, 2^a)$, where a divides $2r$, $q = 2^r$. Since there is an elation group of order $q^2/4$, this means that $a \geq 2r - 2$. Let $ka = 2r$. If $k > 1$, then $2r \geq 2a = 4r - 4$ implying that $4 \geq 2r$, so that $r = 1$ or 2. So, if $r > 2$, we obtain $SL(2, q^2)$, generated by elations groups, implying that there is an elation group of order q^2. When $q = 2$, there are exactly two parallelisms and in this case, a derived conical flock spread becomes Desarguesian again. So assume that $q = 4$, then since there is an elation group of order at least $q^2/4 = 4$, we may obtain $SL(2, 4)$ or we obtain an elation group of order q^2 as above. Such a group $SL(2, 4)$ will fix a regulus net R of Σ and fix all Baer subplanes of it. Hence, we have a group $SL(2, 4)$ that fixes a non-socle plane Σ' and a 2-group has order 4. Thus, we must have a Sylow 2-subgroup of order at least 16. However, this would force an elation subgroup of order at least 8, implying that $SL(2, 4^2)$ is generated. This implies that there is an elation group of order 4^2. Hence, in all cases when $q > 2$, there is an elation subgroup of order q^2. This implies that each non-socle plane is fixed by an elation group of order q that acts on the non-socle plane as a Baer group. As before, we may apply Theorems 8 and 9 to conclude that the non-socle planes are isomorphic derived conical flock planes. This completes the proof of the lemma. ■

4 The Socle Spread is Desarguesian.

We assume the same conditions as in the previous section.

Theorem 17. *The socle spread is Desarguesian.*

The proof follows from the following lemmas.

Lemma 7. *Assume that $q^2 - 1$ has a p-primitive divisor u (an element that divides $q^2 - 1$ but does not divide $p^s - 1$, for $q = p^r$ and $s < 2r$). Let U be a Sylow u-subgroup of G. Then U is Abelian.*

Proof. Let U be a Sylow u-group of G of order u^a. Since U has order u^a and u divides $q^2 - 1$ then U fixes at least two components of the socle, for if it fixes zero or one component, then every non-trivial orbit is divisible by u, forcing u to divide $q^2 + 1$ or q^2, where $q^2 + 1$ is the number of components of the affine socle plane. Let g_u be any element of U and assume that g_u fixes a non-zero point on a fixed component. Let X denote the $GF(p)$-subspace pointwise fixed by g_u. Then there is a Maschke complement C left invariant by g_u on which g_u fixes no non-zero point. Hence, it follows that u divides $|C| - 1$, a contradiction unless $C = 0$. This implies that g_u is an affine homology.

Thus, either an element of U is an affine homology or U acts fixed-point-free on any fixed component. Let L and M be two components fixed by U and let $U_{[Z]}$ denote the subgroup of U fixing Z pointwise, where $Z = L$ or M. Then, since any homology group of odd prime power order is cyclic as such a group is a Frobenius complement, it follows that $U_{[L]}U_{[M]}$ is an Abelian subgroup. Hence, $U/U_{[L]}$ acts faithfully as a fixed-point-free subgroup acting on L. Since a fixed-point-free group is a Frobenius complement then $U/U_{[L]}$ is cyclic. But, either any element of order u^{β} normalizing $U_{[L]}$ must centralize $U_{[L]}$ or u divides $u^t - 1$ for some t. Hence, it follows that U centralizes $U_{[L]}$, so it must be that U is Abelian. ∎

Lemma 8. *Any element g_u in $\Gamma L(4, p^r)$ of order a prime p-primitive divisor u is in $GL(4, p^r)$.*

Proof. If u divides r then consider the element

$$\theta : (x_1, x_2, y_1, y_2) \longmapsto (x_1^{p^{r/u}}, x_2^{p^{r/u}}, y_1^{p^{r/u}}, y_2^{p^{r/u}})$$

of order u. This element fixes each vector (x_1, x_2, y_1, y_2), for $x_i, y_i \in GF(p)$. The argument of the previous lemma then would state that θ must fix each element on both $x_1 = 0 = x_2$ and $y_1 = 0 = y_2$, a contradiction. ∎

Remark 1. *G acts as a collineation group on the socle plane Σ. Recall that when we say that G contains an elation, it is meant that the collineation acts on Σ as an elation.*

Lemma 9. *Suppose that G contains an elation with axis L and there is a p-primitive divisor u of $q^2 - 1$. Let U denote a Sylow u-subgroup of the full group acting on the parallelism. If U leaves L invariant and does not centralize the elation subgroup with axis L then the socle is a semifield plane.*

Proof. Let S_p be a Sylow p-subgroup of G, so of order $\geq q$. Let E denote the elation subgroup. Since U then normalizes E, it follows that either U centralizes E or induces a non-trivial fixed-point-free group on E, implying that E has order q^2, that is, the socle is a semifield plane. ∎

Lemma 10. *Suppose that G contains an elation with axis L and there is a p-primitive divisor u of $q^2 - 1$. If a Sylow u-group U of the full group acting on the parallelism leaves L invariant then the socle is a semifield plane.*

Proof. Certainly, U normalizes the elation subgroup E, the full elation group with axis L. By the previous lemma, we may assume that U centralizes the elation group E. Since U fixes two components L and M, U also fixes the images of M under E. Hence, by Theorem 6, there is a Desarguesian affine plane Σ consisting of the components fixed by any fixed element g_u of U of prime order. Moreover, the normalizer of $\langle g_u \rangle$ is a collineation group of Σ in $\Gamma L(2, q^2)$ and the centralizer is in $GL(2, q^2)$. Hence, U is in the centralizer of $\langle g_u \rangle$, and since U fixes three components, it follows that U is a group of kernel homologies of Σ. Assume that the socle is not Σ and let W be a component of $\Sigma - \pi$. It follows that W is a Baer subplane of π that is fixed by U. However, this means that U fixes the unique spread containing W, since U acts on the parallelism. However, it would then be impossible for the group G to act transitively on $q(q+1)$ components. ∎

Lemma 11. *If the socle is a semifield plane and the associated elation group acts as a collineation group of the parallelism then the socle is Desarguesian.*

Proof. When the socle Σ is a semifield plane, let E be an elation group of order q^2 acting on $q(q+1) = q^2 + q$ remaining spreads. Since the group G is transitive, it follows that there must be an elation subgroup E^- of order q that fixes a second spread Σ_2. However, the axis of E, say $x = 0$, is a Baer subplane of the second spread, implying that E^- fixes a Baer subplane pointwise in Σ_2. This implies by Theorem 8 that E^- is regulus-inducing. Hence, Σ is a conical flock plane by Theorem 7. If Σ is not Desarguesian, E^- is normal in the full collineation group of Σ, implying that E^- must be centralized by the p-primitive elements. Since E is elementary Abelian, we may consider E as a $GF(p)$-vector space with the group U acting on E and fixing E^- elementwise (i.e. commuting with E^-). But, then E^- has a Maschke complement E^* of order q such that U fixes E^* and $E = E^- \oplus E^*$. It now follows directly that E^* must be elementwise fixed by U, implying that U commutes with E, contrary to our assumptions. Hence, Σ is Desarguesian. ■

Lemma 12. *Assume that there exists a p-primitive element u and U is a Sylow u-subgroup. Assume that the socle Σ is not Desarguesian.*

Then U fixes exactly two components of Σ.

Proof. If U fixes three components then there is an associated Desarguesian affine plane Π by Theorem 6, consisting of all g_u-invariant line-size subspaces where g_u has order u. But again U is then a subgroup of $\Gamma L(2, q^2)$ acting on Π and a subgroup of $GL(4, q)$ acting on Σ by Lemma 8. Essentially the same proof shows that U is a subgroup of $GL(2, q^2)$ and fixes at least three components of Π. Hence, it follows that U is a subgroup of the kernel homology group of Π, a contradiction as above if Σ is not Π. ■

Lemma 13. *Suppose there is an elation in G and there is a p-primitive divisor. Assume that the group generated by the elations is non-solvable. Then the socle Σ is Desarguesian.*

Proof. By Lemmas 10, 11 and 12, we may assume that U does not fix any elation axis of any nontrivial elation in G and fixes exactly two components. Hence, we have that the elations generate a normal subgroup N, the non-solvable possibilities of which are listed as follows:

(1) $p = 3$ and $SL(2, 5)$,
(2) $SL(2, p^t)$,
(3) $p = 2$ and $S_z(2^e)$, (see Theorems 10 and 11).

We may assume that U fixes exactly two components of Σ and neither can be elation axes. This implies that u must divide the number of elation axes.

In case (1), there are 10 Sylow 3-subgroups. Hence, $u = 5$. Also, U must normalize $SL(2, 5)$ and since the outer automorphism group of $SL(2, 5)$ has order 2, it follows that U centralizes $SL(2, 5)$, implying that U fixes all elation axes, contrary to our assumptions.

In case (2), since U fixes exactly two components neither of which is an elation axis, it follows that u divides $p^t + 1$. Hence, we obtain $SL(2, q)$ or $SL(2, q^2)$ and Σ is Desarguesian in either case (see e.g. Theorem 5).

In case (3), there are $2^{2e}+1$ elation axes and u dividing this number implies that $2^e = q$, so we have $S_z(q)$ acting in its natural action, implying that the socle is a Lüneburg-Tits plane. However, in this case, U must fix two elation axes, contrary to the above. This proves the lemma. ■

Lemma 14. *If there is a p-primitive divisor of q^2-1 then the socle is Desarguesian.*

Proof. Since there are at least two elation axes the group N generated by the elations is a dihedral group D_s of order $2s$, s odd by Theorem 11.

Since we have an elation, we may choose coordinates so that the elation has the form $(x, y) \longmapsto (x, x + y)$. Considering that we have a 4-dimensional vector space over a field K isomorphic to $GF(q)$, we may further decompose $x = (x_1, x_2)$, where $x_i \in K$, for $i = 1, 2$. We similarly choose $y = (y_1, y_2)$. That σ may be so chosen follows from the choice of the axis as $x = 0$ and an image of $y = 0$ as $y = x$, all of which may be easily accomplished using the standard basis theorem. The reader is directed to the authors' text [2] for additional details. So, we note that $\{(0, x_2, 0, y_2); x_1, y_2 \in K\}$, in particular, is left invariant by σ. But this is a 2-dimensional K-subspace; i.e. a 'line' in the projective space $PG(3, K)$. Since a line lies in exactly one spread of the parallelism, it follows that σ fixes this spread which must be distinct from the socle.

We have $p = 2$ and since the order of a Sylow 2-subgroup S_2 is at least q as there is an orbit of length q^2+q, and there is an element of order 2 stabilizing some spread, it follows that the order of S_2 is strictly larger than q. In the case under consideration, there is a unique elation in S_2. Hence, there is a subgroup of order at least q that fixes exactly a 1-dimensional subspace X_L pointwise on the unique component L fixed by S_2. Let u be a prime p-primitive divisor of q^2-1 and let g_u be an element of G of order u. By Lemma 8, $g_u \in GL(4, q)$. Assume that g_u fixes the elation axis L. Furthermore, on the elation axis, suppose that g_u leaves X_L invariant. Then g_u must fix X_L pointwise and using Maschke complements, it follows that g is an affine homology with axis L. Hence, it is impossible for g_u to centralize the elation group E of order 2. But, g_u must normalize E, a contradiction. Hence, if g_u fixes the elation axis, g_u must move X_L. But, now we have an elation group induced by S_2 on L of order at least q and hence exactly q (thinking of the component L as a Desarguesian affine plane of order q). This implies that we have $SL(2, q)$ induced on L.

Now since we have $SL(2, q)$ induced on L, we may use Theorem 5 which states that $SL(2, q)$ acts on the associated plane Σ. The possible planes (of dimension 2) are Desarguesian, Hall, Ott-Schaeffer, since the order is even and the Dempwolff plane of order 16 has kernel $GF(2)$. Since the Ott-Schaeffer plane does not admit elations, the plane is Desarguesian or Hall. Assume that the plane is Hall. Then, the 2-groups of $SL(2, q)$ fix Baer subplanes of Σ pointwise. However, a Baer subplane must also be a 2-dimensional K-space so it is a line of some spread Σ'. However, then there is a group of order q that fixes Σ', implying that there must be a 2-group of order q^2 in the linear translation complement. However, from Lemma 2, there must be an elation group of order q acting in the Hall plane. But when this occurs in a Hall plane, we know that $q = 2$, a contradiction.

Thus, g_u cannot fix the elation axis L. Consider a Sylow 2-subgroup S_2 of order $2q$. Let τ be an involution of S_2 and assume that $\tau \neq \sigma$, the elation of S_2. Then, τ

is a Baer involution and since S_2 is in $GL(4,q)$, τ fixes a 2-dimensional K-subspace, a line of $PG(3,q)$. Hence, $Fix\tau$ is a Baer subplane that must be fixed by σ, since τ must centralize σ, as σ is the unique non-trivial elation in S_2. However, then $\langle\tau,\sigma\rangle$ has order 4 and fixes a spread Σ' containing $Fix\tau$. Therefore, there must be a Sylow 2-subgroup of order at least $4q$, a contradiction. Hence, the only involution in S_2 is σ. That is, S_2 is a 2-group that has a unique involution so that S_2 is either cyclic or generalized quaternion of order 2^{r+1}, where $2^r = q$. We note that if $g \in S_2$ then g fixes a 1-dimensional K-subspace on L and therefore acts like an affine elation of the associated Desarguesian affine plane formed from the spread of 1-K-subspaces on L. Then, g^2 fixes L pointwise so that $g^2 \in \langle\sigma\rangle$, implying that $g^4 = 1$. Hence, either $2^{r+1} = 4$, so that $q = 2$, implying that Σ is Desarguesian or S_2 is generalized quaternion. However, in this case, there is a cyclic subgroup of order 2^r containing σ, so that the quotient by $\langle\sigma\rangle$ is elementary Abelian. Thus, $2^{r-1} \leq 2$ since the quotient group is cyclic and elementary Abelian. So, $q \leq 4$.

Hence, we are finished or the plane has order 16.

All planes of order 16 are known and these are the Desarguesian, Hall and a semifield plane. In the last case, there is a unique elation axis. In the Hall case, there is a linear Baer 2-group of order 4 acting on the plane and there is an elation in the Sylow 2-group containing a Baer 2-group. However, this implies that the Sylow 2-group must have order at least $4q = 16$, since the Baer 2-group must fix a spread Σ' distinct from Σ. Thus, the proof of the lemma is complete. ∎

Lemma 15. *Every elation group of order p in a translation plane of order p^2 is regulus-inducing.*

Proof. Consider the group generated by an element $\sigma : (x,y) \to (x, x+y)$, then the images union the axis of a component $y = xM$ are of the form:

$$x = 0, y = xiM; i \in GF(p),$$

which is clearly a regulus in $PG(3,p)$. ∎

Lemma 16. *If there is an elation in G then either the socle is Desarguesian or $q = 8$.*

Proof. We may assume from the Lemma 14 that either $q = p$ and $p + 1 = 2^a$ or $q = 2^6$. If q is not 8, we are reduced to the consideration of when q is prime. In this setting, since there are elations, and every elation group of order p is regulus-inducing, we have that Σ is a conical flock plane by Theorem 7. Moreover, since any elation must also fix a Baer subplane, it follows that the order of the p-group is strictly larger than p, so it can only be that the order is p^2, since there can be no Baer p-elements by Theorem 15, as there are elations. Now we have a linear group of order p^2 with a regulus-inducing elation group contained in it. Let E denote the elation group with axis L. If E has order p^2 then Σ is a semifield plane which is necessarily Desarguesian since the order is a prime square. Hence, we may assume that E has order p and that a Sylow p-group fixes a 1-dimensional subspace X_L on L pointwise. If L is moved then $SL(2,p)$ is generated by the elations, implying that the plane is Desarguesian by Theorem 5. Thus, L is invariant. If X_L is not invariant then $SL(2,p)$ is induced on L. Using Theorem 5, it follows that $SL(2,p)$ is induced as a collineation group on Σ. Hence, Σ is Desarguesian, Hall, Hering or Walker of order 25. But, since we have elations, only the Desarguesian plane survives.

Thus, X_L and L are both invariant under the full collineation group of the parallelism. The group induced on L is a subgroup of $\Gamma L(2,p) = GL(2,p)$, acting on L as a Desarguesian affine plane of order p. However, X_L is a fixed component and the stabilizer of a component of L has order dividing $p(p-1)^2$. But, we have a group of order divisible by $p^2(p+1)$ fixing L and X_L. Hence, there must be an affine homology group with axis L of order at least $(p+1)/4$. An affine homology cannot commute with any elation of E, implying that $(p+1)/4$ must divide $(p-1)$, a contradiction unless $(p+1)/4$ is 1 or 2. Hence, we have a contradiction unless possibly $p = 3$. But, then the plane must be Desarguesian or Hall and the Hall plane does not contain affine elations. ∎

Remark 2. *If $q = 8$ and it is assumed that G is a subgroup of $GL(4,q)$, then since 3 is a q-primitive divisor of $q^2 - 1$, the above proof may be utilized to show that Σ is Desarguesian, if G contains an elation. But, G must contain an elation.*

Proof. If G contains no elations then each involution is Baer. However, this implies that the order of G is strictly larger than q. But, then Lemma 2 shows that there is an elation in G. ∎

Henceforth, we shall assume that q is not 2 or 8.

Lemma 17. *If Σ is not Desarguesian then the order of a Sylow p-subgroup is q.*

Proof. Let S_p be a Sylow p-subgroup. Then since S_p is assumed to be within $GL(4,q)$, it follows that S_p fixes a component L and induces a subgroup of $GL(2,q)$ on it. If there are no elations then S_p has order less than or equal to q. But, since S_p has order at least q, this completes the proof, so the order of S_p is forced to be q. ∎

Lemma 18. *If Σ is not Desarguesian there are no Baer p-elements or elations.*

Proof. Note that a Baer p-element will fix a spread containing the fixed point space, forcing the Sylow p-group to have order strictly larger than q, so that elations are generated using Lemma 2. ∎

Lemma 19. *If Σ is not Desarguesian, then q is odd and the Sylow p-subgroups are quartic.*

Proof. There is an elementary Abelian subgroup of order exactly q. However, if q is even, any involution is Baer or an elation. Any element of order p in $GL(4,q)$ is either an elation, Baer or quartic. Since there are no Baer or elations, each element of a Sylow p-subgroup is quartic. ∎

Lemma 20. *If Σ is not Desarguesian then Σ is Hering or a Walker plane of order 25 and $SL(2,q)$ is a subgroup of G.*

Proof. Assume that there are at least two quartic groups of orders q with distinct centers. We may then apply Theorem 16.

Thus, assume that there is a unique quartic group or two quartic groups with the same center. In the latter case, if there are two disjoint quartic groups with the same center, the Sylow p-subgroup has order at least q^2, a contradiction. Hence, assume that there is a unique quartic group of order q.

So, a Sylow p-subgroup T is quartic, fixes a unique 1-dimensional $GF(q)$-subspace X pointwise, the center, which lies on a unique component L_X, the axis of T. If there is a unique quartic group then there is an invariant quartic axis L_X and center X. If there is a p-primitive element u then there is an element g_u that fixes X and L_X and hence must be an affine homology with axis L_X but then T must fix L_X and move the coaxis, so that there is an induced elation, a contradiction. If there is not p-primitive element then $q = p$ and $(p + 1) = 2^a$. In this case, there is a linear subgroup H of order 2^{a-1}, which must fix X and fix a Maschke complement Y on L_X. Since $(2^{a-1}, q - 1) = 2$, as $a > 1$, there is a subgroup of H of order divisible by 2^{a-2} that fixes X pointwise. Since the above argument shows that there are no affine homologies, it follows that 2^{a-2} divides $q - 1$. Hence, $a - 2 = 0$ or 1 so that $p + 1 = 4$ or 8, implying that $p = 3$ or 7. Quartic groups do not exist if $p = 3$, so assume that $p = 7$.

Since there kernel group of order $q - 1$ fixes all spreads, we may assume that G has order divisible by $q(q^2 - 1)$, when acting on the associated translation plane π_Σ. Thus, there exists a 2-group of order $2(p+1) = 16$ fixing X and fixing a complement Y on L_X. So, there is a group of order 8 in the linear translation complement. Hence, we have a group of order 4 fixing X pointwise which must fix Y and since there are no homologies in G, 4 must divide $7 - 1$, a contradiction. ∎

Lemma 21. *The socle plane Σ cannot be a Hering plane.*

Proof. Assume that Σ is a Hering plane and the proof of the previous theorem shows that G contains $SL(2, q)$ acting on Σ.

Let T be a quartic group of $SL(2, q)$ of order q and let T fix X pointwise on a component L. We recall that there are exactly $q(q + 1)$ 2-dimensional K-subspaces containing X, none of which are components of Σ. Each such space is in exactly one non-socle spread. There are $q + 1$ such quartic groups and $q + 1$ fixed points spaces. Let the fixed point spaces be denoted by X_i, for $i = 1, 2, ..., q + 1$. Then, there is a unique spread $\Sigma_{k,j}$ distinct from Σ containing $\langle X_k, X_j\rangle$, for $k \neq j$, as a component.

Choose any two distinct subspaces X_k and X_j, fixed pointwise, respectively, by the quartic groups T_k and T_j. Then there is a subgroup $H_{k,j}$ of order $q - 1$ that normalizes T_k and T_j, since $SL(2, q)$ acts doubly transitive on the quartic axes.

Assume that $\Sigma_{k,j} = \Sigma_{k^*,j^*}$, for $\{k^*, j^*\} \neq \{k, j\}$. That is, assume that the two spreads are equal, then we would also have corresponding groups $H_{k,j}$ and H_{k^*,j^*} that fix X_k and X_j, and X_{k^*} and X_j^*, respectively. If the two spreads are equal then they contain $\langle X_k, X_j\rangle$ and $\langle X_{k^*}, X_{j^*}\rangle$ as components. Consider $\langle H_{k,j}, H_{k^*,j^*}\rangle$ a subgroup of $SL(2, q)$, and note that this subgroup leaves $\Sigma_{k,j}$ invariant. There cannot be a p-element in $\langle h, h^*\rangle$, since the order of a Sylow 2-subgroups of G is q and G is transitive.

Assume that $(q - 1)/2$ is not 2. Therefore, by the structure of subgroups of $PSL(2, q)$, it follows that $\langle H_{k,j}, H_{k^*,j^*}\rangle\, Z\,(SL(2, q))\,/Z(SL(2, q))$ is a subgroup of a dihedral group of order dividing $2(q-1)$. Both $H_{k,j}$ and H_{k^*,j^*} contains the central involution and induce groups of order $(q-1)/2$ on the remaining set of $(q-1)$ quartic axes, permuting these in two orbits of length $(q-1)/2$. The cardinality of $H_{k,j}H_{k^*,j^*}$ is $2((q-1)/2)^2 = (q-1)^2/2$ and induces in the quotient a group of order $(q-1)^2/4$ so that $q-1$ divides 8, implying that $q = 3, 5, 9$. But, since there are no quartic elements in characteristic 3, this forces $q = 5$.

Hence, assume that $q = 5$.

Since we know that $\{k^*, j^*\} \cap \{k, j\} = \phi$, as $\langle X_k, X_j \rangle$ and $\langle X_{k^*}, X_{j^*} \rangle$ are distinct components of the same spread, then the four 1-dimensional K-subspaces are on four distinct components of Σ. There are exactly $q+1 = 6$ components that are quartic axes. Since $H_{k,j}$ fixes exactly two components and has two orbits of length 2, it follows that $\langle H_{k,j}, H_{k^*,j^*} \rangle$ has an orbit of length at least four and hence must have an orbit of length 6. Thus, we have a group of order 12 in G that fixes a non-socle spread. This forces the group G to have order divisible by 3.5.6, so there is a 3-group of order 9. Hence, there is a element g of order 3 in G that fixes a quartic axis L. If this element g leaves the quartic center X in L invariant, it fixes X pointwise and fixes a complement pointwise, implying that there is an elation in Σ, a contradiction. Hence, g must move X on L. But, then $SL(2, q)$ must be generated on L, a contradiction by Theorem 5 as this would force the plane to be Hall. There are exactly $\begin{pmatrix} q+1 \\ 2 \end{pmatrix} = (q+1)q/2$ spreads $\Sigma_{k,j}$. Note that $SL(2, q)$ will permute this set of $q(q+1)/2$ spreads transitively. Suppose that $SL(2, q)$ is not normal in G. Then there is another Sylow p-subgroup not in $SL(2, q)$, which must be quartic, a contradiction. Hence, G permutes the set of $q+1$ quartic centers and hence permutes the spreads $\Sigma_{k,j}$. But, this means that G cannot act transitively on the non-socle spreads, a contradiction. This shows that the plane cannot be a Hering plane. ∎

Lemma 22. Σ *cannot be a Walker plane of order* 25.

Proof. Now assume that $q = 5$ and the plane is a Walker plane of order 25. We note that one of the three Walker planes of order 25 is actually a Hering plane. But, here we are considering the group action to be reducible but not completely reducible (there are two groups isomorphic to $SL(2, 5)$ acting on the Hering plane/Walker plane of order 25). Hence, we have $SL(2, 5)$ acting on Σ, where the 5-elements are quartic.

Then the group $SL(2, 5)$ is reducible and furthermore, we know from the structure of the three Walker planes of order 25 that there is a unique 2-dimensional subspace W that is $SL(2, 5)$-invariant and W is a component L. Consider an element g of $SL(2, 5)$ of order 3. Then it is known that there is an associated Desarguesian spread Δ such that g fixes each line of Δ (see the section in Lüneburg [21] on the three Walker planes of order 25). Moreover, the three Walker planes of order 25 share $5, 8$ or 11 lines of Δ. In any case, this means that g fixes any non-socle spread of the parallelism containing a line of $\Delta - \Sigma$. Since Σ is Walker, g fixes at least one non-socle spread. Hence, 3^2 divides the order of G, since G is transitive on 5.6 non-socle spreads. Let S_3 be a Sylow 3-subgroup. We know that the order of a Sylow 5-subgroup is exactly 5 and the Sylow 5-subgroups are quartic. If $SL(2, 5)$ is not normal then we are back to the Hering case. Hence, S_3 must normalize $SL(2,5)$ (the group generated by the 5-elements) so that there is an element g^* of order 3, which centralizes $SL(2,5)$. Note that the normalizer of $SL(2,5)$ is contained in $\Gamma L(2,5)$. Also, a group H_3 of order 9 is Abelian, so centralizes g, implying that H_3 is a collineation group of the Desarguesian plane whose spread is Δ. This implies that H_3 is a subgroup of $\Gamma L(2,5^2)$. Since g^* centralizes $SL(2,5)$, g^* fixes each 1-dimensional $GF(5)$-subspace and since 3 does not divide $5-1$, it follows that g^* is a homology with axis L and coaxis say M. Since L is $SL(2,5)$-invariant but M is not, this implies by André's theorem [1] that there is an elation in G with axis L. Hence, we have an elation group E of order 5, which is necessarily regulus-inducing, implying that Σ is the Walker plane of order 25 that corresponds to a flock of a quadratic cone. But, also 3 is a 5-primitive divisor and we have a 3-group of order 3^2. Any 3-group must centralize the full elation group by Lemma 10, a contradiction (alternatively, also we know that there are no elations in G). ∎

5 The Main Theorems.

Theorem 18. *Let $q = p^r$, for p a prime . Let $\mathcal{P}$ be a parallelism in $PG(3,q)$ admitting an automorphism group G that fixes one spread (the socle) and acts transitively on the remaining spreads. Assume that the Sylow p-subgroups of G are in $PGL(4,q)$ or, if $q = 8$, that G itself is a subgroup of $PGL(4,q)$.*

Then

(1) the socle is Desarguesian,

(2) the associated group G contains an elation group of order q^2 acting on the socle and

(3) the remaining spreads of the parallelism are isomorphic derived conical flock spreads.

Theorem 19. *Let $\mathcal{P}$ be a parallelism in $PG(3,q)$, for $q \neq 8$, admitting an automorphism group G that fixes one spread (the socle) and acts transitively on the remaining spreads. If $q = p^r$, for p a prime and $(r,q) = 1$, then the following occurs:*

(1) the socle is Desarguesian,

(2) the associated group G contains an elation group of order q^2 acting on the socle and

(3) the remaining spreads of the parallelism are isomorphic derived conical flock spreads.

Proof. The previous section shows that the socle plane Σ is Desarguesian and the remarks of the previous section show part (2) and (3).This then completes the proof of our main theorem. The second listed theorem follows if $(r,q) = 1$, for $q = p^r$, since a Sylow p-subgroup of order p^t will necessarily lie within $GL(3,q)$. ∎

6 Final Remarks.

We have shown that we may determine the spreads in a deficiency one partial parallelism in $PG(3,q)$ provided there is a collineation group G of $\Gamma L(4,q)$ that acts transitively on the spreads, provided the Sylow p-subgroups are linear, except when $q = 8$.

In this case, we have shown that there is always an elation group of order q^2 in the spread extending (uniquely) the partial parallelism to a parallelism and the axis of this group L is fixed by the full group acting on the parallelism. The 'socle' plane Σ becomes Desarguesian and the full group on Σ has order divisible by $q^2(q+1)$, where $q = p^r$, for p a prime, and the group now must sit within $P\Gamma L(2, q^2)$.

If the group is sharply 2-transitive on the components of $\Sigma - \{L\}$, so of order $q^2(q^2-1)$, it is possible to determine the spreads without any assumption on the group G. This is done in Johnson and Pomareda [20] and the reader is directed to this article for the 'shape' of the spreads.

Furthermore, not all transitive deficiency one partial parallelisms admit such collineation groups as the group could be much smaller in order. For example, there are examples where the non-socle planes are derived Knuth semifield planes and in this case, the group is a subgroup of $GL(4, q)$ where the order is not $q^2(q^2-1)$. The reader is directed to Johnson [14] for the construction of these parallelisms.

In Johnson and Pomareda [19], there is a construction of what are called 'nearfield parallelism-inducing' groups that provide examples of groups in $\Gamma L(4, q)$ but not in $GL(4, q)$ that fix a Desarguesian spread and act transitively on the remaining spreads of the parallelism. In this case, the non-socle spreads are Hall. However, the Sylow p-subgroups (there is only one) are in $GL(4, q)$.

Hence, our assumptions on the group G are valid in all of the known examples. Of course, what we have not considered are groups where it is not assumed that the Sylow p-subgroups are linear.

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Dipartimento di Matematica Università di Lecce Via Arnesano 73100 Lecce, Italy;
email : Biliotti@ilenic.unile.it

Mathematics Dept., Caledonian University,Cowcadden Road,Glasgow, Scotland;
email : V.Jha@gcal.ac.uk

Mathematics Dept., University of Iowa, Iowa City, Iowa 52242
email : njohnson@math.uiowa.edu