Parameters of translation generalized quadrangles

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Abstract

In this note we show that the algebraic parameters of a linear translation generalized quadrangle are not restricted. This is done with a free construction of fourgonal families on vector spaces. Secondly we prove that a compact translation generalized quadrangle can only have the topological parameters $(1,t)$, $(2,2)$, $(3,4t)$ or $(7,8t)$ for $t \in \mathbb{N}$. This is achieved by determining the possible dimensions of the elements of continuous partial spreads which satisfy a certain planarity condition.

1 Introduction

1.1 Elation generalized quadrangles.

A *generalized quadrangle* is an incidence geometry $\mathcal{G} = (P, L, F)$ with *point* set P, *line* set L and *flag* set $F \subseteq P \times L$ such that for every anti-flag $(p, l) \in (P \times L) \setminus F$ there is a unique flag $(\pi(p, l), \lambda(p, l)) \in F$ such that $(p, \lambda(p, l))$ and $(\pi(p, l), l)$ are flags; furthermore it is required that any point and any line is incident with at least three lines or points, respectively. Usually we simply write (P, L) for $\mathcal{G}$ if F is given by the element relation. As for any incidence geometry we can regard $\mathcal{G}$ as a graph in the following way.

If P and L are disjoint (this is no restriction), then $(V, E) := (P \,\dot{\cup}\, L, \{\{p, l\} : (p, l) \in F\})$ is a graph, the *incidence graph* of $\mathcal{G}$. Denote by $d : V^2 \to \mathbb{N}_0 \cup \{\infty\}$ the

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graph theoretic *distance*. For $n \in \mathbb{N}$ and $v \in V$ set $D_n(v) := \{w \in V : d(v,w) = n\}$. The number $\sup d(V^2)$ is called the *diameter* of (V, E). A set of l distinct vertices $v_1, \ldots, v_l = v_0$ satisfying $\{v_{i-1}, v_i\} \in E$ for $i = 1, \ldots, l$ is called a *circle* of length l, and the least number $l \geq 3$ (or infinity) such that there is such a circle is called the *girth* of (V, E).

The language of graph theory (see [7] for notions not defined here) provides a different way of defining generalized quadrangles (see [34, Section 1.3]): an incidence geometry is a *generalized n-gon* if its incidence graph has diameter n and girth $2n$ and if there is a circle of length $2n + 2$. Circles of length $2k$ in a (bipartite) graph are also called *ordinary k-gons*. With this terminology the latter definition reads as follows: any two vertices are contained in an ordinary n-gon, there are no k-gons for $k < n$, and there is an ordinary $(n+1)$-gon. This is where the name generalized polygon comes from. A generalized 3-gon is the same thing as a projective plane. The notion of generalized polygons was introduced by Jacques Tits, and it first appeared in [33].

The quadrangle $\mathcal{G}$ is called an *elation generalized quadrangle* if there is a point $\infty \in P$, the *center*, and a group E of *collineations* (i.e., bijections that leave the flag set F invariant) which fixes every line of $D_1(\infty)$ and acts sharply transitively on the set $D_4(\infty)$ of vertices at distance 4 of ∞. If the *elation group* E is abelian, then $\mathcal{G}$ is called a *translation generalized quadrangle*. For a line $l \in D_1(\infty)$ the subgroups $E_l := \{e \in E : e$ fixes every line in $D_2(l)\}$ and $T_{E_l} := \{e \in E : e$ fixes every point in $D_1(l)\}$ are defined; it is easy to see that $E_l \leq T_{E_l}$ and that $E_l \cap T_{E_{l'}} = \{0\}$ for different lines $l, l' \in D_1(\infty)$; so these groups determine l, and T_{E_l} is well-defined. Furthermore these subgroups constitute a fourgonal family in the following sense; see [15] and [2] as well as [3] for the independence of the axioms.

1.2 Fourgonal families.

Let $(E, +)$ be a group, $\mathcal{O}$ be a set of non-trivial subgroups of E and let T be a map from $\mathcal{O}$ to the set of subgroups of E mapping each $A \in \mathcal{O}$ to a subgroup T_A containing A properly. The triple $(E, \mathcal{O}, T)$ is called a *fourgonal family* if the following axioms hold.

(FF1) $T_A + B = E$ for all $A, B \in \mathcal{O}$ with $A \neq B$.

(FF2) $T_A \cap B = \{0\}$ for all $A, B \in \mathcal{O}$ with $A \neq B$.

(FF3) $(A + B) \cap C = \{0\}$ for all $A, B, C \in \mathcal{O}$ with $A \neq C \neq B$.

(FF4) $E = T_A \cup \bigcup_{B \in \mathcal{O}} (B + A)$ for each $A \in \mathcal{O}$.

Note that usually the group E is written multiplicatively and the subgroup T_A is written as A^*. We deviate from this standard notation, because we want to allude to examples arising in vector spaces; see Section 1.4.

The above process can be reversed; i.e., a fourgonal family defines an elation generalized quadrangle: in fact it is easy to check that $(E, \mathcal{O} + E)$ is an affine quadrangle, which can be completed to a generalized quadrangle as described below.

1.3 Affine quadrangles.

The following axiom system is a generalization of the one for affine planes. Here two types of parallel relations and two parallel axioms are needed. These correspond to the two types of vertices at infinity. Let $\mathcal{A} = (P, L, F)$ be an incidence geometry and define the relations

$$g \parallel h :\Longleftrightarrow \forall\, p \in D_1(g), q \in D_1(h) : d(p, h) = d(q, g) \quad \text{and}$$
$$g \parallel\!| \, h :\Longleftrightarrow d(g, h) \in \{0, 6\}.$$

for $g, h \in L$. The geometry $\mathcal{A}$ is called an *affine quadrangle* if the following axioms hold.

(A1) Some point row has at least 2, and some line pencil has 3 elements.
(A2) The girth of $\mathcal{A}$ is greater than 6 and $d(L^2) \leq 6$.
(A3) For every $(p, h) \in P \times L$ there is a unique $l \in D_1(p)$ such that $l \parallel h$.
(A4) For $(g, h) \in L^2$ with $g \nparallel h$ there is a unique $l \in D_2(g)$ such that $l \parallel\!| \, h$.

If (P, L) is a generalized quadrangle and $\infty \in P$, then $(D_4(\infty), D_3(\infty))$ is an affine quadrangle. Furthermore every affine quadrangle can be obtained in this way; i.e., every affine quadrangle has a completion to a generalized quadrangle; see [29], and cf. [27] and [32] for variations of this definition. This is completely analogous to the case of affine and projective planes, and as there the vertices at infinity are given by parallel classes and a further vertex ∞.

1.4 Linearity.

If $\mathcal{G}$ is a translation generalized quadrangle (i.e., the elation group E is abelian), then $K(\infty) := \{\alpha \in \mathrm{End}(E) : \alpha(E_l) \subseteq E_l \text{ for all } l \in D_1(\infty)\}$ is called the *kernel* of $\mathcal{G}$. It is an integral domain (see [12, 3.8]), and it has been conjectured in [13, 4.2] that $K(\infty)$ is always a skew field, but this could only be shown in some special cases: for finite translation generalized quadrangles, for planar translation generalized quadrangles, for translation generalized quadrangles with a strongly regular translation center and for compact connected translation generalized quadrangles, which will be defined in Section 4; see [26, 8.5.1], [13, 4.7], [13, 4.9] and [14, 5.4], respectively. The translation generalized quadrangle is called *linear*, if the kernel $K(\infty)$ is a skew field. In this case E is a vector space over the kernel, and all subgroups E_l and T_{E_l} are subspaces of E. By (FF1) and (FF2) all subspaces E_l and and all spaces T_{E_l}/E_l have the same dimensions $s, t \in \mathbb{N} \cup \{\infty\}$, respectively, and we have $\dim E = 2s + t$. By (FF3) we have $s \leq t$. The numbers s and t are called the *parameters* of the linear translation generalized quadrangle $\mathcal{G}$. Note that for finite generalized quadrangles the numbers $|K(\infty)|^s$ and $|K(\infty)|^t$ are often called the parameters.

A fourgonal family $(E, \mathcal{O}, T)$ such that the group E is a vector space and the subgroups in $\mathcal{O}$ and $T(\mathcal{O})$ are subspaces of E is called *linear*. In the case understood best the subspaces in $\mathcal{O}$ are one-dimensional and the ones in $T(\mathcal{O})$ are hyperplanes; then the axioms of a fourgonal family simply mean that $\mathcal{O}$ is an ovoid in the projective space over E with tangent hyperplanes $T(\mathcal{O})$. In this case the related translation generalized quadrangle is called a Tits-quadrangle. If any three subspaces of $\mathcal{O}$ span E, then $\mathcal{O}$ is called a pseudo-oval and these objects were treated in [22]. The translation generalized quadrangles which define fourgonal families that are pseudo-ovals are called planar in [13, 4.6]. These are related to Laguerre planes.

2 Construction of translation generalized quadrangles

The following theorem shows that there are many translation generalized quadrangles. Note that for finite translation generalized quadrangles the parameters s and t are further restricted by $t \leq 2s$ by Higman's inequality, which holds for all finite generalized quadrangles; see [26, 1.2.3].

We denote the Graßmann space of all k-dimensional subspaces of a vector space E by $\mathcal{G}_k(E)$.

Theorem 2.1. *Let $s, t \in \mathbb{N}$ with $s \leq t$, and let E be a vector space over an infinite skew field such that $\dim E = 2s+t$. Then there is a linear fourgonal family $(E, \mathcal{O}, T)$ satisfying $\mathcal{O} \subseteq \mathcal{G}_s(E)$ and $T(\mathcal{O}) \subseteq \mathcal{G}_{s+t}(E)$.*

Furthermore finitely many of the elements of $\mathcal{O}$ and their images under T can be chosen arbitrarily subject to the conditions (FF1) *to* (FF3)*.*

The proof relies on the following lemma.

Lemma 2.2. *Let E be a vector space over a skew field K. Assume $\mathcal{U}$ is a set of proper subspaces of E with bounded finite dimensions and $|\mathcal{U}| < |K| + 1$.*

(a) *Then $\bigcup \mathcal{U} \neq E$.*

(b) *If X is a subspace with $X \cap \bigcup \mathcal{U} = \{0\}$, then there is a complement Y of some $U \in \mathcal{U}$ such that $X \subseteq Y$ and $Y \cap \bigcup \mathcal{U} = \{0\}$.*

Proof. Part (a) is Theorem 3 of [28]. For (b) we employ Zorn's lemma to get a maximal subspace Y of E such that $X \subseteq Y$ and $Y \cap \bigcup \mathcal{U} = \{0\}$. Then an application of (a) to E/Y and $\{(U+Y)/Y : U \in \mathcal{U}\}$ shows that Y has indeed a complement in $\mathcal{U}$. ∎

The following proof of Theorem 2.1 can also be used to show that (FF1) is independent of the axioms (FF2) to (FF4). To that end one simply has to choose the parameters s and t such that $2s+t < \dim E$, and the proof remains valid; cf. [3].

Proof of Theorem 2.1. Let us call a partial map $T : \mathcal{G}_s(E) \to \mathcal{G}_{s+t}(E)$ a partial fourgonal family if the axioms (FF1) to (FF3) are satisfied for $(E, \operatorname{dom} T, T)$. We regard such a map $T : \operatorname{dom} T \to \mathcal{G}_{s+t}(E)$ as a subset of $\mathcal{G}_s(E) \times \mathcal{G}_{s+t}(E)$. In the following we will define a partial fourgonal family by transfinite recursion. For background on ordinals and cardinals see [9]. The sets

$$\mathcal{Z}_{q,T} := \{q + r : r \in \operatorname{dom} T\} \cup \{T_q\} \quad \text{and} \quad \mathcal{Z}_T := \bigcup_{q \in \operatorname{dom} T} \mathcal{Z}_{q,T}$$

defined for any partial fourgonal family T and $q \in \operatorname{dom} T$ play a crucial role in the proof.

Let κ be the cardinality of E, which is the cardinality of the skew field over which E is defined. Then $E \times \mathcal{G}_s(E)$ also has this cardinality; so let $f : \kappa \to E \times \mathcal{G}_s(E)$ be a bijection, and set $(x^\sigma, p^\sigma) := f(\sigma)$ for $\sigma \in \kappa$.

Let $\alpha \in \kappa$. Assume that T^σ is a partial fourgonal family for every $\sigma \in \alpha$ such that $T^\sigma \subseteq T^{\sigma'}$ if $\sigma \in \sigma' \in \alpha$, assume that $T^{\{\}}$ is finite and assume that we have the following condition for $(x^\sigma, p^\sigma) = f(\sigma)$ and all $\sigma \in \alpha$.

$$(*) \quad \begin{array}{l} \text{If } p^\sigma \in \operatorname{dom} T^\sigma, \text{ then } x^\sigma \in \bigcup \mathcal{Z}_{p^\sigma, T^\sigma}, \text{ and} \\ \text{if } p^\sigma \notin \operatorname{dom} T^\sigma, \text{ then } p^\sigma \cap \bigcup \mathcal{Z}_{T^\sigma} \neq \{0\}. \end{array}$$

This condition guarantees that in every 'step' σ of the construction, any element of $\mathcal{G}_s(E)$ is either an element of $\operatorname{dom} T^\sigma$ or will not become one in any future step.

Now set $(x, p) := (x^\alpha, p^\alpha) = f(\alpha)$, and define a partial fourgonal family $T' := \bigcup_{\sigma \in \alpha} T^\sigma$. If p intersects a member of $\mathcal{Z}_{T'}$ non-trivially, then define $T'' := T'$. If p has trivial intersection with all members of $\mathcal{Z}_{T'}$, then there is a subspace $T_p \in \mathcal{G}_s(E)$ by Lemma 2.2(b) such that $T'' := T' \cup \{(p, T_p)\}$ is a partial fourgonal family.

In order to ensure condition $(*)$ for $\sigma = \alpha$ we may need to enlarge the partial fourgonal family T'' further. If $p \in \operatorname{dom} T''$ and $x \in \bigcup \mathcal{Z}_{p,T''}$ or if $p \notin \operatorname{dom} T''$, then condition $(*)$ is satisfied, and we set $T^\alpha := T''$.

If $p \in \operatorname{dom} T''$ and $x \notin \bigcup \mathcal{Z}_{p,T''}$, then we have $\langle x, p\rangle \nsubseteq U$ for all $U \in \mathcal{Z}_{T''}$ by (FF3). So $\langle x, p\rangle \cap U$ is a proper subspace of $\langle x, p\rangle$ for all $U \in \mathcal{Z}_{T''}$, and we can choose $y \in \langle x, p\rangle \setminus \bigcup \mathcal{Z}_{T''}$ by Lemma 2.2(a), since the cardinality of $\mathcal{Z}_{T''}$ is smaller than κ. Now Lemma 2.2(b) yields a subspace $q \in \mathcal{G}_s(E)$ which has trivial intersection with all subspaces in $\mathcal{Z}_{T''}$ and satisfies $\langle y\rangle \subseteq q$. We have $x \in p+q$. A further application of Lemma 2.2(b) yields a complement $T_q \supseteq q$ of all subspaces in $\operatorname{dom} T''$, and condition $(*)$ is satisfied for $T^\alpha := T'' \cup \{(q, T_q)\}$.

By transfinite recursion we have defined a partial fourgonal family T^α satisfying condition $(*)$ for all $\alpha \in \kappa$. Finally set $T := \bigcup_{\alpha \in \kappa} T^\alpha$. Then T is a partial fourgonal family, and (FF4) follows from the surjectivity of f: Let $p \in \operatorname{dom} T$ and $x \in E$. Then there is a $\sigma \in \kappa$ such that $(p, x) = f(\sigma)$. Since $p \notin \operatorname{dom} T^\sigma$ would contradict (FF2) or (FF3) for T by $(*)$, we have $p \in \operatorname{dom} T^\sigma$, and $(*)$ yields (FF4). So $(E, \operatorname{dom} T, T)$ is a fourgonal family. ■

Corollary 2.3. *For every $s, t \in \mathbb{N}$ with $s \leq t$ and every infinite skew field K there is a linear translation generalized quadrangle with parameters (s, t) and kernel K.*

Proof. Let E be a left vector space of dimension $2s+t$ over K, and use the notation of the previous proof. For $\mathcal{O} \subseteq \mathcal{G}_s(E)$ consider the right vector subspace $K_\mathcal{O} := \{\alpha \in \operatorname{End}(E) : \alpha(A) \subseteq A \text{ for all } A \in \mathcal{O}\}$ of $\operatorname{End}(E)$. We will show that there is a finite subset $\mathcal{O} \subseteq \mathcal{G}_s(E)$ such that (FF3) is satisfied and such that $K_\mathcal{O} = \mathrm{id}_E \cdot K$. Then we can use Lemma 2.2(b) to define T on $\mathcal{O}$ such that (FF1) and (FF2) are satisfied, and an application of Theorem 2.2 yields a translation generalized quadrangle with kernel $K_\mathcal{O} \cong K$ and parameters (s, t).

Now let $\mathcal{O}$ be a finite subset of $\mathcal{G}_s(E)$ such that $K_\mathcal{O}$ has minimal dimension, and let $\alpha \in K_\mathcal{O}$. Then α leaves all subspaces $A \in \mathcal{G}_s(E)$ invariant which have trivial intersection with $X := \bigcup\{B+C : B, C \in \mathcal{O}\}$, because otherwise there were an A such that $\alpha \notin K_{\mathcal{O}\cup\{A\}}$ and this subspace would have smaller dimension than $K_\mathcal{O}$. This implies that any one-dimensional subspace $\langle v\rangle$ of E which has trivial intersection with X is left invariant; indeed by a twofold application of Lemma 2.2(b) there are two s-dimensional subspaces whose intersection is $\langle v\rangle$ and both of which have trivial intersection with X. Now again by Lemma 2.2(b) there are $\dim E+1$ such invariant one-dimensional subspaces in general position. This implies $\alpha \in \mathrm{id}_E \cdot K$. ■

Given $\dim E+1$ one-dimensional subspaces in general position it is also possible to construct a finite family $\mathcal{O} \subseteq \mathcal{G}_s(E)$ which satisfies (FF3) and contains the given one-dimensional subspaces in its lattice span. But this seems more complicated.

Remark 2.4. *As mentioned earlier it is an open problem, whether every translation generalized quadrangle is linear. The above construction method might extend to modules and thus solve this problem in the negative. However, if R is a commutative integral domain such that every submodule of R^n has a complement, then R is a field. Thus Lemma* 2.2(b) *on which the construction relies heavily does not hold for modules.*

3 Parameters of continuous spreads

Let E be a vector space over a skew field. A set $\mathcal{S}$ of non-trivial proper subspaces of E is called a *partial spread* on E if the intersection of different elements of $\mathcal{S}$ is trivial (note that sometimes it is assumed that the subspaces of a partial spread have the same dimension). A partial spread $\mathcal{S}$ of E is called a *spread* if $\bigcup \mathcal{S} = E$. A partial spread $\mathcal{S}$ is called *planar at* $U_0 \in \mathcal{S}$, if $U + U_0 = E$ for all $U \in \mathcal{S} \setminus \{U_0\}$. A *planar* partial spread $\mathcal{S}$ is a partial spread which is planar at every element of $\mathcal{S}$.

A planar spread defines an affine translation plane $(E, \mathcal{S}+E)$, and every translation plane is obtained in this way; see [23]. For linear translation generalized quadrangles there is also a connection to spreads, which is not quite as strong. Every such quadrangle defines a linear fourgonal family $(E, \mathcal{O}, T)$, which defines a spread $\{(A+B)/A : B \in \mathcal{O}\setminus\{A\}\} \cup \{T_A/A\}$ on E/A for every $A \in \mathcal{O}$; it is planar at T_A/A. We will use this spread later in order to obtain properties of the quadrangle.

Now assume that E is a real finite-dimensional vector space. Then the Graßmann space $\mathcal{G}_k(E)$ becomes a compact manifold of dimension $k(\dim E - k)$. Let $\mathcal{S}$ be a spread on E which is planar at U_0. Then $\mathcal{S} \setminus \{U_0\} \subseteq \mathcal{G}_{\dim E - \dim U_0}(E)$ carries a natural topology, and we call $\mathcal{S}$ *continuous*, if $E \setminus U_0 \to \mathcal{S} \setminus \{U_0\}, x \mapsto X \ni x$ is a continuous map. If $\mathcal{S}$ is planar, then $\mathcal{S} \subseteq \mathcal{G}_k(E)$ for $2k = \dim E$. So in the planar case $\mathcal{S}$ carries a topology.

Our next aim is to give a class of examples of continuous spreads. Let $e_1, \ldots, e_k$ be the standard basis of $\mathbb{R}^k$. The real Clifford algebra Cl_k is the 2^k-dimensional vector space over $\mathbb{R}$ with basis

$$\{e_{i_1} e_{i_2} \ldots e_{i_l} : l = 0, \ldots, k \text{ and } 1 \leq i_1 < \cdots < i_l \leq k\}$$

together with the unique multiplication making Cl_k into a real algebra and satisfying $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ for all $i, j = 1, \ldots, k$, $i \neq j$. For a skew field $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ let $F(n)$ denote the $\mathbb{R}$-algebra of $(n \times n)$-matrices over F. The Clifford algebras can be described with such matrix algebras, as the following table shows for $k = s + 8r$ with $0 \leq s \leq 7$; see [20, I4.3].

s	0	1	2	3
Cl_k	$\mathbb{R}(16^r)$	$\mathbb{C}(16^r)$	$\mathbb{H}(16^r)$	$\mathbb{H}(16^r) \oplus \mathbb{H}(16^r)$
module	$\mathbb{R}^{16^r}$	$\mathbb{C}^{16^r}$	$\mathbb{H}^{16^r}$	$\mathbb{H}^{16^r}$

s	4	5	6	7
Cl_k	$\mathbb{H}(2 \cdot 16^r)$	$\mathbb{C}(4 \cdot 16^r)$	$\mathbb{R}(8 \cdot 16^r)$	$\mathbb{R}(8 \cdot 16^r) \oplus \mathbb{R}(8 \cdot 16^r)$
module	$\mathbb{H}^{2 \cdot 16^r}$	$\mathbb{C}^{4 \cdot 16^r}$	$\mathbb{R}^{8 \cdot 16^r}$	$\mathbb{R}^{8 \cdot 16^r}$

The last row shows the respective irreducible modules of Cl_k; see [20, I5.8]. Their real dimensions can be calculated as $2^{\delta(k)}$ where

$$\delta(k) := \left|\{l \in \{1, \dots, k\} : l \equiv 1, 2, 4, 8 \mod 8\}\right| \quad \text{for all } k \in \mathbb{N}_0.$$

This means that there is a representation $\mathrm{Cl}_k \to \mathrm{End}(U_0)$ of real algebras for a real vector space U_0 if and only if $2^{\delta(k)} \mid \dim U_0$. The map δ is a monotone function and $\delta(k)$ is roughly $k/2$; more precisely we have $k - 1 \leq 2\delta(k) \leq k + 2$ for all $k \in \mathbb{N}_0$ and $k \leq 2\delta(k)$ for even k.

Consider the $\mathbb{R}$-linear span of the algebra elements $e_0 := 1, e_1, \dots, e_k$. Its non-zero elements are invertible, as $(r + \sum_{i=1}^k r_i e_i)(r - \sum_{i=1}^k r_i e_i) = r^2 + \sum_{i=1}^k r_i^2$. Let $\sigma : \mathrm{Cl}_k \to \mathrm{End}(U_0)$ be a representation of real algebras. Then

$$\{\langle (e_i, \sigma(e_i)(v)) : i = 0, \dots, k\rangle : v \in U_0\} \cup \{\{0\} \times U_0\}$$

is a spread of $\langle e_0, \dots, e_k\rangle_{\mathbb{R}\text{-linear}} \times U_0$, which is planar at U_0. The continuity of σ implies that this spread is continuous.

Using the representation σ again we will now describe a cross section. Denote by $\mathrm{V}_k^*(U_0)$ the set of linearly independent k-tuples of U_0^k. As an open subset the space $\mathrm{V}_k^*(U_0)$ is a manifold of the same dimension. (The compact submanifold of orthonormal vectors, the Stiefel manifold, is a deformation retract, as can be seen with Gram–Schmidt orthogonalization.) The projection $\pi_1 : \mathrm{V}_k^*(U_0) \to U_0 \setminus \{0\}$ to the first component is a locally trivial bundle. Now the map $v \mapsto (v, \sigma(e_1)(v), \dots, \sigma(e_k)(v))$, which was used above to define a continuous spread from the representation σ, is a continuous cross section of the bundle $\pi_1 : \mathrm{V}_{k+1}^*(U_0) \to U_0 \setminus \{0\}$.

As we have seen, it is possible to define a spread as well as a cross section from a representation of a Clifford algebra. This is no coincidence. Given a continuous spread we will define a continuous cross section in order to apply the following celebrated result by Adams; see [1].

Theorem 3.1. *There is a continuous cross-section of* $\pi_1 : \mathrm{V}_k^*(\mathbb{R}^n) \to \mathbb{R}^n \setminus \{0\}$ *if and only if* $2^{\delta(k-1)} \mid n$.

Note that this result is better known as an existence result about continuous vector fields on spheres: there are $k - 1$ linearly independent vector fields on the sphere $\mathbb{S}_{n-1}$ if and only if $2^{\delta(k-1)} \mid n$. For the connection of Stiefel manifolds and vector fields see [11, Section 1].

Let $\mathcal{S}$ now be an arbitrary spread of E which is planar at $U_0 \in \mathcal{S}$, and set $\mathcal{S}^* := \mathcal{S} \setminus \{U_0\}$. Let $U \in \mathcal{S}^*$, and define the map

$$\alpha : \mathcal{S}^* \to \mathrm{L}(U, U_0) \quad \text{satisfying} \quad \{\alpha(X)(u)\} = (X - u) \cap U_0$$

where $\mathrm{L}(U, U_0)$ denotes the set of linear maps from U to U_0; it is well-defined, because $\mathcal{S}$ is a partial spread which is planar at U_0. Note that $\alpha(U) = 0$ and that $\alpha(X)$ is injective for $X \in \mathcal{S}^* \setminus \{U\}$. We will be particularly interested in the map

$$\alpha(\,\cdot\,)(u) : \mathcal{S}^* \to U_0 \quad \text{and its inverse} \quad \beta_u : U_0 \to \mathcal{S}^*, u_0 \mapsto X \ni u + u_0$$

for $u \in U \setminus \{0\}$; it is bijective, because $\mathcal{S}$ is a spread.

Now assume that $\mathcal{S}$ is continuous. Then β_u is continuous. By definition the map $\alpha(\,\cdot\,)(\,\cdot\,)$ is also continuous. Now let $\{u_1, \dots, u_k\}$ be a basis of U, and define the continuous map

$$\gamma : U_0 \setminus \{0\} \to \mathrm{V}_k^*(U_0),\ u \mapsto \Big(\alpha(\beta_{u_1}(u))(u_1), \dots, \alpha(\beta_{u_1}(u))(u_k)\Big).$$

Note that the first coordinate of $\gamma(u)$ equals u; so γ is a cross-section of the bundle $\pi_1 : \mathrm{V}_k^*(U_0) \to U_0 \setminus \{0\}$. This implies that $2^{\delta(k-1)} \mid \dim U_0$ by Theorem 3.1. On the other hand there are examples in these dimensions, as we have seen above. So we have proved the following theorem.

Theorem 3.2. *There is a continuous spread of $\mathbb{R}^n$ which is planar at an element of dimension l if and only if $2^{\delta(n-l-1)} \mid l$.*

We have the following well known corollary; see [5, p. 47], for example.

Corollary 3.3. *Compact planar spreads of $\mathbb{R}^n$ only exist for $n \in \{2, 4, 8, 16\}$.*

Proof. Let $\mathcal{S}$ be a compact planar spread of $\mathbb{R}^{2k}$. Then mapping a non-zero element $v \in \mathbb{R}^{2k}$ to the subspace of $\mathcal{S}$ containing v is continuous, because this map has a closed graph and $\mathcal{S}$ is compact. Thus Theorem 3.2 yields $2^{\delta(k-1)} \mid k$, which implies for $k \geq 6$ that

$$k \geq 2^{\delta(k-1)} \geq 2^{k/2-1} = 2^2(1+1)^{k/2-3} \geq 2^2(1 + k/2 - 3) = 2k - 8.$$

Thus $k \leq 8$. If $k \geq 2$, then k is even. Finally the case $k = 6$ is not possible, because $2^{\delta(5)} = 8 \nmid 6$. Hence $k \in \{1, 2, 4, 8\}$. ■

4 Compact translation generalized quadrangles

A *compact generalized polygon* is a generalized polygon $\mathcal{G} = (P, L, F)$ such that P and L carry compact topologies and the flag space F is a closed subset of $P \times L$. In this case the geometric operations are continuous, because they have closed graphs. There is a certain converse: if P and L are locally compact, locally connected and non-discrete, then $\mathcal{G}$ is in fact a compact polygon with connected point and line spaces; see [17, 2.5.5]. Knarr and Kramer have shown that compact n-gons whose point and line spaces have positive finite topological dimensions exist only for $n = 3$, 4 or 6; see [16] and [17, 3.3.6]. The point-rows and line-pencils are homology spheres. Their respective dimensions are called the *topological parameters* of the compact polygon. For $n = 3$ and 6 it has been shown that these parameters are equal and divide 8 and 4, respectively; see [21] and [17, 3.3.6].

For a compact connected generalized quadrangle with topological parameters $s \leq t$ it has been conjectured that $(s,t) = (2,2)$, $(4,5)$ or $2^{\delta(s-1)} \,|\, s+t+1$. Examples for these parameters are provided by quadrangles defined from hermitian forms of Witt index 2 or by representations of real Clifford algebras; see [19, 10.3] and [4] for a nice introduction to the subject. Examples for the parameters $(1,t)$ for $t \in \mathbb{N}$ and $(2,2)$ are also given by Tits-quadrangles and Laguerre planes, respectively. These constitute translation generalized quadrangles; see Section 1.4. In [24] Markert proves the following somewhat weaker result.

Theorem 4.1. *For a compact generalized quadrangle with topological parameters (s,t) such that $m := \min\{s-1, t-s\} \geq 1$ we have $2^{\delta(m)} \mid s+t+1$.*

We will use this theorem together with the result about continuous spreads to obtain restrictions on the topological parameters of a compact translation generalized quadrangle.

Before we start some remarks about this theorem are in order. It originates from work by Stolz about Dupin hypersurfaces which are generalizations of isoparametric hypersurfaces; see [31]. An isoparametric hypersurface is a closed connected submanifold of $\mathbb{S}_n$ of codimension one such that all its principal curvatures, i.e., the eigenvalues of the shape (or Weingarten) operator, are constant. Münzner showed in [25, Satz 1(b)] that the number g of distinct principal curvatures equals 1, 2, 3, 4 or 6 and that the multiplicities are determined by two of the g multiplicities. For the case $g = 4$ Stolz showed that the two multiplicities s and t are related in the following fashion: if $s \leq t$, then $(s,t) = (2,2)$, $(4,5)$ or $2^{\delta(s-1)} \mid s+t+1$. In the last case examples are given by representations of Clifford algebras.

Immervoll has shown in [10] that every isoparametric hypersurface with $g = 4$ defines a smooth generalized quadrangle. Now Markert's approach (she attributes some of the ideas in her work to Kramer and Stolz) is to adapt Stolz's proof to generalized quadrangles. His proof splits into two parts. In the first part it is shown that, if $X := D_2(p) \wedge D_2(l)$ desuspends m-times for $m \leq s-1$, then one has $2^{\delta(m)} \,|\, s+t+1$. This part of the proof can be modified using Knarr's embedding (see [16, 2.8]), and the statement remains valid in the general case. In the second part it is shown that the space X desuspends $(s-1)$-times. In the smooth case $D_2(p)$ is a Thom space of a vector bundle over $\mathbb{S}_s$ and the Bott periodicity theorem can be used. In the general case the vector bundle is only known to be a bundle, so only Freudenthal's suspension theorem is available. As a consequence one only obtains that the space X desuspends $\min\{s-1, t-s\}$ times. We prove the following result.

Theorem 4.2. *The topological parameters of a compact connected translation generalized quadrangle are $(1,t)$, $(2,2)$, $(3,4t)$ or $(7,8t)$ for $t \in \mathbb{N}$.*

Proof. Let (P,L) be a compact translation generalized quadrangle with center ∞, translation group E and parameters (s,t) such that $s > 1$ and $(s,t) \neq (2,2)$. By [14, 5.4 and 5.5] the kernel of (P,L) is $\mathbb{R}$ and for the groups of the corresponding fourgonal family we have $E_l \cong \mathbb{R}^s$ and $T_{E_l}/E_l \cong \mathbb{R}^t$. In particular $s \leq t$. The case $s = t$ only occurs for $s \in \{1,2\}$ (see [6, 3.6] or [30, 1.2]), because then (P,L) is a planar translation generalized quadrangle and defines a $2s$-dimensional locally compact Laguerre plane; see [14, 3.7]. Thus we have $1 < s < t$ and an application of Theorem 4.1 yields $2^{\delta(m)} \mid t+s+1$ where $m := \min\{s-1, t-s\} \geq 1$.

Let $(E, \mathcal{O}, T)$ be the fourgonal family defined by (P, L). We can identify $D_4(\infty)$ with E and $D_3(\infty)$ with $\mathcal{O} + E$. For $A \in \mathcal{O}$ the set $\mathcal{S} := \{(A+B)/A : B \in \mathcal{O} \setminus \{A\}\} \cup \{T_A/A\}$ defines a spread which is planar at T_A/A. The map $(E/A) \setminus (T_A/A) \to \mathcal{S} \setminus \{T_A/A\}, A + x \mapsto (\lambda(x, A) - \pi(x, A) + A)/A$ is well defined, since A consists of translations of the quadrangle. So the spread is continuous, as π and λ are continuous. We apply Theorem 3.2 and obtain $2^{\delta(s-1)} \mid t$, since $\dim E = 2s + t$, $\dim A = s$ and $\dim T_A = s + t$.

This implies $2^{\delta(m)} \mid s + 1$. The parameter s is odd, as $m \geq 1$. If $s = 5$, then $8 = 2^{\delta(4)} \mid t$, and if $s = 9$, then $16 = 2^{\delta(8)} \mid t$. So in both cases $t - s \geq 3$. Thus $m \geq 3$ and we have $8 = 2^3 \mid s + 1$, a contradiction. We have excluded the cases $s = 5$ and $s = 9$. Now if $s \geq 11$, we have

$$t \geq 2^{\delta(s-1)} \geq 2^{\frac{s-1}{2}} = 2^5(1+1)^{\frac{s-1}{2}-5} \geq 2^5\left(1 + \frac{s-1}{2} - 5\right) = 16s - 144 \geq 2s$$

which implies $t - s > s - 1$, so $m = s - 1$ and we have $2s \leq 2^{\delta(s-1)} \leq s + 1$, again a contradiction. ∎

As mentioned above a large class of generalized quadrangles can be constructed from representations of Clifford algebras. We want to determine whether there are translation generalized quadrangles among these. The point and line spaces of a Clifford quadrangle are disjoint compact submanifolds of $\mathbb{S}_{2n-1} \subseteq \mathbb{R}^{2n}$. A point and a line are incident if their scalar product is $1/\sqrt{2}$. The map $-\mathrm{id}_{\mathbb{R}^{2n}}$ induces a collineation which maps any vertex to a non-incident vertex. Now a translation generalized quadrangle with two elation centers at distance 4 is a Moufang quadrangle as shown in [12, Proposition 5.2]. So every Clifford quadrangle which is a translation generalized quadrangle (up to duality) is a Moufang quadrangle. And indeed, there is a single Moufang quadrangle with parameters $s = 3$ or $s = 7$ which is a translation generalized quadrangle (for a list of compact connected Moufang quadrangles see [8, Table 1]). It has parameters $(3, 4)$ and is the dual of the smallest hermitian quadrangle over $\mathbb{H}$; see [34, 4.9.8 and Table 5.1] or [34, p. 213, bottom]. The above argument also applies to the octonion hermitian quadrangles $H_n\mathbb{O}$ which are related to Clifford quadrangles; for a definition see [18]. But as none of them is a Moufang quadrangle, none of them is (up to duality) a translation generalized quadrangle. The author does not know whether there are compact connected translation generalized quadrangles with parameters $s = 3$ and $t \geq 8$ or with $s = 7$.

Finally we remark that Theorem 4.2 cannot be improved on just by sharpening the conclusion of Theorem 4.1 to $2^{\delta(s-1)} \mid s + t + 1$, as the obtained parameters satisfy this condition.

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