

Primitive symmetric spaces

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Abstract

In this note, we study primitive symmetric spaces, namely discrete symmetric spaces whose main group acts primitively. We give three families of examples of such spaces, and we prove that every primitive symmetric space belongs to one of these families.

1 Introduction

1.1

In the appendix of [Bru77], it is proved that *if G is a simple Lie group which is connected and of finite center, then any maximal compact subgroup K of G is abstractly maximal*. In other words, this means that the group G acts primitively on the symmetric space $X = G/K$, namely that there is no non trivial partition of X which is invariant under the action of G . In this note, we are interested in a possible converse for this statement, and propose an answer to the following question : *What can we say about the group G if we assume that it acts primitively on a symmetric space $X = G/K$?* It turns out that this question may be tackled in a very general context, which is much wider than the strict framework of differential geometry. Let us introduce some definitions in order to make the statements more precise.

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1.2

We define a **discrete symmetric space** to be a set of points X with a mapping $\phi : X \rightarrow \text{Sym}(X)$ satisfying the following conditions:

- (i) x is a fixed point of ϕ_x ,
- (ii) ϕ_x has order 2,
- (iii) $\phi_x \circ \phi_y \circ \phi_x(z) = \phi_{\phi_x(y)}(z)$.

Each permutation ϕ_x is called the **symmetry** corresponding to the point x . The group G generated by all of the symmetries is called the **main group** of the discrete symmetric space (X, ϕ) , and the group D generated by all of the products of two symmetries is called the **transvection group**. A (discrete) symmetric space is called **primitive** if its main group acts primitively ; in particular G is transitive.

This definition of a discrete symmetric space is actually obtained from the usual definition of a symmetric space (see for example [Loo69], p. 63) by just omitting the requirement that X is a manifold. The concept so obtained is very rough ; primitivity is introduced as a possible way of refinement.

1.3

Observe, as a consequence of the aforementioned result of Brun, that a symmetric space, in the usual sense, which is of non-compact type and whose main group is a simple Lie group, must be primitive. Nevertheless, it should be noted that a usual symmetric space may admit a partition in blocks of imprimitivity, given for example by an equivariant fibration when it exists, or by pairs of antipodal points in the case of the sphere. However, it follows from the definition that such blocks are necessarily of empty interior.

Before stating the main result, we still need additional terminology.

1.4

A **morphism** between two discrete symmetric spaces (X, ϕ) and (Y, ψ) is a mapping $\alpha : X \rightarrow Y$ such that $\alpha \circ \phi_x = \psi_{\alpha(x)} \circ \alpha$ for every $x \in X$. **Automorphisms** and **isomorphisms** are defined as usual. If $X' \subset X$ is invariant under $\phi_{x'}$ for all $x' \in X'$ then $(X', \phi|_{X'})$ is called a **subspace** of (X, ϕ) .

The main result of this paper is the following proposition. The proof of it is naturally in the spirit of the O'Nan-Scott theorem (see for example Chapter 4 in [DM96]). However, note that the groups considered here are possibly infinite.

1.5

Main theorem. *A group G is isomorphic to the main group of a primitive symmetric space if, and only if, G possesses an involution σ and satisfies the following properties:*

- (i) $\mathbf{Z}(G) = 1$,
- (ii) $\langle \sigma^G \rangle = G$,
- (iii) $\mathbf{C}_G(\sigma)$ is a maximal subgroup of G ,

where σ^G denotes the conjugacy class of σ in G .

For such groups, one of the following assertions holds.

(P1) G is simple;

(P2) G is a semi-direct product $G = \langle \sigma \rangle \rtimes D$, where D is simple and σ is an outer automorphism of order 2 of D ;

(P3) G is a semi-direct product $G = \langle \sigma \rangle \rtimes D$, where $D = S \times S$ is a direct product of two copies of a non-abelian simple group S ; the action of σ on D is described by

$$\sigma : D \rightarrow D : (g, h) \mapsto (h, g).$$

Moreover, for each abelian simple group D , there is, up to isomorphism, a unique primitive symmetric space with main group of type (P2) and transvection group isomorphic to D .

Similarly, for each non-abelian simple group S , there is, up to isomorphism, a unique primitive symmetric space (X, ϕ) with main group of type (P3) and transvection group isomorphic to $S \times S$. Furthermore, every primitive symmetric space with transvection group isomorphic to S , is isomorphic to a subspace of (X, ϕ) .

2 Preliminaries

In this section, we collect preliminary observations, which will be used in the proof of the main theorem, given in the next section. Throughout, we keep the notation of §1.2.

2.1

Lemma. *The mapping $\phi : X \rightarrow \text{Sym}(X)$ is injective.*

Proof. Fix a point $e \in X$ and define $\Delta = \{x \in X \mid \phi_x = \phi_e\}$. Then it readily follows that Δ is a block, namely that $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$ for each $g \in G$. Therefore, the primitivity of G implies that Δ is reduced to a single point, as was to be proved. ■

Since it is clear by definition that the set of all symmetries is a conjugacy class of involutions of the main group G , this lemma allows us to identify each point with the corresponding symmetry. Axiom 1.2(iii) insures that the structure of discrete symmetric space is included in the group structure of G . More precisely, we have the following result.

2.2

Lemma. *Let G be the main group of a primitive symmetric space (X, ϕ) , and fix a point $e \in X$. Let $Q = \{\phi_x \mid x \in X\} \subset G$. Then Q is a conjugacy class of involutions of G . By defining ψ_x to be the restriction to Q of the conjugation by ϕ_x in G , we have $(X, \phi) \simeq (Q, \psi)$. Moreover $\mathbf{Z}(G) = 1$ and $\mathbf{C}_G(\phi_e)$ is a maximal subgroup of G .*

Conversely, let G be a group and suppose that G possesses an involution σ and satisfies Properties (i), (ii) and (iii) as in the statement of the main theorem. For each $x \in X$, define ϕ_x to be the restriction to X of the conjugation by x in G . Then (X, ϕ) is a primitive symmetric space, and its main group is isomorphic to G .

Proof. Assume that G is the main group of the primitive symmetric space (X, ϕ) . Axiom 1.2(iii) and Lemma 2.1 insure that the action of G on X is equivalent to its action on Q by conjugation, and thus that $(X, \phi) \simeq (Q, \psi)$. It follows that $\mathbf{Z}(G)$ is the kernel of the action of G on X , whence $\mathbf{Z}(G) = 1$. The maximality of $\mathbf{C}(G)(\phi_x)$ for $x \in X$ is a consequence of the primitivity of G (see [Hal76], Theorem 5.6.1 pp. 64-65).

The converse statement of the lemma follows by similar arguments. \blacksquare

Thus, a primitive symmetric space is always identifiable with a conjugacy class of involutions in the main group. The following result highlights subsets of the transvection group which may be canonically identified with the symmetric space as well.

2.3

Lemma. Fix a point $e \in X$. Define $t_x = \phi_x \phi_e \in D$. The set $\{t_x | x \in X\}$ generates D . Moreover, we have the following equivalence:

$$\phi_x(y) = z \Leftrightarrow t_x t_y^{-1} t_x = t_z. \quad (*_1)$$

Proof. The first statement is a direct consequence of the definition of D and the following equality:

$$\phi_x \phi_y = \phi_x \phi_e \phi_{\phi_e(y)} \phi_e.$$

For the second statement, we compute that $t_x t_y^{-1} t_x = t_{\phi_x(y)}$. The result now follows from Lemma 2.1. \blacksquare

Lemma 2.2 makes it easy to produce examples of primitive symmetric spaces, starting from certain abstract groups.

2.4

Example 1. Let G be a simple group, and suppose that G possesses an involution σ such that $\mathbf{C}_G(\sigma)$ is a maximal proper subgroup of G . Clearly, G is non-abelian, whence $\mathbf{Z}(G) = 1$. Moreover, a simple group is generated by anyone of its nontrivial conjugacy classes. Therefore, it follows from 2.2 that the conjugacy class of σ in G has a canonical structure of primitive symmetric space, whose main group coincides with the transvection group and is isomorphic to G . The main group is thus of **type (P1)**, with the notation of the main theorem.

It should be pointed out that not all of the simple groups possess an involution with maximal centralizer. Indeed, the group $Alt(5)$ is already a counter-example.

On the other hand, it may be checked (see for example [CCN⁺85]) that each sporadic finite simple group possesses such an involution, except the Mathieu Groups M_{23} and M_{24} .

2.5

Example 2. Let D be a simple group, and suppose that D possesses an outer automorphism τ of order 2. Set $G := \langle \tau \rangle \rtimes D$ and suppose also that $C_G(\tau)$ is a maximal subgroup of G . It is easy to see that the conjugacy class X of τ in G generates G . Now it is a consequence of Lemma 2.2 and Lemma 2.6 below, that X has a canonical structure of primitive symmetric space. The transvection group of it is isomorphic to D , the main group is isomorphic to G , and we have $[G : D] = 2$. Here, the main group has **type (P2)**.

Notice also that a sufficient condition for the maximality of $C_G(\tau)$ in G is that $Fix(\tau) = C_D(\tau)$ is a maximal subgroup of D .

2.6

Lemma. *Let D be a simple group of order at least 3, and let τ be an involutory automorphism of D . Set $G := \langle \tau \rangle \rtimes D$. Then $Z(G) = 1$ if, and only if, τ is an outer automorphism of D .*

Proof. If D is abelian, then it is cyclic of odd prime order, and it possesses an unique involutory automorphism τ , which is outer and given by $g \mapsto g^{-1}$. One easily computes that $Z(G) = 1$ as required.

Suppose now that τ is inner, namely that it corresponds to the conjugation by an involution $t \in D$. Then, it is easily seen that $(\tau, t) \neq 1$ is central in G . Conversely, suppose that (σ, g) is central in G , where $\sigma \in \{1, \tau\}$ and $g \in D$. Since D is not abelian, we must have $\sigma = \tau$. Expressing the fact that (τ, g) centralizes $(1, h)$ for each $h \in D$, one obtains $h^\tau = ghg^{-1}$, which means that τ is inner. ■

2.7

Example 3. In [Loo69] (see p. 65), a canonical process for producing symmetric spaces from Lie groups, is given. This construction has a straightforward adaptation to our situation.

Let X be a group of order at least 3, and define $\phi : X \rightarrow Sym(X)$ by

$$\phi_x(y) = xy^{-1}x. \quad (*_2)$$

A direct computation shows that (X, ϕ) is a discrete symmetric space. Let G be its main group and D its transvection group. We have the following.

2.8

Lemma. *(X, ϕ) is primitive if, and only if, X is simple. In that case, we have $[G : D] = 2$, and either X is abelian, G has **type (P2)** and $D \simeq X$, or X is*

non-abelian, G has **type (P3)** and $S \simeq X$.

Proof. Assume X is not simple. Let N be a nontrivial normal subgroup of X . Then for all $x, y \in X$, we have $\phi_x(Ny) = xy^{-1}Nx = Nxy^{-1}x$, and the cosets of N are thus permuted among themselves by G . This contradicts the primitivity of X .

Now assume that X is simple. Choose $1 \in X$ as base point and define $t_x = \phi_x\phi_1$. We have $t_x(y) = xyx$, and by Lemma 2.3, the t_x 's generate D .

In a first case, suppose that X is abelian. Then it is cyclic of odd prime order, and $t_x(y) = x^2y$. Therefore, D consists of all of the left translations of X , and thus $D \simeq X$, namely D is simple. Since X has no nontrivial subgroups, it follows that D is primitive, and so is also G . Here, G has even order, while the order of D is odd, whence D is a proper subgroup of G and we have indeed $[G : D] = 2$.

In a second case, suppose that X is not abelian. Notice that

$$t_{x^{-1}y^{-1}}t_x t_y(z) = (x^{-1}y^{-1}xy)z$$

for each $z \in X$. Therefore, $L = \langle t_{x^{-1}y^{-1}}t_x t_y | x, y \in X \rangle \leq D$ consists of all of the left translations of X by elements of the derived group X' . Since X is simple and non-abelian, we have $X = X'$ and all of the left translations of X are contained in L . Notice also that $L \triangleleft D$. Similarly, $R = \langle t_{x^{-1}y^{-1}}t_y t_x | x, y \in X \rangle \triangleleft D$ is the group of all of the right translations of X . One easily checks that $L \cap R = 1$, $LR = D$ and $L \simeq X \simeq R$ whence $D = L \times R \simeq X \times X$. Suppose now that $\Delta \subset X$ is a proper block of imprimitivity for G which contains $1 \in X$. Then Δ is invariant under the left translations by its elements, and hence it is a subgroup of X . On the other hand, the group $D_1 = \{g \in D | g(1) = 1\}$ also stabilizes Δ . But it is clear that if $z \in X$ then $c_z : X \rightarrow X : x \mapsto zxz^{-1}$ belongs to $D = L \times R$ since $c_z = l_z \circ r_{z^{-1}}$, where l_z (resp. r_z) denotes the left (resp. right) multiplication by z . Therefore, the invariance of Δ under D_1 implies that the subgroup Δ of X is normal. Since X is simple and $\Delta \neq X$, we have $\Delta = 1$ and so (X, ϕ) is primitive. Finally, notice that $L^{\phi_1} = R$, whence L is not normal in G , while we already know it is normal in D . Hence, D is proper subgroup of G and we have indeed $[G : D] = 2$. This completes the proof. \blacksquare

2.9

Notice that it is a consequence of $(*_1)$ in Lemma 2.3 and $(*_2)$ in 2.7 that every primitive symmetric space with transvection group isomorphic to X , is isomorphic to a subspace of (X, ϕ) .

3 Proof of the main theorem

The first assertion of the theorem is a consequence of Lemma 2.2.

For the other assertions, we suppose that G is the main group of a primitive symmetric space (M, ϕ) with transvection group D .

Assume that G has not type (P1), namely that G is not simple. Let N be a nontrivial normal subgroup of G . Let $x \in X$. Since $N \neq 1$ is normal in G , it does not stabilize x and hence, N must be transitive on X by the primitivity of G . Moreover $\phi_x \notin N$ because $(\phi_x)^G$ generates G and N is properly contained in G . By the transitivity of N , we have $G = \langle \phi_x \rangle N$, using again the fact that $(\phi_x)^G$ generates G . Therefore

$$G/N = \langle \phi_x \rangle N/N \simeq \langle \phi_x \rangle / (N \cap \langle \phi_x \rangle) = \langle \phi_x \rangle.$$

This shows that N has index 2 in G . But now, $\phi_y \phi_z \in N$ for all $y, z \in X$ and thus $N = D$ is the transvection group of G . Hence, we have proved that the transvection group is the only nontrivial normal subgroup of G .

Suppose now that the transvection group D is abelian. Then for all $x, y, z \in X$, we have $\phi_z \phi_y \phi_z \phi_x = \phi_z \phi_x \phi_z \phi_y$ and so $(\phi_y \phi_z)^{\phi_x} = \phi_z \phi_y$. This implies that $g^{\phi_x} = g^{-1}$ for each $g \in D$ and each $x \in X$. Therefore, each subgroup of D is normal in G , and so D has no nontrivial subgroups by the last assertion of the previous paragraph. Hence D is simple. Obviously, D has order at least 3, whence D is cyclic of odd prime order. Since G has even order, we have $[G : D] = 2$ and G is of type (P2), with $\sigma = \phi_e$ for some fixed point $e \in X$. The fact that in this case, the discrete symmetric space is uniquely determined by its transvection group follows from the fact that a cyclic group of odd prime order has a unique automorphism of order 2. Moreover, for each abelian simple group D of order at least 3, we have already constructed a discrete symmetric space with transvection group isomorphic to D (see 2.7).

We now assume that D is not abelian.

If the transvection group D is simple then Lemma 2.6 shows that ϕ_x is an outer automorphism of D for each point x , and thus that G has type (P2), with $\sigma = \phi_e$ for some fixed point $e \in X$.

We now suppose that D is not simple. Let L be a nontrivial normal subgroup of D . Since D is the only non trivial normal subgroup of G , we see that L is not normal in G . On the other hand, D has index 2 in G and normalizes L , which shows that L has exactly one distinct conjugate R in G . Clearly, $R \triangleleft D$. Moreover, we have $L \cap R \triangleleft G$ and so $L \cap R = 1$. Similarly, as $\{L, R\}$ is a conjugacy class of subgroups of G , we have $LR \triangleleft G$ and so $LR = D$. This shows that $D = L \times R$. Since D is not abelian, $L \simeq R$ is not abelian.

Now let $S \neq 1$ be a normal subgroup of L . Since S is centralized by R , it is normal in $D = L \times R$, and we may repeat the arguments of the preceding paragraph to obtain that $D = S \times T$ where $T \triangleleft R$ is the unique subgroup of G distinct to S and conjugate to it in G . Hence we must have $S = L$ and $T = R$, which means that L and R are isomorphic copies of a non-abelian simple group S .

We have thus shown that $D = L \times R$, and that conjugation by ϕ_x switches L and R for each $x \in X$. Since $g \in D$ fixes the point x if, and only if, $g^{\phi_x} = g$, we deduce that $L \cap G_x = 1 = R \cap G_x$, namely that both L and R act semi-regularly on X . On the other hand, for $x \in X$, the intersection $L(x) \cap R(x)$ of the orbits of x under L and R respectively, is a block of imprimitivity for the action of G on X . We must thus have $L(x) \cap R(x) = X$, and both L and R are sharply transitive on X .

We now fix a point $e \in X$. Each element $d \in D$ can be written in a unique way as a product $d = gh^{\phi_e}$ for some $g, h \in L$. Therefore, by choosing $\sigma = \phi_e$ and by

writing gh^{ϕ_e} as (g, h) , we have $(g, h)^\sigma = (h, g)$.

Moreover, for each $g \in L$, ϕ_e commutes with gg^{ϕ_e} , which means that $gg^{\phi_e}(e) = e$ and so that $g^{\phi_e}(e) = g^{-1}(e)$. Therefore, for all $x, y \in L$ we have

$$\begin{aligned}\phi_{x(e)}(y(e)) &= x\phi_e x^{-1}y(e) \\ &= x(x^{-1}y)^{\phi_e}(e) \\ &= xy^{-1}x(e).\end{aligned}$$

This shows that (X, ϕ) is isomorphic to the primitive symmetric space constructed on the simple group L as in 2.7, and hence that if the main group has type (P3), then the primitive symmetric space is determined up to isomorphism by the simple group S . The fact that S is arbitrary in the class of non-abelian simple groups follows from the construction of 2.7.

It just remains to prove that every primitive symmetric space with transvection group isomorphic to S , is isomorphic to a subspace of (X, ϕ) . But this was pointed out in 2.9. ■

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